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On "integration" of non-integrable vector-valued functions

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A new definition of the integral for functions with values in Banach spaces is presented. The new integrability is a weaker property than the Bochner integrability but stronger than the Pettis one. This new definition leads naturally to the notion of the limit set of integral sums, which may be considered as a "generalized integral" for non-integrable functions. This set is shown to be always convex and non-empty when the function has an integrable majorant and the space is separable or reflexive.

Introduction

Let X be a Banach space, $f : [0;1] \to X$ be a bounded function (not necessarily measurable). For every partition of $[0;1]$ into a union of disjoint intervals $\{\Delta_k\}_{k=1}^n$ and every choice of points $t_k \in \Delta_k$ one can define Riemann integral sums $S = S(f, \{\Delta_k\}, \{t_k\}) = \sum_{k=1}^n f(t_k)|\Delta_k|$. One can consider the set $I_R(f)$ of all limit points of Riemann integral sums, when the maximal length of Δ_k tends to zero. If f is not a Riemann integrable function then $I_R(f)$ in a certain sense plays the role of f's "integral". It is known that in a separable space $I_R(f)$ is non-empty [1] and it is a star-set [3]; in many spaces this set is convex for every bounded

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function: finite-dimensional [2], B-convex, in particular, all L_p with $1 < p < \infty$ [6], see also [4]; reflexive "weakly B-convex" [4]. An extensive survey of the contemporary situation in this field can be found in Appendix to [5]. Unfortunately, this convexity theorem does not hold for l_1 ([6]), and characterization of spaces where $I_R(f)$ is convex for every bounded function is still unknown (for example, we don't know the answer for the space c_0 of all vanishing numerical sequences).

Another negative feature of $I_R(f)$ as a "generalized integral" is that it is based on Riemann sums and thus even for a Lebesgue integrable function $I_R(f)$ may consist of more than one point. Moreover, all the theory makes sense for bounded functions only.

The goal of this paper is to introduce a more general concept of Riemann–Lebesgue (RL) integral sums in such a way that for scalar functions RL integrability coincides with the usual Lebesgue integrability, and the set $I(f)$ of limit points has much better properties than $I_R(f)$ does:

1. $I(f)$ is convex for every function.
2. If f has an integrable majorant and the space is reflexive or separable then $I(f)$ is non-empty.
3. $I(f)$ does not change if instead of the strong convergence one considers the weak convergence.

1. The Riemann–Lebesgue integral

In this section we introduce the new notion of integral sums for vector-valued functions and investigate the basic properties of the corresponding integral.

Let X be a Banach space and let $f : [0;1] \to X$ be an arbitrary function from the unit segment $[0;1]$ into the space X. The Lebesgue measure on the segment will be denoted by μ. Let A be a measurable subset of $[0;1]$. Consider Π, a partition of A into a countable number of disjoint measurable subsets: $\Pi = \{\Delta_i\}_{i=1}^{\infty}$, $\bigcup_{i=1}^{\infty} \Delta_i = A$, $\Delta_i \bigcap \Delta_j = \oslash$ for $i \neq j$. Let $T = \{t_i\}$ be a sequence of sampling points, i.e., $t_i \in \Delta_i$. We can construct a formal series $S(f, \Pi, T) = \sum_{i=1}^{\infty} f(t_i)\mu(\Delta_i)$. If this series is absolutely convergent, we call it an (*absolute*) *Riemann–Lebesgue integral sum* of f with respect to Π and T over A. If it is merely unconditionally convergent, we will call it an *unconditional Riemann–Lebesgue integral sum*. In the sequel we will assume that $A = [0;1]$, unless otherwise specified.

It also seems natural to order the set of partitions in the following way. We will say that the partition Π_1 *follows* partition Π_2, or Π_1 *is inscribed into* Π_2 ($\Pi_1 \succ \Pi_2$), if $\Pi_1$is a finer partition, that is, if Π_1 consists of subsets $\{D_i\}_{i=1}^{\infty}$ and Π_2 consists of $\{E_k\}_{k=1}^{\infty}$, then the set of indices $\mathbb{N}$ can be broken into disjoint subsets I_k, $k \in \mathbb{N}$, $\bigcup_{k=1}^{\infty} I_k = \mathbb{N}$ such that $E_k = \bigcup_{i \in I_k} D_i$.

Now we can introduce a new definition of the integral, which we call, for lack of a better term, the *Riemann–Lebesgue* integral.

Definition 1.1. *A function $f : [0;1] \to X$ is called absolutely Riemann–Lebesgue integrable over a measurable set $A \subset [0;1]$ if there exists a point $x \in X$, such that for any $\epsilon > 0$ there exists such a partition Π of A, that for any finer partition $\Gamma \succ \Pi$ and any set of sampling points T, $||S(f,\Gamma,T) - x|| < \epsilon$, and the sum $S(f,\Gamma,T)$ over A converges absolutely. This point x is called then the absolute Riemann–Lebesgue integral of f and denoted as usual, by $\int_A f(t)dt$. In an analogous way, a function f is called unconditionally Riemann–Lebesgue integrable, if we use unconditional integral sums in the definition above. We will also call an absolutely RL integrable function simply RL integrable.*

Obviously, an RL integrable function is also unconditionally RL integrable, while the converse need not to be true. It is true, however, if the space X is finite-dimensional.

It is not hard to see that a Bochner integrable function is also RL integrable, and the values of both integrals coincide.

Lemma 1.2. *Let a countably-valued function $f : [0;1] \to X$ be Bochner integrable. Then it is RL integrable, and the values of both integrals coincide.*

P r o o f. Since the function f is countably-valued, we can write

$$f = \sum_{k=1}^{\infty} x_n \chi_{A_n} ,$$

where A_n are disjoint sets, whose union gives the entire segment $[0;1]$ and $x_n \in X$. Since f is Bochner integrable, and therefore measurable, A_n's are measurable. Consider the partition $\Pi = \{A_n\}_{n=1}^{\infty}$. For any partition $\Gamma \succ \Pi$ and any set of sampling points T, we will have

$$S(f,\Gamma,\Pi) = \sum_{n=1}^{\infty} x_n \mu(A_n) .$$

Since f is Bochner integrable, this series converges absolutely. Therefore the function f is RL integrable and its RL integral equals the sum of the series, i.e., its Bochner integral. ■

Theorem 1.3. *Any Bochner integrable function is also RL integrable, and the values of both integrals coincide.*

P r o o f. Let $f : [0;1] \to X$ be a Bochner integrable function. Take an arbitrary $\epsilon > 0$. There exists a countably-valued function g, approximating $f : ||f(t) - g(t)|| < \epsilon$ for all $t \in [0;1]$. Then $|| \int f d\mu - \int g d\mu || < \epsilon/3$ (Bochner integrals). According to Lemma 1.2, the function g is RL integrable. Therefore, there exists a partition Π, such that for any finer partition $\Gamma \succ \Pi$ and any set of sampling points T, $||S(g, \Gamma, T) - \int g d\mu|| < \epsilon/3$. Since $||f(t) - g(t)|| < \epsilon/3$, $||S(g, \Gamma, T) - S(f, \Gamma, T)|| < \epsilon/3$. Thus, $||S(f, \Gamma, T) - \int g d\mu|| < 2\epsilon/3$, and therefore $||S(f, \Gamma, T) - \int f d\mu|| < \epsilon$. This proves that Bochner $\int f \, d\mu$ is the RL integral of f. ■

The converse, however, is not true in general, but we can guarantee that an RL integrable *real-valued* function is Lebesgue integrable.

Theorem 1.4. *A real-valued RL integrable function is Lebesgue integrable.*

P r o o f. Let $f : [0;1] \to \mathbb{R}$ be a RL integrable function. We will construct a sequence of Lebesgue integrable countably-valued functions that converges to f is measure.

Fix an index n. Since f is RL integrable, there exists such a partition Π, that for any two finer partitions Π' and Π'' and any sets of sampling points T' and T'', we have $|S(f, \Pi', T') - S(f, \Pi'', T'')| < 1/n$. Let's choose Π_n's for all n such that $\Pi_1 \prec \Pi_2 \prec \Pi_3 \prec \dots$.

Let $\Pi_n = \{\Delta_{n,i}\}_{i=1}^{\infty}$. Fix a $t_{n,i} \in \Delta_{n,i}$ in each $\Delta_{n,i}$ arbitrarily. Define f_n as follows:

$$f_n = \sum_{i=1}^{\infty} f(t_{n,i}) \chi_{\Delta_{n,i}} \,.$$

Obviously, f_n is Lebesgue integrable, $\int f_n d\mu = S(f, \Pi_n, T_n)$, where $T_n = \{t_{n,i}\}_{i=1}^{\infty}$.

Let's denote by $\omega_f(A)$ the oscillation of f on a set A, i.e.,

$$\omega_f(A) = \sup\{|f(x_1) - f(x_2)| : \, x_1, x_2 \in A\} \,.$$

It is clear that if $\omega_f(A) \geq a$, then there exist $x_1, x_2 \in A$, such that $|f(x_1) - f(x_2)| > a/2$.

Fix an arbitrary $\eta > 0$. Let's show that the total measure of those subsets Π_n, where the oscillation of f exceeds η is bounded by $2/(n\eta)$, i.e.,

$$\mu(\bigcup_{i: \omega_f(\Delta_{n,i}) > \eta} \Delta_{n,i}) \leq 2/(n\eta) \,. \tag{1.1}$$

Indeed, if this were not so, then by moving sampling points in these $\Delta_{n,i}$'s, we could obtain two integral sums that differ by at least $(\eta/2) \cdot 2/(n\eta) = 1/n$, which is impossible due to the choice of Π_n. Condition 1.1 implies that $\mu^*(\{t :$

$|f(t) - f_n(t)| > \eta\}) < 2/(n\eta) \to 0$ as $n \to \infty$, i.e., $\{f_n\}$ converge to f in measure. Therefore there exists a subsequence $\{f_{n_k}\}$ that converges to f almost everywhere. Note that all f_n have a Lebesgue integrable majorant. Indeed, due to the choice of Π_1, f is bounded on each subset $\Delta_{1,i}$, and

$$\sum_{i=1}^{\infty} \sup_{\Delta_{1,i}} f \cdot \mu(\Delta_{1,i}) < \infty .$$

Therefore the function $F = \sum_{i=1}^{\infty} \sup_{\Delta_{1,i}} f \cdot \chi_{\Delta_{1,i}}$ is a Lebesgue integrable majorant of f_1 as well as all f_n's, since $\Pi_n \succ \Pi_1$ for all n. Thus we can apply the Lebesgue dominated convergence theorem to prove that f is Lebesgue integrable. ∎

To demonstrate that the class if RL integrable functions is indeed strictly larger than that of Bochner integrable, we now present an example of non-measurable RL integrable function.

Example 1.5. Consider the space $l_2([0;1])$. It consists of all such functions $f : [0;1] \to \mathbb{R}$ that $\sum_{t\in[0;1]} |f(t)|^2 < \infty$ (it follows that these functions take non-zero values on countable subsets of $[0;1]$). The norm on $l_2([0;1])$ is given by $||f|| = (\sum_{t\in[0;1]} |f(t)|^2)^{1/2}$. It is easy to see that $l_2([0;1])$ is a non-separable Hilbert space. Its orts are given by $e_t = \chi_{\{t\}}$.

Now consider a function $f : [0;1] \to l_2([0;1])$: $f(t) = e_t$. This function is not measurable, since it is not even "almost separable-valued". However, it is RL integrable. To see that, note that for any partition $\Pi = \{\Delta_i\}$ and any sample points $T = \{t_i\}$,

$$\begin{aligned} ||S(f,\Pi,T)|| &= ||\sum f(t_i)\mu(\Delta_i)|| = ||\sum e_{t_i}\mu(\Delta_i)|| \\ &= (\sum \mu(\Delta_i)^2)^{1/2} \le (\sum d(\Pi)\cdot\mu(\Delta_i))^{1/2} \\ &= \sqrt{d(\Pi)}(\sum \mu(\Delta_i))^{1/2} = \sqrt{d(\Pi)}\,, \end{aligned}$$

where $d(\Pi)$ denotes $\sup\{\mu(\Delta_i)\}$. Now take an arbitrary $\epsilon > 0$. Fix a partition Π with $d\Pi < \epsilon$. Then for any $\Pi' \succ \Pi$, any T', $||S(f,\Pi',T')|| \le \sqrt{d\Pi'} \le \sqrt{d\Pi} \le \epsilon$. Thus f is RL integrable and the integral equals 0.

However, even unconditional RL integrability implies a weaker kind of integrability, namely Pettis integrability. To show this, let us first consider the following simple lemma.

Lemma 1.6. *Let a function f be (unconditionally) RL integrable over a set $A \subset [0;1]$. Then for any measurable subset $B \subset A$, f is (unconditionally) RL integrable over B.*

P r o o f. We will prove the lemma for absolute sums. The proof for unconditional sums is analogous.

Assume that on the contrary, f is not RL integrable over B. Fix an $\epsilon > 0$. Since f is RL integrable over A, there exists a partition $\Pi_A = \{\Delta_i\}$ of A, such that

1) for any finer partition $\Gamma_A \succ \Pi_A$ and any set of sampling points T_A, the series $S(f, \Gamma_A, T_A)$ is absolutely convergent;

2) for any two finer partitions Γ'_A and Γ''_A and any sets of sampling points T'_A and T''_A, $||S(f, \Gamma'_A, T'_A) - S(f, \Gamma''_A, T''_A)|| < \epsilon$.

Consider a partition Π_B of B formed by all non-empty intersections of Δ_i's with B. Since f is not RL integrable over B, there either exists a partition $\Gamma_B \succ \Pi_B$ and a set of sampling points T_B, such that the series $S(f, \Gamma_B, T_B)$ is not absolutely convergent, or there exist two partitions $\Gamma'_B \succ \Pi_B$ and $\Gamma''_B \succ \Pi_B$ of B with their corresponding sets of sampling points T'_B and T''_B, such that $||S(f, \Gamma'_B, T'_B) - S(f, \Gamma''_B, T''_B)|| \geq \epsilon$. In the former case, complement Γ_B by all non-empty intersections of the form $\Delta_i \cap (A \setminus B)$, and T_B by arbitrarily selected sampling points in those intersections, to get a partition $\Gamma_A \succ \Pi_A$ of A with sampling points T_A, such that $S(f, \Gamma_A, T_A)$ is not absolutely convergent (since any subseries of an absolutely convergent series has to be absolutely convergent). In the latter case, complement both Γ'_B and Γ''_B by the same intersections $\Delta_i \cap (A \setminus B)$, and T'_B and T''_B by arbitrarily but identically selected sampling points in those intersections, to get two partitions $\Gamma'_A \succ \Pi_A$ and $\Gamma''_A \succ \Pi_A$ of A with their corresponding sets of sampling points T'_A and T''_A, such that $||S(f, \Gamma'_A, T'_A) - S(f, \Gamma''_A, T''_A)|| = ||S(f, \Gamma'_B, T'_B) - S(f, \Gamma''_B, T''_B)|| \geq \epsilon$. In both cases we arrive at a contradiction, which proves the lemma. ■

Now, let $f : [0; 1] \to X$ be an unconditionally RL integrable function and $F \in X^*$ a continuous linear functional on X. Since $F(S(f, \Pi, T)) = S(F \circ f, \Pi, T)$, it is easy to see that $F \circ f$ is RL integrable and $\int_A F \circ f d\mu = F \int_A f d\mu$ for any measurable $A \subset [0; 1]$. But $F \circ f$ is a real-valued function, therefore its RL integral coincides with its Lebesgue integral. Thus $F \int_A f d\mu$ (RL integral) is the Pettis integral of f over A.

Now we will present an example of a Pettis integrable function which is not unconditionally RL integrable, thus showing that the property of RL integrability lies strictly between the Bochner and Pettis integrabilities.

Example 1.7. Again, consider the non-separable Hilbert space $l_2(\Gamma)$, where Γ has the cardinality of continuum. Let us denote by $\mathcal{P}$ the set of all countable collections of non-negligible disjoint F_σ-subsets of $[0; 1]$. For every partition $\Pi = \{\Delta_i\}_1^\infty$ we can find a $\Pi' \in \mathcal{P}$, $\Pi' = \{\Delta'_i\}_1^\infty$, such that $\Delta'_i \subset \Delta_i$ and $\mu(\Delta_i \setminus \Delta'_i) = 0$. Because of this property, we can identify $\mathcal{P}$ with the set of all partitions for the purposes of creating integral sums . Since the set of all F_σ sets has the continuum cardinality, the set $\mathcal{P}$ has the continuum cardinality too. Therefore there exists a bijective map $\alpha : \mathcal{P} \to \Gamma$. We will now construct a function $f : [0;1] \to l_2(\Gamma)$, with the following properties:

1. $f([0;1]) \subset \{e_\gamma\} \cup \{0\}$;
2. for each e_γ, $f^{-1}(e_\gamma)$ is countable;
3. for each $\Pi \in \mathcal{P}$ there exists an ort e_γ ($\gamma = \alpha(\Pi)$) such that for any element Δ of Π, there is a $t \in \Delta$ with $f(t) = e_\gamma$.

Such function f is quickly seen to be Pettis integrable. Indeed, as $l_2(\Gamma)$ is a Hilbert space, any linear functional F on $l_2(\Gamma)$ is the scalar product with an element of $l_2(\Gamma)$, which, in turn, has a decomposition in terms of e_γ's. But $< f(t), e_\gamma >= 0$ almost everywhere (the scalar product is not equal to zero only in that countable number of points t where $f(t) = e_\gamma$). Therefore, for any $F \in l_2(\Gamma)$, $F \circ f = 0$ a.e., and thus Pettis integral of f equals 0.

However, f is not unconditionally RL integrable. Indeed, consider an arbitrary $\Pi \in \mathcal{P}$, $\Pi = \{\Delta_i\}$. If $\gamma = \alpha(\Pi)$, we can choose the sampling point t_i within each Δ_i, such that $f(t_i) = e_\gamma$ and therefore have $S(f, \Pi, T) = e_\gamma$. Thus, for each partition we can have finer partitions with their integral sums equal to e_γ's, and, by virtue of (2), among these e_γ's there are different ones. This would be impossible if f were unconditionally RL integrable.

Now we will prove that such a function f exists. In order to construct f, we will exercise transfinite induction. The set $\mathcal{P} = \{\Pi\}$ has the continuum cardinality. There exists an ordinal number Ω, which is the smallest ordinal of the continuum cardinality. Therefore, $\mathcal{P}$ can be completely ordered and numbered with ordinal numbers smaller than Ω: $\mathcal{P} = \{\Pi_1, \Pi_2, \ldots, \Pi_\omega, \ldots\}$.

We will construct a collection of disjoint countable subsets $A_\sigma \subset [0;1]$, indexed with ordinals $\sigma < \Omega$, with the following property: for every partition $\Pi_\sigma = \{\Delta_i\}$ with $\sigma < \Omega$, the set A_σ contains exactly one point of each Δ_i. Such construction could be easily accomplished by transfinite induction, since for every σ the set $\bigcup_{\beta<\sigma} A_\sigma$ is of less than continuum cardinality.

Now we set $f(t) = e_{\alpha(\Pi_\sigma)}$ for $t \in A_\sigma$ and $f(t) = 0$ for $t \in [0;1] \setminus \bigcup_{\sigma<\Omega} A_\sigma$. The properties (1)–(3) are evident for this function. ■

We have shown that Bochner integrable, RL integrable, unconditionally RL integrable and Pettis integrable functions, in this order, form a strictly increasing sequence of function classes. However, if we restrict ourselves to separable Banach spaces only, the first two members of the sequence, as well as the last two, coincide.

Theorem 1.8. *Let X be a separable Banach space. Then for X-valued functions RL integrability coincides with Bochner integrability.*

Proof. In view of theorem 1.3 we only need to show that an RL integrable function is Bochner integrable. Let $f : [0;1] \to X$ be such a function. It is easy to see that for any functional $F \in X^*$, $F \circ f$ is a real-valued function, which is RL integrable, and therefore simply Lebesgue integrable. This means that $F \circ f$ is measurable, and therefore f is weakly measurable. Due to X's separability, this is equivalent to strong measurability of f.

Now to prove that f is Bochner integrable, it is sufficient to show that $||f||$ is a Lebesgue integrable function. Let's find an integrable majorant for $||f||$. Since f is RL integrable, there exists a partition $\Pi = \{\Delta_i\}_{i=1}^{\infty}$, such that for any finer partition $\Pi' \succ \Pi$ and any set of sampling points $T = \{t_i\}_{i=1}^{\infty}$, the series $S(f, \Pi', T)$ is absolutely convergent. In particular, it means that for any choice of sampling points for the partition Π, the series $S(f, \Pi, T)$ is absolutely convergent. But this implies that the series

$$\sum_{i=1}^{\infty} \sup_{\Delta_i} ||f(t_i)||\mu(\Delta_i)$$

is convergent. This, in turn, means that the countably-valued scalar function

$$g = \sum_{i=1}^{\infty} \sup_{\Delta_i} ||f(t_i)||\chi_{\Delta_i}$$

is Lebesgue integrable. But g clearly majorates $||f||$. Therefore $||f||$ is Lebesgue integrable, and thus f is Bochner integrable. ■

Theorem 1.9. *Let X be a separable Banach space. Then for X-valued functions unconditional RL integrability coincides with Pettis integrability.*

Proof. We have already shown that RL integrability implies Pettis integrability in the general case. Let's show the converse implication. Let $f : [0;1] \to X$ be a Pettis integrable function and Pettis $- \int_0^1 f d\mu = x \in X$.

Fix an $\epsilon > 0$. Our first goal is to cover the image of f by a countable number of disjoint Borel sets B_i, such that $\mathrm{diam} B_i < \epsilon$. Indeed, since the space X is separable, we can cover $f([0;1])$ by a countable number of balls B_i' with $\mathrm{diam} B_i' < \epsilon$. Now let $B_1 = B_1'$, $B_2 = (B_2' \setminus B_1)$, $B_3 = (B_3' \setminus (B_1 \cup B_2))$, etc. Since $B_i \subset B_i'$, $\mathrm{diam} B_i < \epsilon$. Obviously, B_i's are Borel and they form a disjoint covering of $f[0;1]$.

Let $A_i = f^{-1}(B_i)$. Clearly A_i's are disjoint and their union gives the entire segment $[0;1]$. Since f is Pettis integrable, it is weakly measurable, which is equivalent to being measurable, in view of X's separability. Therefore, the sets A_i are measurable.

Consider the partition $\Pi = \{A_i\}$. Take an arbitrary $\Gamma = \{\Delta_i\} \succ \Pi$ with arbitrary sampling points $T = \{t_i\}$. Let's show that $||S(f,\Gamma,T) - x|| < \epsilon$ and $S(f,\Gamma,T)$ converges unconditionally. Let $x_i = \text{Pettis} - \int_{\Delta_i} f d\mu$. Since the Pettis integral is a countably-additive function of the set, the series $\sum_{n=1}^{\infty} x_i$ converges unconditionally and its sum equals x. Pick a Δ_i. There exists an A_n, such that $\Delta_i \subset A_n$. Since $f(\Delta_i) \subset f(A_n) \subset B_n$, we have $x_i \in \mu(\Delta_i) \cdot \text{conv} f(\Delta_i) \subset \mu(\Delta_i) \cdot \text{conv} B_n$. Now the condition $\text{diam}\,\text{conv} B_i = \text{diam} B_i < \epsilon$ allows us to conclude that $||f(t_i)\mu(\Delta_i) - x_i|| < \epsilon\mu(\Delta_i)$ for any i. But this implies that the series $S(f,\Gamma,T) = \sum f(t_i)\mu(\Delta_i)$ converges unconditinally and

$$\begin{aligned} ||S(f,\Gamma,T) - x|| &= ||\sum f(t_i)\mu(\Delta_i) - \sum x_i|| \\ &= ||\sum (f(t_i)\mu(\Delta_i) - x_i)|| \\ &\le \sum ||f(t_i)\mu(\Delta_i) - x_i|| \le \epsilon \sum \mu(\Delta_i) = \epsilon\,. \end{aligned}$$

This proves that f is inconditionally RL integrable. ■

Theorems 1.8. and 1.9. provide an interesting justification to the widely known informal statement that "the difference between Bochner and Pettis integrabilities is the same as that between absolute and unconditional convergences".

2. The limit set of integral sums

The notion of RL integral introduced above naturally leads to a related notion of the limit set of integral sums.

Definition 2.1. *Let $f : [0;1] \to X$ be an arbitrary function with values in a Banach space X. We say that a point $x \in X$ belongs to the limit set $I(f)$ if for every $\epsilon > 0$ and any partition Π there exists a partition $\Gamma \succ \Pi$, and a set of sampling points T, such that $||S(f,\Gamma,T) - x|| < \epsilon$ and $S(f,\Gamma,T)$ is an absolute integral sum. In other words, $I(f)$ is the set of all partial limits of f's absolute integral sums.*

One can also consider an analogous notion of unconditinal limit set, where unconditional RL integral sums are used.

If f is RL integrable, then $I(f)$ obviously consists of just one point – the f's integral. However, even if f is not integrable, the limit set $I(f)$ can be non-empty and play the role of f's "generalized integral". We will show that $I(f)$ is non-empty if the space X is separable or reflexive and f has a scalar Lebesgue integrable majorant. We will also show that $I(f)$ is always convex. This is surprisingly different from the case of Riemann $I_R(f)$, where non-convex examples for l_1-valued functions have been constructed (see [6]). Note also that $I(f)$ is obviously a closed set.

To approach the convexity theorem we will need some lemmas.

Lemma 2.2. *For any function $f : [0;1] \to X$ and any partition Π of $[0;1]$, the set $\{S(f,\Gamma,T) : \Gamma \succ \Pi\}$ is convex (both for absolute and unconditional sums).*

P r o o f. Throughout this proof the word "convergence" will mean either absolute or unconditional convergence. The proof is identical for both cases.

Let $S(f,\Pi',T')$ and $S(f,\Pi'',T'')$ be two convergent RL integral sums, where $\Pi' \succ \Pi$, $\Pi'' \succ \Pi$. Let's show that for any $\lambda \in [0;1]$ there exists a partition $\Gamma \succ \Pi$ and a set of sampling points T, such that $S(f,\Gamma,T) = \lambda S(f,\Pi',T') + (1-\lambda)S(f,\Pi'',T'')$.

Let $\Pi = \{\Delta_k\}_{k=1}^{\infty}$. Consider a particular Δ_N. Let $\{\Delta_i'\}$ and $\{\Delta_i''\}$ be the elemens of Π' and Π'', which are subsets of Δ_N, i.e., $\bigcup_i \Delta_i' = \bigcup_i \Delta_i'' = \Delta_N$, and let $\{t_i'\}$ and $\{t_i''\}$ be their corresponding sampling points. Denote $S' = \sum_i f(t_i')\mu(\Delta_i')$ and $S'' = \sum_i f(t_i'')\mu(\Delta_i'')$, segments of sums $S(f,\Pi',T')$ and $S(f,\Pi'',T'')$ respectively, located on Δ_N. Let's construct such a partition of Δ_N that the integral sum over it equals $\lambda S' + (1-\lambda)S''$.

Let's enumerate the elements of the denumerable set $\{t_i''\} \cup \{t_i''\}$ as $\{a_1, a_2, \ldots\}$. The expression $\lambda S' + (1-\lambda)S''$ can be written as $\sum_k \beta_k f(a_k)$, where β_k are certain real coefficients, namely:

- $\beta_k = \lambda\mu(\Delta_j')$, if $a_k = t_j'$ and $a_k \notin \{t_i''\}$;
- $\beta_k = (1-\lambda)\mu(\Delta_j'')$, if $a_k = t_j''$ and $a_k \notin \{t_i'\}$;
- $\beta_k = \lambda\mu(\Delta_j') + (1-\lambda)\mu(\Delta_l'')$, if $a_k = t_j' = t_l''$.

Therefore

$$\sum_k \beta_k = \sum_i \lambda\mu(\Delta_j') + \sum_l (1-\lambda)\mu(\Delta_l'') = \lambda\mu(\Delta_N) + (1-\lambda)\mu(\Delta_N) = \mu(\Delta_N)\,.$$

In view of this fact, we can partition the set Δ_N into certain disjoint measurable sets c_k' (their number being equal to that of a_k's), such that $\mu(c_k') = \beta_k$. Now if we denote $c_k = c_k' \cup \{a_k\} \setminus \{a_i\}_{i \neq k}$, then still $\mu(c_k) = \beta_k$, c_k's are disjoint, but $a_k \in c_k$. Therefore we can consider $\{c_k\}$ as a partition of Δ_N with sampling points a_k. The integral sum over it equals $\sum_k f(a_k)\mu(c_k) = \sum_k \beta_k f(a_k) = \lambda S' + (1-\lambda)S''$.

We can construct such partitions for each Δ_k in Π. Their union will give us a partition of the entire segment $[0;1]$, which is finer than Π, and its integral sum equals $\lambda S(f,\Pi',T') + (1-\lambda)S(f,\Pi'',T'')$. This proves the lemma. ∎

Lemma 2.3. *The limit set $I(f)$ equals the intersection of the closures of the sets $\{S(f,\Gamma,T) : \Gamma \succ \Pi\}$ over all partitions Π (both for absolute and unconditional sums).*

P r o o f. Indeed, by the definition, a point x belongs to $\bigcap_\Pi \mathrm{cl}\{S(f,\Gamma,T) : \Gamma \succ \Pi\}$ if and only if

$$\forall \Pi\, \forall \epsilon\, \exists \Gamma \succ \Pi, \exists T : \, ||S(f,\Gamma,T) - x|| < \epsilon$$

which is exactly the definition of $x \in I(f)$. ■

Theorem 2.4. *The limit set $I(f)$ is always convex.*

P r o o f. It is an immediate consequence of Lemmas 2.2 and 2.3, and the following obvious facts: the closure of a convex set is convex; the intersection of a family of convex sets is convex. ■

Now we proceed to investigate when the limit set $I(f)$ is non-empty. In order to do that we will introduce the set of weak limits of RL integral sums.

Definition 2.5. *Let $f : [0;1] \to X$ be an arbitrary function. We will say that a point $x \in X$ belongs to the weak limit set $WI(f)$ of f if for any weak neighborhood U of x and any partition Π of $[0;1]$ there exists a finer partition $\Gamma \succ \Pi$ and a set of sampling points T, such that $S(f,\Gamma,T) \in U$. Depending on whether $S(f,\Gamma,T)$ denotes an absolute or unconditional RL integral sum, we can distinguish the absolute and the unconditional $WI(f)$.*

It is obvious that $I(f) \subset WI(f)$. We will show now that in fact $I(f) = WI(f)$.

Lemma 2.6. *The weak limit set $WI(f)$ equals the intersection of the weak closures of the sets $\{S(f,\Gamma,T) : \Gamma \succ \Pi\}$ over all partitions Π.*

P r o o f. Just as in Lemma 2.3, note that by the definition, a point x belongs to $\bigcap_\Pi \mathrm{wcl}\{S(f,\Gamma,T) : \Gamma \succ \Pi\}$ if and only if

$$\forall \Pi\, \forall U - \text{weak neighborhood of } x\; \exists \Gamma \succ \Pi, \exists T : \, S(f,\Gamma,T) \in U$$

which is exactly the definition of $x \in WI(f)$. ■

Theorem 2.7. *The weak limit set $WI(f)$ coincides with the strong limit set $I(f)$ for all functions $f : [0;1] \to X$.*

P r o o f. According to Mazur theorem, the weak and the strong closures of the convex (Lemma 2.2) set $\{S(f,\Gamma,T) : \Gamma \succ \Pi\}$ coincide. The claim of the theorem now follows from the Lemmas 2.3 and 2.6. ■

Theorem 2.8. *Let the space X be reflexive and let a function $f : [0;1] \to X$ have an integrable scalar majorant. Then the set $WI(f)$ is non-empty.*

P r o o f. Let $g : [0;1] \to \mathbb{R}$ be the majorant of f, i.e., $||f(t)|| \leq g(t)$ for all $t \in [0;1]$. Since g is integrable (for scalar functions we don't have to distringuish between Lebesgue and RL integrabilities), i.e., its integral sums have a limit, there exists a partition Π, such that the set $\{S(g,\Gamma,T) : \Gamma \succ \Pi\}$ is bounded. Since $||S(f,\Gamma,T)|| \leq S(||f||,\Gamma,T) \leq S(g,\Gamma,T)$, the sets $\{S(f,\Gamma,T) : \Gamma \succ \Pi'\} \subset X$ are also bounded for any $\Pi' \succ \Pi$. Since X is reflexive, these sets are relatively weakly compact. Therefore the intersection of their weak closures $\bigcap_{\Pi'} \mathrm{wcl}\{S(f,\Gamma,T) : \Gamma \succ \Pi'\}$ is non-empty. According to the Lemma 2.6, this intersection equals $WI(f)$. This proves the theorem. ■

Theorems 2.7 and 2.8 give us the following important corollary.

Corollary 2.9. *Let the space X be reflexive and let a function $f : [0;1] \to X$ have an integrable scalar majorant. Then the limit set $I(f)$ is non-empty.*

Our next goal is to prove a much more complicated fact: if the space X is separable and f has an integrable scalar majorant then $I(f)$ is non-empty.

Theorem 2.10. *Let X be a separable Banach space. Let $f : [0;1] \to X$ be a function that has a Lebesgue integrable scalar majorant. Then $I(f) \neq \oslash$.*

To prove this theorem we will need several lemmas. In the sequel, g will denote f's integrable majorant, i.e., $||f(t)|| \leq g(t)$ for all $t \in [0;1]$. Note that since g is scalar, it's Lebesgue integrability is equivalent to RL integrability. Therefore there exists a partition Π_0, such that for any $\Pi \succ \Pi_0$ and any set of sampling points T, the series $S(g,\Pi,T)$ are convergent. Since g majorates f, then $S(f,\Pi,T)$ is also absolutely convergent. Without loss of generality, we can consider only partitions that are finer than Π_0 and thus guarantee that f's intergal sums over these partitions are absolutely convergent.

Now we introduce the notion of ϵ-approximable and ϵ-spread partitions and consider their properties.

Definition 2.11. *We will call a partition Γ with sampling points T ϵ-approximable (for the given function f), if for any finer partition $\Gamma' \succ \Gamma$ there exists a set of sampling points T, such that $||S(f,\Gamma,T) - S(f,\Gamma',T')|| < \epsilon$.*

We will call a partition Γ with sampling points T ϵ-spread if

1. *each* Δ_i *contains a (not necessarily measurable) subset* P_i*, such that* $\mu_*(\Delta_i \setminus P_i) = 0$ *and* $\mathrm{diam} f(P_i) < \epsilon$*;*

2. *the sampling point* t_i *of* Δ_i *lies in* P_i*.*

Lemma 2.12. *An* ϵ*-spread partition that is finer than* Π_0 *is* ϵ*-approximable.*

P r o o f. Let $\Gamma = \{\Delta_k\}_{k=1}^{\infty}$ be an ϵ-spread partition with sampling points $T = \{t_k\}_{k=1}^{\infty}$ ($\Gamma \succ \Pi_0$). Let $\Gamma' = \{c_i\}_{i=1}^{\infty}$ be an arbitrary partition, inscribed into Γ. For any i we will denote by $k(i)$ the index k for which $c_i \subset \Delta_k$. Note that $c_i \bigcap P_{k(i)} \neq \oslash$, since otherwise $\Delta_{k(i)} \setminus c_i$ would be a measurable set containing $P_{k(i)}$ whose measure is strictly less than that of $\Delta_{k(i)}$ – which contradicts the property 1 of ϵ-spread partitions. Choose the set of sampling points $T' = \{t'_i\}$ for Γ' by picking $t'_i \in c_i \bigcap P_{k(i)}$. Since $\Gamma' \succ \Gamma \succ \Pi_0$, $S(f, \Gamma, T) < \infty$ and $S(f, \Gamma', T') < \infty$.

We have $S(f, \Gamma, T) = \sum_{k=1}^{\infty} f(t_k)\mu(\Delta_k) = \sum_{i=1}^{\infty} f(t_{k(i)})\mu(c_i)$. Therefore

$$
\begin{aligned}
||S(f, \Gamma', T') - S(f, \Gamma, T)|| &= ||\sum_i f(t'_i)\mu(c_i) - \sum_i f(t_{k(i)})\mu(c_i)|| \\
&\leq \sum_i ||f(t'_i) - f(t_{k(i)})||\mu(c_i) < \epsilon
\end{aligned}
$$

since $t'_i, t_{k(i)} \in P_{k(i)}$, and $\mathrm{diam} P_{k(i)} < \epsilon$. This proves the lemma. ■

Now we will describe how to construct an ϵ-spread partition inscribed into a given one. In order to do this, consider an arbitrary measurable set $\Omega \subset [0; 1]$ and its not necessarily measurable subset $P \subset \Omega$, such that $\mu_*(\Omega \setminus P) = 0$. We will construct an ϵ-spread partition of Ω, such that the P_i's, mentioned in the definition of the ϵ-spread partition, are subsets of P.

Let's partition the set $f(P)$ into a countable number of disjoint subsets $\{B_i\}_{i=1}^{\infty}$, such that $\mathrm{diam} B_i < \epsilon$. This is possible because the space X is separable.

Take $P_1 = f^{-1}(B_1)$. There exists a measurable set $E_1 \supset P_1$, such that $\mu(E_1) = \mu^*(P_1)$. Then, denote $P_2 = f^{-1}(B_2) \setminus E_1$. There exists a measurable set $E_2 \supset P_2$, such that $E_2 \bigcap E_1 = \oslash$ and $\mu(E_1) = \mu^*(P_2)$. Continue this process. At the kth step we take $P_k = f^{-1}(B_k) \setminus \bigcup_{i=1}^{k-1} E_i$ and $E_k \supset P_k$, such that $E_k \bigcap (\bigcup_{i=1}^{k-1} E_i) = \oslash$ and $\mu(E_k) = \mu^*(P_k)$. In this manner we construct P_k and E_k for all $k \in \mathbb{N}$.

Note that $\bigcup_{k=1}^{n} E_k \supset \bigcup_{k=1}^{n} f^{-1}(B_k)$ for any n and $\bigcup_{k=1}^{\infty} f^{-1}(B_k) = P$, therefore $\bigcup_{k=1}^{\infty} E_k \supset P$. But $\bigcup_{k=1}^{\infty} E_k$ is a measurable set, therefore $\bigcup_{k=1}^{\infty} E_k = \Omega$. Thus $\{E_k\}_{k=1}^{\infty}$ is a partition of Ω. It is obvious from the construction that this partition is ϵ-spread.

Lemma 2.13. *Let the partition Γ with the sampling points T be ϵ-spread, $\Gamma \succ \Pi_0$. Then for any $\epsilon' < \epsilon$ there exists a ϵ'-spread partition Γ' with sampling points T', such that $||S(f,\Gamma,T) - S(f,\Gamma',T')|| < \epsilon$.*

P r o o f. Let $\Gamma = \{\Delta_k\}_{k=1}^{\infty}$ and let $P_k \subset \Delta_k$ be the subsets mentioned in the definition of an ϵ-spread partition. Apply the partitioning procedure described above to each Δ_k, taking $P = P_k$. We will obtain an ϵ'-spread partition of each Δ_k. Their union will give us an ϵ'-spread partition of the entire segment $[0;1]$, inscribed into Γ. Denote it by $\Gamma' = \{\Delta_i'\}$ and its sampling points by $T' = \{t_i'\}$. Let's estimate $||S(f,\Gamma,T) - S(f,\Gamma',T')||$. Just like in Lemma 2.12, denote by $k(i)$ the index k, for which $\Delta_i' \subset \Delta_k$. By applying the same argument as in Lemma 2.12, we'll get that $||S(f,\Gamma,T) - S(f,\Gamma',T')|| < \epsilon$ due to $\mathrm{diam} P_{k(i)} < \epsilon$. ∎

Lemma 2.14. *Assume that for each $k \in \mathbb{N}$ there exists an ϵ_k-approximable partition $\Pi_k \succ \Pi_0$ with sampling points T_k. Assume that $\epsilon_k \to 0$ and $\exists\, x = \lim_{k\to\infty} S(f,\Pi_k,T_k) \in X$. Then $x \in I(f)$.*

P r o o f. Let Π be an arbitrary partition of the segment $[0;1]$, and $\epsilon > 0$. Since $\epsilon_k \to 0$ and $S(f,\Pi_k,T_k) \to x$, there exists an index n, such that $\epsilon_n < \epsilon/2$ and $||S(f,\Pi_n,T_n) - x|| < \epsilon/2$. Take an arbitrary partition Π', which is finer than any of Π, Π_0 and Π_n. Since the partition Π_n with the sampling points T_n is ϵ_n-approximable, there exists a set of sampling points T', such that $||S(f,\Pi',T') - S(f,\Pi_n,T_n)|| < \epsilon_n < \epsilon/2$. But $||S(f,\Pi_n,T_n) - x|| < \epsilon/2$. Thus we obtain that $||S(f,\Pi',T') - x|| < \epsilon/2 + \epsilon/2 = \epsilon$. So, for an arbitrary partition Π and $\epsilon > 0$ we have found a finer partition $\Pi' \succ \Pi$ and sampling points T, such that $||S(f,\Pi',T') - x|| = \epsilon$. This means that $x \in I(f)$. ∎

Now we can complete the proof of the Theorem 2.10. We will construct a sequence of ϵ_k-approximable partitions Π_k with sampling points T_k, such that $\Pi_k \succ \Pi_0$, $\epsilon_k \to 0$ and there exists a limit of $S(f,\Pi_k,T_k)$. By Lemma 2.14 this limit will belong to $I(f)$.

Take $\epsilon_k = 2^{-k}$. By Lemma 2.13 we can construct an ϵ_1-spread partition Π_1 with sampling points T_1, inscribed into Π_0. By Lemma 2.12 it will be ϵ_1-approximable. Then by Lemma 2.13 there exists an ϵ_2-spread (and therefore ϵ_2-approximable) partition Π_2 with sampling points T_2, inscribed into Π_1, such that $||S(f,\Pi_2,T_2) - S(f,\Pi_1,T_1)|| < \epsilon_1$. By the same lemma, there exists an ϵ_3-approximable partition Π_3 with sampling points T_3, inscribed into Π_2, such that $||S(f,\Pi_3,T_3) - S(f,\Pi_2,T_2)|| < \epsilon_2$. Continuing this process, we will construct the sequence $\{\Pi_k\}$, such that $||S(f,\Pi_{k+1},T_{k+1}) - S(f,\Pi_k,T_k)|| < \epsilon_k$. Since $\sum_{k=1}^{\infty} \epsilon_k < \infty$, the sequence $\{S(f,\Pi_k,T_k)\}$ is convergent. This proves the theorem. ∎

If the space X is neither separable nor relfexive, one cannot guarantee that the limit set $I(f)$ isn't empty even for a bounded function. The following example demonstrates this.

Example 2.15. Consider the space $l_1[0;1]$. Consider the function $f : [0;1] \to l_1([0;1])$ identical to that of Example 1.5, that is, $f(t) = e_t$. It is easy to see that the l_1-norm of any integral sum $S(f, \Gamma, T)$ of f is 1. Suppose that there exists an $x \in I(f)$. Then $||x||_{l_1} = 1$. Consider coordinate functionals $\delta_t \in (l_1[0;1])^*$: $\delta_t(g) = g(t)$. For every $t \in [0;1]$ we have $\delta_t(f(\tau)) = 0$ for almost all $\tau \in [0;1]$ (in fact, for all $\tau \neq t$). But because $x \neq 0$, there exists such $t_0 \in [0;1]$, that $\delta_{t_0}(x) \neq 0$. But

$$\delta_{t_0}(x) \in \delta_{t_0}(I(f)) \subset I(\delta_{t_0} \circ f) = \{0\}\,.$$

This contradiction shows that $I(f) = \oslash$.

The characterization of those Banach spaces X where $I(f) \neq \oslash$ for all functions f is yet unknown to the authors.

We have established that $I(f)$ is always a convex closed set. It is also easy to see that $I(f)$'s cardinality is not greater than continuum. Indeed, it was already noted that we may restrict ourselves to consider only partitions into F_σ-sets, and the number of such sets is continuum. Now it is natural to ask, if an arbitrary convex closed set of no more than continuum cardinality is the $I(f)$ set for a certain function f. We can answer this question in positive.

Theorem 2.16. *For an arbitrary convex closed set S of no more than continuum cardinality in a Banach space X there exists a function $f : [0;1] \to X$, such that $I(f) = S$.*

P r o o f. Our construction will be very close to that of Example 1.7. We introduce the same set $\mathcal{P}$ and consider its elements instead of partitions. We will construct a function f with the following properties:

1) $f([0;1]) \subset S$;

2) for any $\Pi \in \mathcal{P}$, any $\Delta \in \Pi$ and any $x \in S$ there exists a $t \in \Delta$, such that $f(t) = x$.

It is clear that all points $x \in S$ belong to $I(f)$. On the other hand, since f does not take values outside the set S and any integral sum is a limit of convex combinations of f's values, $I(f) \subset S$. Thus $I(f) = S$.

Just like in Example 1.7, we introduce the smallest ordinal number Ω of continuum cardinality and index $\mathcal{P}$ with ordinals $\sigma < \Omega$. Now we consider the set of all pairs (Π, x), where $\Pi \in \mathcal{P}$ and $x \in S$. This set has continuum cardinality, and therefore there exists a bijective map $\alpha : \mathcal{P} \to \mathcal{P} \times S$. We denote by β and γ the superpositions of this map with the natural projections from $\mathcal{P} \times S$ onto $\mathcal{P}$ and onto S respectively. Thus β maps $\mathcal{P}$ onto $\mathcal{P}$, γ maps $\mathcal{P}$ onto S.

Then we construct a family of mutually disjoint sets A_σ, indexed with ordinal numbers $\sigma < \Omega$ with the following property: for any $\sigma < \Omega$, A_σ, meets each subset of the partition $\Pi = \beta(\Pi_\sigma)$ at exactly one point. The construction is accomplished in the way, analogous to Example 1.7. Once the sets A_σ are constructed, we put $f(t) = \gamma(\Pi_\sigma)$ if $t \in A_\sigma$ and let $f(t)$ be an arbitrary element of S if $t \in [0;1] \setminus \bigcup A_\sigma$. The function f satisfies the properties we need. Indeed, for each $\Pi_\sigma \in \mathcal{P}$ and each $x \in S$ consider the ordinal ν, which is the index of $\Pi_\nu = \alpha^{-1}(\Pi_\sigma, x)$. Then the set A_ν meets each subset of $\beta(\Pi_\nu) = \Pi_\sigma$ and f equals $\gamma(\Pi_\nu) = x$ at each point of A_ν. This completes the proof.

Note that in the process of the proof we have found a continuum-cardinality collection of disjoint non-measurable sets $f^{-1}(x)$, $x \in S$, each having its outer measure equal the measure of the entire segment $[0;1]$. ■

For the purposes of clarity we considered functions f defined on the unit segment $[0;1]$ with the regular Lebesgue measure. However, all our constructions make sense on an arbitrary measure space (Ω, Σ, μ). All the main facts remain valid, as long as the measure μ is atomless.

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Об "интегрировании" неинтегрируемых векторнозначных функций

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Представлено новое определение интеграла для функций со значениями в банаховом пространстве. Новое свойство интегрируемости является более слабым свойством, чем интегрируемость по Бохнеру, но более сильным, чем интегрируемость по Петтису. Это определение интеграла естественным образом ведет к понятию множества пределов интегральных сумм, которое может рассматриваться как "обобщенный интеграл" для неинтегрируемых функций. Показано, что данное множество всегда выпукло и непусто, если функция имеет интегрируемую мажоранту, а пространство сепарабельно или рефлексивно.

Про "інтегрування" неінтегровних векторнозначних функцій

В.М. Кадець, Л.М. Цейтлін

Представлено нове визначення інтеграла для функцій зі значеннями в банаховому просторі. Нова властивість інтегровності є слабкішою властивістю за інтегровність за Бохнером, але сильнішою, ніж інтегровність за Петтісом. Це визначення інтеграла природним чином веде до поняття множини границь інтегральних сум, яке може розглядатися як "узагальнений інтеграл" для неінтегровних функцій. Показано, що дана множина завжди є опуклою, а також непустою, якщо функція має інтегровну мажоранту, а простір є сепарабельним або рефлексивним.