Matematicheskaya fizika, analiz, geometriya
2000, v. 7, No. 2, p. 219–230

Abstract interpolation problem and four–block problem

S. Kupin

Mathematics Division, B. Verkin Institute for Low Temperature Physics and Engineering
National Academy of Sciences of Ukraine
47 Lenin Ave., Kharkov, 61164, Ukraine
E-mail: kupin@ilt.kharkov.ua

Received September 8, 1998
Communicated by I.V. Ostrovskii

We treat a four-block problem as an abstract interpolation problem with an appropriately chosen data. This approach enables us to obtain a description of all solutions for the four-block problem.

In this note we suggest a way of the inclusion of the four-block problem [6, Ch. IX, Sect. 4] into the framework of the abstract interpolation problem [8–10]. We would like to note that one can investigate the four-block problem with the help of the commutant lifting theorem [6, 13, Ch. II, Sect. 2]. This approach is presented in the monograph [6] and in the papers [5, 7].

We should mention here that the relation between general commutant lifting and a coordinate-free version of the abstract interpolation problem is discussed in [3].

The reasoning we are going to present is quite similar to that of [3, 11].

1. Abstract interpolation problem

1.1. We would like to begin the exposition with a brief recalling of the concept of the abstract interpolation problem (or the (AI)-problem, in abbreviated form). The formulation of the problem and its detailed discussion can be found in [8–10]. The second paper contains also the applications of the problem to the generalized bitangential Schur–Nevanlinna–Pick problem and j-inner-outer factorization as well as a wide bibliography on the question. We would like to mention the paper [4] that deals with a very close circle of ideas.

Let us have a complex linear space X and a nonnegative quadratic form D on the space, $D(x,x)\geq 0, \forall x\in X$. Let us assume that the identity holds

$$D(T_2x,T_2y)-D(T_1x,T_1y)=(M_1x,M_1y)_{\mathcal{E}_1}-(M_2x,M_2y)_{\mathcal{E}_2}, \forall x,y\in X\,. \quad (1.1.1)$$

Here T_1 and T_2 are some operators, defined on X, $\mathcal{E}_1$ and $\mathcal{E}_2$ are separable Hilbert spaces, M_1 and M_2 are linear maps, acting from X to $\mathcal{E}_1$ and from X to $\mathcal{E}_2$, respectively. For the sake of brevity we call the set $\{X, \mathcal{E}_1, \mathcal{E}_2; D, T_1, T_2, M_1, M_2\}$ an interpolation data of the (AI)-problem.

We should give some definitions to introduce the notion a solution of the problem. Let $\omega(\zeta)$ be a $[\mathcal{E}_1, \mathcal{E}_2]$-valued analytic contractive function on the unit disk $\mathbf{D} = \{\zeta \in \mathbf{C} : |\zeta| < 1\}$. We consider an operator-valued function

$$\begin{bmatrix} 1_{\mathcal{E}_1} & \omega^* \\ \omega & 1_{\mathcal{E}_2} \end{bmatrix} : \begin{bmatrix} L^2(\mathcal{E}_1) \\ L^2(\mathcal{E}_2) \end{bmatrix} \to \begin{bmatrix} L^2(\mathcal{E}_1) \\ L^2(\mathcal{E}_2) \end{bmatrix}.$$

It is almost obvious that the defined map is nonnegative, we denote its square root by $\Sigma_\omega, \Sigma_\omega = \begin{bmatrix} 1_{\mathcal{E}_1} & \omega^* \\ \omega & 1_{\mathcal{E}_2} \end{bmatrix}^{1/2}$. Next, we define a subspace L^2_ω of $L^2(\mathcal{E}_1 \oplus \mathcal{E}_2)$ by the equality

$$L^2_\omega = \overline{\Sigma_\omega \begin{bmatrix} L^2(\mathcal{E}_1) \\ L^2(\mathcal{E}_2) \end{bmatrix}}.$$

We set

$$H_\omega = L^2_\omega \ominus \left\{ \Sigma_\omega \begin{bmatrix} H^2(\mathcal{E}_1) \\ 0 \end{bmatrix} \oplus \Sigma_\omega \begin{bmatrix} 0 \\ H^2_-(\mathcal{E}_2) \end{bmatrix} \right\} \tag{1.1.2}$$

now. Here $H^2_-(\mathcal{E}_2)$ stands for $L^2(\mathcal{E}_2) \ominus H^2(\mathcal{E}_2), H^2(\mathcal{E}_2)$ is the standard Hardy space of vector-valued functions on the unit disk $\mathbf{D}$. The space H_ω is called de Branges–Rovnyak space, associated with function ω.

Let us observe that a vector $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ from L^2_ω lies in H_ω if and only if the vector $\begin{bmatrix} f_- \\ f_+ \end{bmatrix} = \Sigma_\omega f$ is in $\begin{bmatrix} H^2_-(\mathcal{E}_1) \\ H^2(\mathcal{E}_2) \end{bmatrix}$. The assertion can be easily deduced from the equality

$$0 = \left(\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \Sigma_\omega \begin{bmatrix} h_+ \\ h_- \end{bmatrix} \right) = \left(\Sigma_\omega \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} h_+ \\ h_- \end{bmatrix} \right),$$

where $h_+ \in H^2(\mathcal{E}_1), h_- \in H^2(\mathcal{E}_2)$.

By a solution of the (AI)-problem we mean a pair (ω, F), where ω is a $[\mathcal{E}_1, \mathcal{E}_2]$-valued contractive analytic on $\mathbf{D}$ function and F is a linear map from X to H_ω, possessing the properties

$$1) \quad \bar{t}\left(FT_1 x \oplus \Sigma_\omega \begin{bmatrix} M_1 x \\ 0 \end{bmatrix} \right) = FT_2 x \oplus \Sigma_\omega \begin{bmatrix} 0 \\ \bar{t} M_2 x \end{bmatrix}, \ \forall x \in X, \tag{1.1.3}$$

$$2) \quad ||Fx||_2^2 \le D(x,x), \ \forall x \in X. \tag{1.1.4}$$

The map F is called the Fourier representation of the (AI)-problem. We would like to comment on relation (1.1.3). It can be readily seen that the summands in both sides of the equality are really orthogonal one to another. Indeed, FT_1x and FT_2x lie in H_ω while $\Sigma_\omega \begin{bmatrix} M_1x \\ 0 \end{bmatrix} \in \Sigma_\omega \begin{bmatrix} H^2(\mathcal{E}_1) \\ 0 \end{bmatrix}$ and $\Sigma_\omega \begin{bmatrix} 0 \\ \bar{t}M_2x \end{bmatrix} \in \Sigma_\omega \begin{bmatrix} 0 \\ H^2_-(\mathcal{E}_2) \end{bmatrix}$, the rest is clear from the definition of H_ω. Further, we rewrite equality (1.1.3) in the form

$$\bar{t}\left(\begin{bmatrix} F_- \\ F_+ \end{bmatrix} T_1x \oplus \begin{bmatrix} 1_{\mathcal{E}_1} & \omega^* \\ \omega & 1_{\mathcal{E}_2} \end{bmatrix} \begin{bmatrix} M_1x \\ 0 \end{bmatrix}\right) = \begin{bmatrix} F_- \\ F_+ \end{bmatrix} T_2x \oplus \begin{bmatrix} 1_{\mathcal{E}_1} & \omega^* \\ \omega & 1_{\mathcal{E}_2} \end{bmatrix} \begin{bmatrix} 0 \\ \bar{t}M_2x \end{bmatrix}, \tag{1.1.5}$$

or, equivalently,

$$\begin{bmatrix} F_- \\ F_+ \end{bmatrix} T_1x = t \begin{bmatrix} F_- \\ F_+ \end{bmatrix} T_2x - \begin{bmatrix} 1_{\mathcal{E}_1} & \omega^* \\ \omega & 1_{\mathcal{E}_2} \end{bmatrix} \begin{bmatrix} M_1x \\ -M_2x \end{bmatrix}, \ \forall x \in X. \tag{1.1.6}$$

Here $\begin{bmatrix} F_- \\ F_+ \end{bmatrix} x$ stands for $\Sigma_\omega Fx, x \in X$, and $t \in \mathbf{T} = \{\zeta \in \mathbf{C} : |\zeta| = 1\}$ stands for the independent variable on the unit circle.

1.2. We need more details about the inner machinery of the (AI)-problem. Firstly, it is possible to build a new Hilbert space that preserves, in essential, the structure of the space X. We define an antilinear functional $\bar{x}$ with the help of an arbitrary vector x from X by formula

$$\bar{x}(y) = D(x, y), \forall y \in X.$$

Denote the linear space of the functionals by $\bar{X}$. The space $\bar{X}$ can be equipped with a scalar product in the following way:

$$(\bar{x}, \bar{y})_D = D(x, y), \quad \forall \bar{x}, \bar{y} \in \bar{X}\,.$$

The scalar product is well-defined because of the nonnegativity of D. Now we should only take a closure of $\bar{X}$ in the metric to get a Hilbert space, $H = clos_{(.,.)_D} \bar{X}$.

We are going to define a certain isometry, acting on the Hilbert space H. The solutions of the (AI)-problem will be characterized in terms of unitary extensions of this isometry. Let us put

$$d_V = \{\overline{T_1x} \oplus M_1x : x \in X\} \subset H \oplus \mathcal{E}_1,$$

$$\Delta_V = \{\overline{T_2x} \oplus M_2x : x \in X\} \subset H \oplus \mathcal{E}_2.$$

Identity (1.1.1) states that the map, defined by the formula

$$V : d_v \to \Delta_V, \quad V(\overline{T_1x} \oplus M_1x) = \overline{T_2x} \oplus M_2x, \quad x \in X, \tag{1.2.1}$$

is an isometry. We refer to this operator as to the (AI)-isometry, also it is convenient to denote its defect subspaces by

$$N_{d_V} = (H \oplus \mathcal{E}_1) \ominus d_V \text{ and } N_{\Delta_V} = (H \oplus \mathcal{E}_2) \ominus \Delta_V. \qquad (1.2.2)$$

1.3. In this subsection we demonstrate how a unitary extension of the (AI)-isometry generates a solution of (AI)-problem .

Let K be a Hilbert space that contains H as its subspace. By a unitary extension of an isometry $V, V : H \oplus \mathcal{E}_1 \to H \oplus \mathcal{E}_2$, we mean a unitary operator U acting from $K \oplus \mathcal{E}_1$ to $K \oplus \mathcal{E}_2$ in such a way that $U|_{d_V} = V$.

We prefer to treat the unitary operator as a unitary colligation with coefficient spaces $\mathcal{E}_1, \mathcal{E}_2$ and the state space K. In other words, we decompose the operator U into the blocks

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} K \\ \mathcal{E}_1 \end{bmatrix} \to \begin{bmatrix} K \\ \mathcal{E}_2 \end{bmatrix}.$$

Following [1], we define the characteristic function of the colligation by the formula

$$\omega(\zeta) = D + \zeta C(I - \zeta A)^{-1} B. \qquad (1.3.1)$$

Evidently, the function is analytic on the unit disk $\mathbf{D}$. Moreover, one can prove that the function is contractive [1, 2, 10]. Further, we define a map $\mathbf{F}$ with the help of the relations

$$\mathbf{F}(\zeta) = \begin{bmatrix} \mathbf{F}_-(\zeta) \\ \mathbf{F}_+(\zeta) \end{bmatrix} = \begin{bmatrix} \bar{\zeta} B^*(I - \bar{\zeta} A^*)^{-1} \\ C(I - \zeta A)^{-1} \end{bmatrix} : K \to \begin{bmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \end{bmatrix}. \qquad (1.3.2)$$

The map is called a Fourier representation of the state space K. The following proposition describes the main properties of the representation.

Proposition 1.1. *([2, 8–10]) A Fourier representation, defined by (1.3.2), has the properties:*

1) *for every k from K the vector $\mathbf{F}k$ lies in $\Sigma_\omega H_\omega$, so the map $\Sigma_\omega^{(-1)}\mathbf{F}$, acting from K to H_ω, is well-defined.*

2)

$$\bar{t}\left(\mathbf{F}P_K \oplus \begin{bmatrix} 1_{\mathcal{E}_1} & \omega^* \\ \omega & 1_{\mathcal{E}_2} \end{bmatrix} \begin{bmatrix} 1_{\mathcal{E}_1} \\ 0 \end{bmatrix} P_{\mathcal{E}_1}\right) = \mathbf{F}P_K U \oplus \begin{bmatrix} 1_{\mathcal{E}_1} & \omega^* \\ \omega & 1_{\mathcal{E}_2} \end{bmatrix} \begin{bmatrix} 0 \\ 1_{\mathcal{E}_2} \end{bmatrix} P_{\mathcal{E}_2} U.$$

3)

$$||\Sigma_\omega^{(-1)}\mathbf{F}k||_2^2 \le ||k||_K^2, \forall k \in K.$$

It seems to be relevant to compare 2) with (1.1.5) and (1.2.1).

Assign $Fx = \Sigma_\omega^{(-1)}\mathbf{F}\bar{x}, x \in X$. Since properties 2) and 3) of the previous proposition turn into (1.1.3) and (1.1.4), we deduce that the pair (ω, F) is a solution of the (AI)-problem.

The cornerstone of the approach is that, conversely, every solution (ω, F) of the (AI)-problem arises in this way (see [8, 10]).

1.4. Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be copies of the defect subspaces of the (AI)-isometry. This time we obtain a unitary extension of the isometry by enlarging the coefficient spaces instead of the state space. Namely, we consider a unitary operator U_0 mapping $H \oplus (\mathcal{E}_1 \oplus \mathcal{N}_2)$ onto $H \oplus (\mathcal{E}_2 \oplus \mathcal{N}_1)$. Note that

$$H \oplus (\mathcal{E}_1 \oplus \mathcal{N}_2) = (H \oplus \mathcal{E}_1) \oplus \mathcal{N}_2 = d_V \oplus N_{d_V} \oplus \mathcal{N}_2,$$

$$H \oplus (\mathcal{E}_2 \oplus \mathcal{N}_1) = (H \oplus \mathcal{E}_2) \oplus \mathcal{N}_1 = \Delta_V \oplus N_{\Delta_V} \oplus \mathcal{N}_1.$$

This remark permits to introduce the universal extension of the (AI)-isometry (see [1, 8, 9]) as

$$U_0|_{d_V} = V, \quad U_0|_{N_{d_V}} = id : N_{d_V} \to \mathcal{N}_1, \quad U_0|_{\mathcal{N}_2} = id : \mathcal{N}_2 \to N_{\Delta_V}.$$

Let us consider the characteristic function and the Fourier representation of the unitary colligation

$$S(\zeta) = D_0 + \zeta C_0 (I - \zeta A_0)^{-1} B_0,$$

$$G(\zeta) = \begin{bmatrix} G_-(\zeta) \\ G_+(\zeta) \end{bmatrix} = \begin{bmatrix} \bar\zeta B_0^*(I - \bar\zeta A_0^*)^{-1} \\ C_0 (I - \zeta A_0)^{-1} \end{bmatrix} : H \to \Sigma_S H_S,$$

where

$$U_0 = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} : \begin{bmatrix} H \\ \mathcal{E}_1 \oplus \mathcal{N}_2 \end{bmatrix} \to \begin{bmatrix} H \\ \mathcal{E}_2 \oplus \mathcal{N}_1 \end{bmatrix}.$$

It is convenient to introduce the block decompositions of the just defined maps:

$$G_+ = \begin{bmatrix} G_{+2} \\ G_{+1} \end{bmatrix} : H \to \begin{bmatrix} \mathcal{E}_2 \\ \mathcal{N}_1 \end{bmatrix}, G_- = \begin{bmatrix} G_{-1} \\ G_{-2} \end{bmatrix} : H \to \begin{bmatrix} \mathcal{E}_1 \\ \mathcal{N}_2 \end{bmatrix},$$

$$S = \begin{bmatrix} s_0 & s_2 \\ s_1 & s \end{bmatrix} : \begin{bmatrix} \mathcal{E}_1 \\ \mathcal{N}_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{E}_2 \\ \mathcal{N}_1 \end{bmatrix}.$$

Before stating the theorem, let us denote by $\mathbf{B}[\mathcal{N}_1, \mathcal{N}_2]$ the set of all $[\mathcal{N}_1, \mathcal{N}_2]$-valued analytic contractive functions on the unit disk $\mathbf{D}$.

Theorem 1.1. *([8–10]) All solutions of the (AI)-problem are parametrized by functions ε from $\mathbf{B}[\mathcal{N}_1, \mathcal{N}_2]$. The correspondence is given by formulas:*

$$\omega = s_0 + s_2 (1_{\mathcal{N}_2} - \varepsilon s)^{-1} \varepsilon s_1 \,, \tag{1.4.1}$$

$$\begin{bmatrix} F_-(\zeta) x \\ F_+(\zeta) x \end{bmatrix} = \begin{bmatrix} G_{-1} + s_1^* (1_{\mathcal{N}_1} - \varepsilon^* s^*)^{-1} \varepsilon^* G_{-2} \\ G_{+2} + s_2 (1_{\mathcal{N}_2} - \varepsilon s)^{-1} \varepsilon G_{+1} \end{bmatrix} : H \to \Sigma_\omega H_\omega \,. \tag{1.4.2}$$

2. Formulation of the four-block problem

2.1. Let E_1, E_1', E_2, E_2', F be separable Hilbert spaces. Let

$$\Omega_0 = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{2.1.1}$$

be a fixed function from $L^\infty(E_1 \oplus E_2, E_1' \oplus E_2')$. The operator-valued function maps $\begin{bmatrix} L^2(E_1) \\ L^2(E_2) \end{bmatrix}$ into $\begin{bmatrix} L^2(E_1') \\ L^2(E_2') \end{bmatrix}$ via multiplication. Let a $[F, E_1']$-valued inner function be given, $\theta \in H^\infty(F, E_1')$. By the four-block problem [6, Ch. IX, Sect. 4] one understands the problem of finding δ,

$$\begin{aligned} \delta = & \inf || \begin{bmatrix} A + \theta H^\infty(E_1, F) & B \\ C & D \end{bmatrix} ||_\infty \\ = & \inf\{|| \begin{bmatrix} A + \theta\Psi & B \\ C & D \end{bmatrix} ||_\infty : \Psi \in H^\infty(E_1, F)\}, \end{aligned}$$

and characterizing all optimal Ψ_* from $H^\infty(E_1, F)$.

A detailed discussion of the problem from the commutant lifting theorem point of view can be found in the already mentioned monograph [6, Ch. IX, Sect. 4]. The same source contains applications of the problem to the robust control theory as well as the history of the question.

In particular, the construction permits to calculate the value δ as a norm of a certain Hankel type operator. We need some definitions to formulate the result.

We define an inner function by the formula

$$\Phi = \begin{bmatrix} \theta \\ 0 \end{bmatrix} : L^2(F) \to \begin{bmatrix} L^2(E_1') \\ L^2(E_2') \end{bmatrix}, \Phi^*\Phi = 1_F .$$

Then we consider spaces

$$H = \begin{bmatrix} H^2(E_1) \\ L^2(E_2) \end{bmatrix} \text{ and } H' = \begin{bmatrix} L^2(E_1') \\ L^2(E_2') \end{bmatrix} \ominus \Phi H^2(F). \tag{2.1.2}$$

We denote the operator of the orthogonal projection from $L^2(E_1') \oplus L^2(E_2')$ to H' by $P_{H'}$. Further we set operators T and T' to be

$$Th = th, h \in H, \text{ and } T'h' = P_{H'}th', h' \in H'.$$

Let us define an operator Γ, mapping H to H' by the formula

$$\Gamma h = P_{H'}\Omega_0 h, h \in H. \tag{2.1.3}$$

Clearly, the operator is in the commutant of T and T'. Indeed,

$$\Gamma T h = \Gamma t h = P_{H'}\Omega_0 t h = P_{H'} t \Omega_0 h = P_{H'} t P_{H'} \Omega_0 h = T'\Gamma h, \forall h \in H.$$

Theorem 2.2. *([6, Ch. IX, Sect. 4]) Let* Ω_0 *be an operator-valued function defined by (2.1.1). Then*

$$\inf ||\Omega_0 + \begin{bmatrix} \theta H^\infty(E_1, F) & 0 \\ 0 & 0 \end{bmatrix} ||_\infty = ||\Gamma||.$$

2.2. We would like to study a question that slightly differs from the problem discussed in the previous subsection. Assuming that $||\Gamma|| \le 1$, we are interested in describing all operator-valued functions Ω from $L^\infty(E_1 \oplus E_2, E_1' \oplus E_2')$ which define the same operator Γ via (2.1.3) while their norms satisfy the inequality $||\Omega||_\infty \le 1$.

More precisely, let an operator Γ, acting from H to H', be given. We suppose that

$$\Gamma T = T'\Gamma \text{ on } H. \tag{2.2.1}$$

The problem is to parametrize the set of operator-valued functions $\Omega \in L^\infty(E_1 \oplus E_2, E_1' \oplus E_2')$ such that

1) $\Gamma h = P_{H'}\Omega h, \forall h \in H,$ (2.2.2)
2) $||\Omega||_\infty \le 1.$ (2.2.3)

Hereafter we will refer to the problem as to the four-block problem (or, to be brief, to the (4B)-problem). The function Ω possessing properties (2.2.2) and (2.2.3), will be called a solution of the (4B)-problem .

2.3. The purpose of this subsection is to put the (4B)-problem into the framework of the scheme of the abstract interpolation.

It can be easily done by exploiting commutativity relation (2.2.1). We have

$$\Gamma Th = T'\Gamma h = P_{H'}t\Gamma h = t\Gamma h - \Phi M_2 h \tag{2.3.1}$$

by definition of $P_{H'}$ (see Subsect. 2.1). Here M_2 is an operator from H to $H^2(F)$. We rewrite (2.3.1) in the equivalent form

$$t\Gamma h = \Gamma th + \Phi M_2 h$$

and we note that $\Gamma Th \perp \Phi M_2 h$, because $\Gamma Th \in H'$ and $\Phi M_2 h \perp H'$. This remark immediately implies that

$$(t\Gamma h, \Phi g) = (\Phi M_2 h, \Phi g) = (M_2 h, g) \quad \forall h \in H, \forall g \in H^2(F). \tag{2.3.2}$$

In particular, the equality shows that $(\Phi^* t\Gamma h - M_2 h, g) = 0 \ \forall g \in H^2(F)$, or, consequently, $M_2 h = P_+ t\Phi^*\Gamma h \in F$. Furthermore, this enables us to calculate the expression in the following way:

$$\begin{aligned} ((I - \Gamma^*\Gamma)h_1, h_2) \quad &- \quad ((I - \Gamma^*\Gamma)th_1, th_2) = (\Gamma th_1, \Gamma th_2) - (\Gamma h_1, \Gamma h_2) \\ &= \quad (t\Gamma h_1, t\Gamma h_2) - (t\Gamma h_1, \Phi M_2 h_2) - (\Phi M_2 h_1, t\Gamma h_2) \\ &+ \quad (\Phi M_2 h_1, \Phi M_2 h_2) - (\Gamma h_1, \Gamma h_2) = -(M_2 h_1, M_2 h_2)\,. \end{aligned}$$

We have applied relation (2.3.2) with $g = M_2h_2$ and its conjugate with $g = M_2h_1$ to get the latter equality.

Let us assign

$$\begin{aligned} &D = I - \Gamma^*\Gamma \geq 0 \text{ (because } \ ||\Gamma|| \leq 1), \\ &M_2h = P_+t\Phi^*\Gamma h : H \to F, \ M_1h = 0, \\ &T_1 = T = t \cdot \_\_, \ T_2 = id \text{ on } H, \\ &X = H, \ \mathcal{E}_1 = \{0\}, \ \mathcal{E}_2 = F. \end{aligned} \tag{2.3.3}$$

Hence, the new setting transforms the previous equality to identity (1.1.1). It means that we have reduced the given (4B)-problem to an (AI)-problem with appropriate data. We will call the problem the (AI')-problem to stress the particular choice of interpolation data (2.3.3).

We want to finish this subsection with an explicit formula for the operator of the orthogonal projection $P_{H'}$. Arguing as during the computation of M_2h, one can get that

$$P_{H'}g = g - \Phi P_+\Phi^*g, g \in L^2(E_1') \oplus L^2(E_2'), \tag{2.3.4}$$

where P_+ is the Riesz projection acting from $L^2(F)$ to $H^2(F)$ (see also [12, Ch. 2] or [14]). The relation will be useful in the next section.

3. Correspondence between the solutions of the (4B)-problem and the (AI')-problem

3.1. We begin this subsection with the following observation. Let (ω, F) be an arbitrary solution of the (AI')-problem. Since $\mathcal{E}_1 = \{0\}$ and ω is a $[\mathcal{E}_1, \mathcal{E}_2]$-valued function, we automatically obtain that $\omega \equiv 0$ on the unit disk $\mathbf{D}$. Then, by definition (1.1.2)

$$H_\omega = \begin{bmatrix} \{0\} \\ L^2(F) \end{bmatrix} \ominus \begin{bmatrix} \{0\} \\ H^2_-(F) \end{bmatrix} = \begin{bmatrix} \{0\} \\ H^2(F) \end{bmatrix} \simeq H^2(F).$$

Consequently, the first component F_- of the Fourier representation is identically equal to zero, while the second one maps H into $H^2(F)$ and satisfies the relations

$$1) \quad F_+th = tF_+h + M_2h, \ \forall h \in H, \tag{3.1.1}$$

$$2) \quad (F_+h, F_+h) \leq ((I - \Gamma^*\Gamma)h, h), \ \forall h \in H. \tag{3.1.2}$$

To derive them, we substitute the explicit expressions from (2.3.3) into formulas (1.1.4) and (1.1.6).

3.2. The connection between solutions of the (AI')-problem and the (4B)-problem is given by the following proposition.

Proposition 3.2. *The following assertions hold true:*

1) *every solution of the (AI')-problem determines a solution of the (4B)-problem by the formula*

$$\Omega_F h = (\Gamma + \Phi F_+)h, \;\; h \in H; \tag{3.2.1}$$

2) *conversely, an arbitrary solution of the (4B)-problem generates a solution of the (AI')-problem*

$$F_{+,\Omega} h = \Phi^*(\Omega - \Gamma)h, \;\; h \in H; \tag{3.2.2}$$

3) *the mappings are bijective and mutually inverse.*

Let us prove the first assertion of the proposition. In other words, we should demonstrate that Ω_F, defined by (3.2.1), possesses properties (2.2.2) and (2.2.3). We have

$$P_{H'}\Omega_F h = P_{H'}(\Gamma + \Phi F_+)h = P_{H'}\Gamma h = \Gamma h,$$

since $\Phi F_+ h \perp H'$. We proceed with the estimation of the norm of the operator Ω_F. We get

$$\begin{aligned}(\Omega_F h, \Omega_F h) = & \;\; ((\Gamma + \Phi F_+)h, (\Gamma + \Phi F_+)h) \\ = & \;\; (\Gamma h, \Gamma h) + (F_+ h, F_+ h).\end{aligned}$$

Here we have used the orthogonality of Γh to $\Phi F_+ h$, for every h from H, and the equality $\Phi^*\Phi = 1_F$. The latter sum can be estimated with the help of (3.1.2):

$$(\Gamma h, \Gamma h) + (F_+ h, F_+ h) \leq (h, h),$$

hence $||\Omega_F|| \leq 1$. To show that the operator Ω_F is actually the multiplication by a certain operator-valued function, it suffices to verify that

$$\Omega_F t h = t\Omega_F h, \forall h \in H.$$

Using commutativity relations (2.3.1) and (3.1.1), we obtain

$$\begin{aligned}(\Gamma + \Phi F_+)th = & \;\; t\Gamma h - \Phi M_2 h + \Phi(tF_+ h + M_2 h) \\ = & \;\; t(\Gamma + \Phi F_+)h = t\Omega_F h.\end{aligned}$$

This computation completes the proof of the first part of the proposition.

Let us pass to the second part now. The formula for orthogonal projection $P_{H'}$ (see (2.3.4)) implies that

$$\Gamma h = \Omega h - \Phi P_+ \Phi^* \Omega h, \tag{3.2.3}$$

and, hence, by virtue of (3.2.2)

$$F_{+,\Omega} h = P_+ \Phi^* \Omega h \in H^2(F). \tag{3.2.4}$$

We obtain inequality (3.1.2) for $F_{+,\Omega}$ by estimating the norm of the functional l,

$$l(g) = (g, F_{+,\Omega}h), \ g \in L^2(F).$$

The Schwarz–Bunyakovskii inequality immediately yields that

$$|l(g)|^2 = |(\Phi g, (\Omega - \Gamma)h|^2 \leq ((\Omega - \Gamma)h, (\Omega - \Gamma)h)(g, g).$$

It is not difficult to see that

$$((\Omega - \Gamma)h, (\Omega - \Gamma)h) = (\Omega h, \Omega h) - (\Gamma h, \Gamma h),$$

since $(\Omega h, \Gamma h) = (P_{H'}\Omega h, \Gamma h) = (\Gamma h, \Gamma h)$, in accordance with (2.2.2). Hence, we have

$$(\Omega h, \Omega h) - (\Gamma h, \Gamma h) \leq (h, h) - (\Gamma h, \Gamma h) = ((I - \Gamma^*\Gamma)h, h),$$

because of $||\Omega|| \leq 1$. Thus we conclude that

$$||l||^2 = ||F_{+,\Omega}h||^2 \leq ((I - \Gamma^*\Gamma)h, h), \ \forall h \in H,$$

and the second statement is also proved.

Let Ω be an arbitrary solution of the (4B)-problem. We set $F_{+,\Omega}$ by Ω with the help of (3.2.2), or, which is the same, with the help of (3.2.4). Then, combining (3.2.1) and (3.2.3), we obtain

$$\tilde{\Omega}_{F_{+,\Omega}}h = \Gamma h + \Phi F_{+,\Omega}h = \Omega h - \Phi P_+\Phi^*\Omega h + \Phi P_+\Phi^*\Omega h = \Omega h, \ h \in H.$$

Conversely, let a solution of the (AI')-problem, say, F_+, be given. We construct the corresponding solution of the (4B)-problem according to (3.2.1). Then we pass one more time to a solution $\tilde{F}_{\Omega_F}$ of the (AI')-problem by (3.2.2). Summing up, we have

$$\tilde{F}_{\Omega_F} = \Phi^*(\Omega_F - \Gamma) = \Phi^*(\Gamma + \Phi F_+ - \Gamma) = F_+\,.$$

We are done.

3.3. First, being in the setting of the (AI')-problem, we are able to rewrite formula (1.2.2) in the terms of the (4B')-problem. Indeed, taking into account equalities (2.1.1), (1.2.2) and formula (2.3.2), we see that

$$d_V = \{tH^2(E_1) \oplus L^2(E_2)\} \oplus \{0\},$$

and

$$N_{d_V} = \{H^2(E_1) \oplus L^2(E_2)\} \ominus \{tH^2(E_1) \oplus L^2(E_2)\} = E_1 \ominus (E_1 \cap Ker(I - \Gamma^*\Gamma)^{1/2}).$$

Unfortunately, we cannot say anything about the structure of the second defect subspace.

Since all solutions of the (4B)-problem are, in fact, operator-valued functions from $L^2(E_1 \oplus E_2, E_1' \oplus E_2')$, it is sufficient to point out the values of the operators on vectors from $E_1 \oplus E_2$. Comparing the second row of (1.4.2) with (3.2.1), we conclude that the following parametrization takes place

$$\Omega_F \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \left(\Gamma + \begin{bmatrix} \theta \\ 0 \end{bmatrix} G_{+2} + \begin{bmatrix} \theta \\ 0 \end{bmatrix} s_2 (1_{\mathcal{N}_2} - \varepsilon s)^{-1} \varepsilon G_{+1} \right) \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, e_1 \in E_1, e_2 \in E_2,$$

where ε is an arbitrary function from $\mathbf{B}[\mathcal{N}_1, \mathcal{N}_2]$.

Acknowledgments. It is a pleasure to express my gratitude to P.M. Yuditskii for stimulating discussions and the patience he manifested while supervising this work. Also I would like to thank A.Ya. Kheifets for his constant interest to the problem and helpful remarks.

References

[1] *D.Z. Arov and L.Z. Grossman*, Scattering matrices in the extension theory of isometrical operators. — Soviet Math. Dokl. (1983), v. 27, p. 573–578.

[2] *J. Ball and N. Cohen*, De Branges–Rovnyak operator models and system theory: a survey. — Operator Theory: Adv. and Appl. (1991), v. 50, p. 93–136.

[3] *J. Ball and T. Trent*, The abstract interpolation problem and commutant lifting: a coordinat-free approach. (In preparation)

[4] *H. Dym and B. Freidin*, Bitangential interpolation for upper triangular operators. — Operator Theory: Adv. and Appl. (1997), v. 95, p. 105–142.

[5] *A. Feintuch and B.A. Francis*, Distance formulas for operator algebras arising in optimal control problems. — Topics in Operator Theory and Interpolation; Operator Theory: Adv. and Appl. (I. Gohberg, ed.) (1988), v. 29, p. 151–170.

[6] *C. Foias and A.E. Frazho*, The Commutant Lifting Approach to Interpolation Problems. — Operator Theory: Adv. and Appl., Birkhauser–Verlag (1990), v. 44.

[7] *A.E. Frazho*, A note on the commutant lifting theorem and a generalized four block problem. — Integral Equations Operator Theory (1991), v. 14, p. 299–303.

[8] *V.E. Katsnelson, A.Ya. Kheifets, and P.M. Yuditskii*, An abstract interpolation problem and extension theory of isometric operators. — Operators in Function Spaces and Problems in Function Theory: Collected scientific papers, Naukova Dumka, Kyiv (1987), p. 83–96 (Russian); (English transl.: Operator Theory: Adv. and Appl. (1997), v. 95, p. 283–298).

[9] *A.Ya. Kheifets and P.M. Yuditskii*, An analysis and extension of V.P. Potapov's approach to interpolation problems with applications to the generalized bi-tangential Schur–Nevanlinna–Pick problem and j-inner-outer factorization. — Operator Theory: Adv. and Appl. (1994), v. 72, p. 133–161.

[10] *A.Ya. Kheifets*, Parseval equality in abstract interpolation problem. — Teor. Funktsii, Funktsion. Anal. i ikh Prilozhen. (1988), v. 49, p. 112–120; (1988), v. 50, p. 98–103 (Russian); (English transl.: J. Soviet Math. (1990), v. 49, p. 1114–1120, p. 1307–1310.

[11] *S. Kupin*, Lifting theorem as a special case of abstract interpolation problem. — J. Anal. and its Applications (1996), v. 15, No. 4, p. 789–798.

[12] *N.K. Nikolskii*, Treatise on the Shift Operator. Springer–Verlag, Berlin, New York (1986).

[13] *B. Sz.-Nagy and C. Foias*, Harmonic analysis of operators on Hilbert spaces. Amer. Elsevier, New York (1970).

[14] *P.M. Yuditskii*, Lifting problem. — Dep. Ukr. SSR NIINTI 18.04.83, No. 311-Uk-D83 (Russian).

Абстрактная интерполяционная задача и задача о четырех блоках

С.А. Купин

Изучается 4-блок задача как абстрактная интерполяционная задача с подходящим образом подобранными данными. Этот подход позволяет получить описание всех решений произвольной 4-блок задачи.

Абстрактна інтерполяційна задача та задача про чотири блоки

С.О. Купін

Вивчається 4-блок задача як абстрактна інтерполяційна задача з відповідно підібраними даними. Цей підхід дозволяє одержати опис усіх розв'язань довільної 4-блок задачі.