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On the spectra of infinite Hessenberg and Jacobi matrices

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The probability measures on the unit circle are studied in conjunction with multiplication operators acting in appropriate Hilbert spaces. The measure with constant reflection coefficients and the corresponding operator are treated as unperturbed objects. Under certain perturbations of this measure it is shown that the support of perturbed measure contains the support of the original one. More generally, we evaluate the size of gaps inside the support of the perturbed measure. The similar results pertaining to Jacobi and banded matrices are also under consideration.

1. Introduction

In the present paper we continue initiated in [6] study of probability measures μ on the unit circle $\mathbb{T}$ with infinite support, supp μ, in terms of their reflection coefficients $a_n = a_n(\mu) = \Phi_n(\mu, 0)$ based on the spectral theory of unitary operators in the Hilbert spaces. Here Φ_n are monic orthogonal polynomials with respect to μ and $|a_n| < 1,\ n \geq 1$. We assume that μ does not belong to the Szegő class, that is, $\sum_{n=1}^{\infty} |a_n|^2 = \infty$. In that case the unitary multiplication operator $U(\mu)$ which is defined by

$$[U(\mu)]\, f(\zeta) = \zeta f(\zeta), \quad \zeta \in \mathbb{T}, \quad f \in L^2(\mu, \mathbb{T}),$$

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and its matrix representation

$$U(\mu) = \begin{pmatrix} u_{00} & u_{01} & \cdots \\ u_{10} & u_{11} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \qquad u_{kj} = (U(\mu)\varphi_j, \varphi_k)_\mu, \tag{1}$$

where for $j \in \mathbb{Z}^+ \stackrel{\text{def}}{=} \{0, 1, \ldots\}$

$$u_{kj} = \begin{cases} -a_{j+1}\overline{a_k}\prod_{p=k+1}^{j}\left(1-|a_p|^2\right)^{1/2}, & \text{for} \quad k = 0, 1, \ldots, j, \\ \left(1-|a_{j+1}|^2\right)^{1/2}, & \text{for} \quad k = j+1, \\ 0, & \text{for} \quad k \geq j+2 \end{cases} \tag{2}$$

(cf. [7, p. 401]) are in effect. The infinite matrices in which all entries below the subdiagonal vanish are called the *Hessenberg matrices*.

The cornerstone of operator theoretic approach to the theory of orthogonal polynomials on the unit circle results from the fact that the spectral measure of operator U (1)–(2) which acts in ℓ^2 agrees with the orthogonality measure μ. In particular, supp μ coincides with the spectrum $\sigma(U)$ of U. Thereby we can relate two measures, μ and μ' on $\mathbb{T}$ (with reflection coefficients sequences $\{a_n\}$ and $\{a'_n\}$, respectively) by using perturbation theory methods applied to the pair of operators U, U' (1)–(2) acting on the same Hilbert space ℓ^2. More precisely, the closeness of $\{a_n\}$ to $\{a'_n\}$ in a certain sense yields the closeness of U to U' in the sense of operator theory which enables one to compare their spectra.

Throughout the paper we take $\mu' = \mu_a$ having constant reflection coefficients $a_n(\mu_a) = a \neq 0, \quad n = 1, 2, \ldots$ (it is called the *Geronimus measure*). The corresponding operator (1)–(2) takes now the form

$$U_a = \begin{pmatrix} -a & -a\rho & -a\rho^2 & \cdots \\ \rho & -|a|^2 & -|a|^2\rho & \cdots \\ 0 & \rho & -|a|^2 & \cdots \\ \vdots & \vdots & \ddots & \end{pmatrix}, \quad \rho^2 = 1 - |a|^2. \tag{3}$$

For $|\tau_1| = |\tau_2| = 1$ denote by (τ_1, τ_2) ($[\tau_1, \tau_2]$) an open (closed) arc on $\mathbb{T}$ swept out as τ_1 moves to τ_2 counterclockwise. As is known (cf. e.g. [3, formulas (XI.26), (XI.27)]) the spectrum $\sigma(U_a)$ (that is, the support of μ_a) consists of the arc

$$\Delta_\alpha = [\tau, \bar{\tau}], \quad \tau = e^{i\alpha}, \quad 0 < \alpha \stackrel{\text{def}}{=} 2\arcsin|a| < \pi \tag{4}$$

along with at most one eigenvalue (masspoint) off this arc.

As recently as 1941 Ya.L. Geronimus studied the measures on $\mathbb{T}$ which satisfy

$$\lim_{n\to\infty} a_n = a, \qquad 0 < |a| < 1 \tag{5}$$

and established the following two properties of the support (cf. [4, Theorem 1']):

(i) $\Delta_\alpha \subset \operatorname{supp} \mu$,

(ii) $\operatorname{supp} \mu$ on $\Delta'_\alpha = \mathbb{T} \backslash \Delta_\alpha$ is at most countable set with no limit points inside Δ'_α

(for the operator theoretic approach see [7, Theorem 3]).

We will show in Section 2 that (5) can be relaxed as far as (i) goes. Moreover, we will be able to estimate the size of gaps in $\sigma(U)$ on the arc Δ_α. The property (ii) turns out to be more unstable. The example of a measure ν such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} |a_j(\nu) - a| = 0$$

and $\operatorname{supp} \nu = \mathbb{T}$ can be manufactured. In contrast to Theorem 9 below the construction here is based on different ideas and will be presented elsewhere.

The similar results pertaining to measures on the real line, Jacobi and finite-banded matrices are discussed in Section 3.

2. Method of test vectors

The method of test functions is well known in the theory of singular differential operators (cf. [5, Ch. 2]). We adapt this idea in the context of unitary operators (1)–(2).

A vector $h \in \ell^2$ is called *finite* if it has a finite number of nonzero entries.

Lemma 1. *For operator U_a (3) and for each $\zeta \in \Delta_\alpha$ (4) there exists a sequence $\{\eta(n,\zeta)\}_{n\ge 0}$ of finite vectors with $\|\eta(n)\| = 1$ such that*

$$\lim_{n\to\infty} \|(U_a - \zeta I)\, \eta(n,\zeta)\| = 0. \tag{6}$$

P r o o f. According to the definition of (continuous) spectrum the normed sequence which satisfies (6) does exist. In our particular situation we can find the sequence with some additional properties.

We begin with the characteristic equation

$$w^2 - \frac{\zeta+1}{\rho} w + \zeta = 0 \tag{7}$$

which is closely related to the measure μ_a and operator U_a (see [8, Section 2]), and its root $r_1 = r_1(\zeta)$. We know that $|r_1| = 1,\ \zeta \in \Delta_\alpha$. It easily follows from (7) that

$$\rho r_1^{-n+1} - |a|^2 \sum_{k=n}^{\infty} \rho^{k-n} r_1^{-k} = \zeta r_1^{-n}, \qquad n = 1, 2, \ldots . \tag{8}$$

On the other hand, for an arbitrary vector $h = \{h_n\}_{n\geq 0} \in \ell^2$ put $g = U_a h = \{g_n\}_{n\geq 0}$. Then

$$\begin{aligned} g_0 &= -a\sum_{k=0}^{\infty}\rho^k h_k, \\ g_n &= \rho h_{n-1} - |a|^2\sum_{k=n}^{\infty}\rho^{k-n}h_k, \quad n = 1, 2, \ldots . \end{aligned} \tag{9}$$

Hence the vector $r \stackrel{\text{def}}{=} \{r_1^{-n}\}_{n\geq 0}$ is "almost" eigenvector of U_a (the point is that $r \notin \ell^2$). So we take the truncated vector

$$r(p,\zeta) = \{r_n(p,\zeta)\}_{n\geq 0} = \{1, r_1^{-1}, r_1^{-2}, \ldots, r_1^{-p}, 0, 0, \ldots\}$$

and put $\xi(p) = \{\xi_n(p)\}_{n\geq 0} = (U_a - \zeta I)\, r(p)$. It is clear from (9) that

$$\xi_n(p,\zeta) = \begin{cases} -a\sum_{k=0}^{p}\rho^k r_1^{-k} - \zeta, & n = 0, \\ \rho r_1^{-n+1} - |a|^2\sum_{k=n}^{p}\rho^{k-n}r_1^{-k} - \zeta r_1^{-n}, & n = 1, 2, \ldots, p, \\ \rho r_1^{-p}, & n = p+1. \end{cases}$$

and $\xi_n(p) = 0$ for $n \geq p+2$. By (8) we see that

$$\xi_n(p,\zeta) = |a|^2\sum_{k=p+1}^{\infty}\rho^{k-n}r_1^{-k} = \frac{|a|^2}{r_1-\rho}\, r_1^{-p}\rho^{p+1-n}, \quad n = 1, 2, \ldots, p\,.$$

Next, for $q > p$ consider the difference

$$r(p,q;\zeta) = r(q,\zeta) - r(p,\zeta) = \{0, \ldots, 0, r_1^{-p-1}, \ldots, r_1^{-q}, 0, \ldots\}.$$

For $\xi(p,q) = \{\xi_n(p,q)\}_{n\geq 0} = (U_a - \zeta I)\, r(p,q) = \xi(q) - \xi(p)$ we now have

$$\xi_n(p,q;\zeta) = \begin{cases} \frac{|a|^2}{r_1-\rho}(r_1^{-q}\rho^{q+1-n} - r_1^{-p}\rho^{p+1-n}), & n = 1, 2, \ldots, p, \\ \frac{|a|^2 r_1^{-q}}{r_1-\rho}\,\rho^{q+1-n}, & n = p+2, \ldots, q \end{cases}$$

and also

$$\xi_0(p,q;\zeta) = -a\sum_{k=p+1}^{q}\rho^k r_1^{-k}, \quad \xi_{p+1}(p,q;\zeta) = \frac{|a|^2}{r_1-\rho}\, r_1^{-q}\rho^{q-p} - \rho r_1^{-p},$$

$\xi_{q+1}(p,q;\zeta) = \rho r_1^{-q}$ and $\xi_n(p,q) = 0$, $n \geq q+2$. Hence for $\zeta \in \Delta_\alpha$

$$\sum_{n=p+2}^{q}|\xi_n(p,q;\zeta)|^2 \leq \frac{|a|^4}{(1-\rho)^2}\sum_{n=p+2}^{q}\rho^{2(q+1-n)} \leq C_1(a)$$

and similarly

$$\sum_{n=1}^{p} |\xi_n(p,q;\zeta)|^2 \le C_2(a),$$

where $C_k(a)$, $k = 1, 2, \dots$ stand for positive constants which depend only on a. The same inequality is obviously true for the rest of indices. Thus we come to the following relation

$$\|(U_a - \zeta I)\, r(p,q;\zeta)\| \le C_3(a), \quad \zeta \in \Delta_\alpha. \tag{10}$$

Since $|r_1| = 1$ on Δ_α we deduce that $\|r(p,q)\|^2 = q - p$. Define now the sequence $\eta(n)$ announced in Lemma by

$$\eta(n,\zeta) \overset{\text{def}}{=} \frac{r(p_n, q_n; \zeta)}{\sqrt{q_n - p_n}}\,,$$

so that each $\eta(n)$ is a finite vector and $\|\eta(n)\| = 1$. It is clear from (10) that

$$\lim_{n\to\infty} \|(U_a - \zeta I)\, \eta(n,\zeta)\| = 0$$

as long as $q_n - p_n$ tends to infinity. It is also worth mentioning that the system $\eta(n)$ is orthonormal if we take $p_{n+1} > q_n$. ∎

Let μ be a probability measure on $\mathbb{T}$ with reflection coefficients a_n and unitary Hessenberg operator U (1)–(2). The difference

$$V \overset{\text{def}}{=} U - U_a = \{v_{k,j}\}_{k,j\ge 0}$$

which acts in ℓ^2 provides some information about the closeness of supp $\mu = \sigma(\mathrm{U})$ to the arc Δ_α, which constitutes the essential support of μ_a.

Theorem 2. *Assume that a_n are bounded away from the origin*

$$\inf_n |a_n| > 0. \tag{11}$$

For $\zeta \in \Delta_\alpha$ let an arc $\Omega = (\zeta - \tau, \zeta + \tau)$ have an empty intersection with supp μ. *Then*

$$\tau^2 \le C \liminf_{d_n\to\infty} \frac{1}{d_n} \sum_{j=p_n+1}^{q_n} |a_j - a|\,, \quad d_n \overset{\text{def}}{=} q_n - p_n, \tag{12}$$

where C is a positive constant which depends on a and a_n.

P r o o f. Since Ω belongs to the resolvent set of U, the Spectral Theorem for unitary operators states that

$$\|(U-\zeta I)\,h\|^2=\int_{\mathbb{T}\setminus\Omega}|\lambda-\zeta|^2\,d\,(E(\lambda)h,h)\geq 4\sin^2\frac{\tau}{2}\,\|h\|^2$$

for each vector $h\in\ell^2$, and hence

$$2\sin\frac{\tau}{2}\leq\|(U-\zeta I)\,h\|\leq\|(U_a-\zeta I)\,h\|+\|Vh\|\,. \tag{13}$$

To evaluate the right-hand side of (13) we need to handle vector h in appropriate way.

Put $h=\eta(n,\zeta)$. By Lemma 1 the first term goes to zero as $n\to\infty$, so let us focus on the second term

$$\|V\eta(n,\zeta)\|^2=\frac{1}{q-p}\sum_{j=0}^{\infty}\left|\sum_{n=p+1}^{q}v_{jn}r_1^{-n}(\zeta)\right|^2\leq\frac{1}{q-p}\sum_{j=0}^{\infty}\left(\sum_{n=p+1}^{q}|v_{jn}|\right)^2, \tag{14}$$

$p=p_n, q=q_n$. Note that $v_{jn}=0$ for $j\geq n+2$, so that

$$(q-p)\,\|V\eta(n,\zeta)\|^2\leq\sum_{j=0}^{q+1}\left(\sum_{n=p+1}^{q}|v_{jn}|\right)^2.$$

By [6] (see Step 2 in the proof of Theorem 6) we have $v_{j,j-1}=\rho_j-\rho$ for $j=1,2,\dots,$

$$\begin{aligned}|v_{jn}| &\leq (|a_{n+1}-a|+|a_j-a|)\rho^{n-j}+R(j,n),\\ R(j,n) &= \sum_{l=j+1}^{n}\rho_{j+1}\dots\rho_{l-1}|\rho_l-\rho|\rho^{n-l},\qquad n\geq j\geq 1,\end{aligned} \tag{15}$$

Here $\rho_j^2\stackrel{\text{def}}{=}1-|a_j|^2>0$ and $R(j,j)=0$. The inequality (15) holds for $j=0$ as well if we take $a_0=a$. It is clear from (11) that $\sup_n\rho_n=\delta<1$ and

$$|R(j,n)|\leq\sum_{l=j+1}^{n}\delta^{l-j-1}\rho^{n-l}.$$

To estimate $\|V\eta(n)\|$ we split the sum in (14) in three groups.

1. Let $0\leq j\leq p+1$. Since $|a_n-a|<2$, we have

$$\sum_{n=p+1}^{q}|v_{jn}|\leq\sum_{n=p+1}^{q}\left(4\rho^{n-j}+|R(j,n)|\right)\leq\frac{4\,\rho^{p+1-j}}{1-\rho}+\sum_{n=p+1}^{q}\sum_{l=j+1}^{n}\delta^{l-j-1}\rho^{n-l}. \tag{16}$$

Next

$$\sum_{n=p+1}^{q}\sum_{l=j+1}^{n}\delta^{l-j-1}\rho^{n-l}=\sum_{n=p+1}^{q}\left(\sum_{l=j+1}^{p}+\sum_{l=p+1}^{n}\right)\delta^{l-j-1}\rho^{n-l}=S_1+S_2\,,$$

where

$$\begin{aligned} S_1 &= \sum_{l=j+1}^{p}\delta^{l-j-1}\sum_{n=p+1}^{q}\rho^{n-l}\le(1-\rho)^{-1}\sum_{l=j+1}^{p}\delta^{l-j-1}\rho^{p+1-l}, \\ S_2 &= \sum_{l=p+1}^{q}\delta^{l-j-1}\sum_{n=l}^{q}\rho^{n-l}\le K\delta^{p-j}. \end{aligned} \tag{17}$$

Here and in what follows K stands for (different) positive constants which depend on a and a_n. Note also that $S_1\le K$ and hence $S_1^2\le KS_1$. From (16)–(17) we derive

$$\sum_{n=p+1}^{q}|v_{jn}|\le K\left(\rho^{p-j}+\delta^{p-j}+S_1\right),$$

$$\left(\sum_{n=p+1}^{q}|v_{jn}|\right)^2\le K\left(\rho^{2(p-j)}+\delta^{2(p-j)}+S_1\right)$$

and finally

$$\sum_{j=0}^{p+1}\left(\sum_{n=p+1}^{q}|v_{jn}|\right)^2\le K+K\sum_{j=0}^{p+1}\sum_{l=j}^{p+1}\delta^{2(l-j-1)}\rho^{2(p+1-l)}\le K. \tag{18}$$

2. Let now $p+2\le j\le q$. Then

$$\begin{aligned} \sum_{n=p+1}^{q}|v_{jn}| &= \sum_{n=j-1}^{q}|v_{jn}|=|v_{j,j-1}|+\sum_{n=j}^{q}|v_{jn}| \\ &\le |\rho_j-\rho|+\sum_{n=j}^{q}\left(|a_{n+1}-a|+|a_j-a|\right)\rho^{n-j}+\sum_{n=j}^{q}R(j,n) \\ &= S_3+S_4. \end{aligned}$$

To estimate $R(j,n)$ we proceed in a more delicate way. As $|\rho_j-\rho|\le K|a_j-a|$ (cf. [6, formula (38)]) we have

$$S_3 = |\rho_j-\rho|+\sum_{n=j}^{q}\left(|a_{n+1}-a|+|a_j-a|\right)\rho^{n-j}$$

$$\leq K\left(|a_j - a| + \sum_{n=j}^{q} |a_{n+1} - a|\rho^{n-j}\right),$$

$$S_4 = \sum_{n=j}^{q} R(j,n) \leq K \sum_{n=j}^{q}\sum_{l=j}^{n} \delta^{l-j-1}|a_l - a|\rho^{n-l}$$

$$\leq K\sum_{l=j}^{q}\delta^{l-j-1}|a_l - a|\sum_{n=l}^{q}\rho^{n-l} \leq K\sum_{l=j}^{q}\delta^{l-j-1}|a_l - a|.$$

Again, as above, $S_m \leq K,\ m = 3,4$ and $S_m^2 \leq K S_m$. Hence

$$\left(\sum_{n=p+1}^{q} |v_{jn}|\right)^2 \leq K(S_3 + S_4)$$

$$\leq K\left(|a_j - a| + \sum_{n=j}^{q}|a_{n+1} - a|\rho^{n-j} + \sum_{l=j}^{q}|a_l - a|\delta^{l-j-1}\right)$$

and finally

$$\sum_{j=p+2}^{q}\left(\sum_{n=p+1}^{q}|v_{jn}|\right)^2 \leq K\sum_{j=p+2}^{q+1}|a_j - a|. \tag{19}$$

3. For $j = q+1$ the bound is obvious

$$|v_{q+1,q}| = |\rho_{q+1} - \rho| \leq 1. \tag{20}$$

Upon combining (18)–(20) we obtain the following inequality

$$\|V\eta(n,\zeta)\|^2 \leq \frac{1}{q-p}\sum_{j=0}^{q+1}\left(\sum_{n=p+1}^{q}|v_{jn}|\right)^2 \leq \frac{K}{q-p}\left(1 + \sum_{j=p+1}^{q}|a_j - a|\right). \tag{21}$$

Going back to (13) we see that

$$\tau^2 \leq \frac{\pi^2}{2}\left(\|(U_a - \zeta I)\,\eta(n,\zeta)\|^2 + \|V\eta(n,\zeta)\|^2\right).$$

The statement of theorem is now immediate from (21) and Lemma 1. ■

The next result is akin to [5, Ch. 2, Theorem 22].

Theorem 3. *Let there exist a sequence of intervals* $I_n = [p_n, q_n]$ *with* $q_n - p_n \to \infty$ *as* $n \to \infty$ *such that*

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} \sum_{j=p_n+1}^{q_n} |a_j - a| = 0.$$

Assume further that $\liminf_{n\to\infty} |a_n| > 0$. *Then* $\Delta_\alpha \subset \operatorname{supp} \mu$.

P r o o f. Let μ and $\tilde{\mu}$ be measures on $\mathbb{T}$ with the Hessenberg operators U, $\tilde{U}$ (1)–(2) having the same reflection coefficients from some point on

$$a_n = \tilde{a}_n, \quad n = N, N+1, \dots .$$

It is not hard to check (cf. [6, Remark 5]) that

$$u_{k,j} = \tilde{u}_{k,j}, \quad k = N, N+1, \dots ,$$

which means that U is the finite rank perturbation of $\tilde{U}$.

Now, given a_n with $\liminf_{n\to\infty} |a_n| > 0$ we can change finite number of a_n's to end up with the new sequence $\tilde{a}_n$ satisfying (11). By (12) $\tau = 0$, that is, supp $\tilde{\mu}$ contains Δ_α or, in other words, $\Delta_\alpha \subset \sigma_{es}(\tilde{U})$. By H. Weyl's Theorem (cf. [9, Problem 143, p. 91]) $\sigma_{es}(U) = \sigma_{es}(\tilde{U})$ contains Δ_α as needed. ■

Corollary 4. *In the premises of Theorem 3 let*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} |a_j - a| = 0.$$

Then $\Delta_\alpha \subset \operatorname{supp} \mu$.

3. Jacobi and banded matrices

Our objective here is to establish the result similar to Theorem 3 in the setting of Jacobi (or, more generally, finite-banded) matrices.

Consider an infinite complex Jacobi matrix

$$J = \begin{pmatrix} \beta_0 & \alpha_1 & 0 & 0 & \dots \\ \gamma_1 & \beta_1 & \alpha_2 & 0 & \dots \\ 0 & \gamma_2 & \beta_2 & \alpha_3 & \dots \\ 0 & 0 & \gamma_3 & \beta_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \alpha_j, \beta_j, \gamma_j \in \mathbb{C} \tag{22}$$

with bounded entries: $\sup_n (|\alpha_n| + |\beta_n| + |\gamma_n|) < \infty$. It is known to generate a bounded operator in ℓ^2 (which we denote by the same letter J). The spectrum

$\sigma(J)$ is defined as the set of points $\zeta \in \mathbb{C}$ for which the operator $J - \zeta I$ is not invertible.

H. Weyl's Theorem, which is by far one of the fundamental results in operator theory, claims that $[-1,1] \subset \sigma(J)$ whenever

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \gamma_n = \frac{1}{2}, \qquad \lim_{n\to\infty} \beta_n = 0 \tag{23}$$

(cf. e.g. [1] for the application of this theorem to complex Jacobi matrices). It turns out that (23) can be relaxed as follows.

Theorem 5. *Let there exist a sequence of intervals $I_n = [p_n, q_n]$ with $q_n - p_n \to \infty$ as $n \to \infty$ such that*

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} \sum_{j=p_n}^{q_n} \left(\left|\alpha_j - \frac{1}{2}\right|^2 + |\beta_j|^2 + \left|\gamma_j - \frac{1}{2}\right|^2 \right) = 0. \tag{24}$$

Then $[-1,1] \subset \sigma(J)$. In particular, the latter is true provided

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \left(\left|\alpha_j - \frac{1}{2}\right|^2 + |\beta_j|^2 + \left|\gamma_j - \frac{1}{2}\right|^2 \right) = 0. \tag{25}$$

P r o o f. Let

$$J_0 = \begin{pmatrix} 0 & 1/2 & 0 & 0\dots & \\ 1/2 & 0 & 1/2 & 0\dots & \\ 0 & 1/2 & 0 & 1/2\dots & \\ 0 & 0 & 1/2 & 0\dots & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and put $V \stackrel{\text{def}}{=} J - J_0$. As (23) does not in general hold, J is no longer a compact perturbation of J_0. We apply the method of test vectors to $J - \zeta I$ with $\zeta = \cos\theta \in [-1,1]$ by taking

$$r(p,q;\zeta) = \{0, \dots, 0, \cos(p+1)\theta, \dots, \cos q\theta, 0, \dots\}, \quad p < q. \tag{26}$$

Now the vector $(J_0 - \zeta I)r(p,q)$ has at most four nonzero entries, each of which of the form $\pm\frac{1}{2}\cos k\theta$, so that $\|(J_0 - \zeta I)r(p,q)\| \le 1$. On the other hand, for fixed $|\zeta| \le 1$

$$\begin{aligned} \|r(p,q;\zeta)\|^2 &= \sum_{k=p+1}^{q} \cos^2 k\theta = \frac{q-p}{2} + \frac{1}{2} \sum_{k=p+1}^{q} \cos 2k\theta \\ &= \frac{1}{2}\{(q-p) + \frac{\sin(q-p)\theta}{\sin\theta} \cos(p+q+1)\theta\} \ge C(\zeta)(q-p) \end{aligned}$$

with some positive constant $C(\zeta)$ $(C(\pm 1)=1)$. Hence for the vectors

$$\eta(p,q;\zeta)=\frac{r(p,q;\zeta)}{\sqrt{q-p}} \tag{27}$$

we have $\|\eta(p,q)\|\ge C(\zeta)>0$ and $\lim_{q-p\to\infty}\|(J_0-\zeta I)\eta(p,q)\|=0$.

Next, it is clear that

$$\|(J-\zeta I)\eta(p,q;\zeta)\|\le\|(J_0-\zeta I)\eta(p,q;\zeta)\|+\|V\eta(p,q;\zeta)\|\,.$$

It remains to show that the second term on the right-hand side goes to zero for a suitable choice of p,q. Indeed, put $\tau_k \overset{\text{def}}{=} \alpha_k-1/2$, $\omega_k \overset{\text{def}}{=} \gamma_k-1/2$. Then

$$(q-p)\,\|V\eta(p,q;\zeta)\|^2=|\tau_{p+1}|^2\cos^2(p+1)\theta+|\beta_{p+1}\cos(p+1)\theta+\tau_{p+2}\cos(p+2)\theta|^2$$
$$+\sum_{j=p+2}^{q-1}|\omega_j\cos(j-1)\theta+\beta_j\cos j\theta+\tau_{j+1}\cos(j+1)\theta|^2$$
$$+|\omega_q\cos(q-1)\theta+\beta_q\cos q\theta|^2+|\omega_{q+1}|^2\cos^2 q\theta$$

so that

$$\|V\eta(p,q;\zeta)\|^2\le\frac{3}{q-p}\sum_{j=p+1}^{q+1}\left(\left|\alpha_j-\frac12\right|^2+|\beta_j|^2+\left|\gamma_j-\frac12\right|^2\right).$$

We see that for $\eta_n=\eta(p_n-1,q_n-1)$ in hypothesis (24)

$$\lim_{n\to\infty}\|(J-\zeta I)\eta(n,\zeta)\|=0,\qquad \|\eta(n,\zeta)\|\ge C(\zeta)>0,$$

which implies $\zeta\in\sigma(J)$ giving the desired result. ■

R e m a r k 6. Let $\alpha_j=\gamma_j>0$, $\beta_j=\bar\beta_j$.Then J is closely related to a certain measure μ on the real line. Specifically, the three-term recurrence relation

$$xp_n(x)=\alpha_{n+1}p_{n+1}(x)+\beta_n p_n(x)+\alpha_n p_{n-1}(x),\quad p_{-1}=0,\ p_0=1,$$

gives rise to the system of polynomials orthogonal with respect to this measure μ and, what is more to the point, supp $\mu=\sigma(\mathrm{J})$. The restriction on α_n, β_n to be bounded is now not essential. We may assume that J is merely self–adjoint (not necessarily bounded) operator. The latter holds if, for instance, β_n is an arbitrary sequence and $\sum_{n=1}^{\infty}1/\alpha_n=\infty$ (cf. [2, Ch. 7, Theorem 1.3]).

The same line of reasoning leads to the counterpart of Theorem 5 for finite-banded matrices. An infinite matrix $B=\{b_{k,j}\}_{k,j\ge 0}$ is called m–banded if $b_{k,j}=0$ for $|k-j|>m$. We assume that the entries $b_{k,j}$ are bounded so that B generates a bounded operator B in ℓ^2.

Theorem 7. *Let there exist a sequence of intervals* $I_n = [p_n, q_n]$ *with* $q_n - p_n \to \infty$ *as* $n \to \infty$ *such that*

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} \sum_{j=p_n}^{q_n} \left(\left| b_{j,j-m} - \frac{1}{2} \right|^2 + \sum_{k=-m+1}^{m-1} |b_{j,j+k}|^2 + \left| b_{j,j+m} - \frac{1}{2} \right|^2 \right) = 0. \quad (28)$$

Then $[-1,1] \subset \sigma(J)$. *In particular, the latter is true provided*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \left(\left| b_{j,j-m} - \frac{1}{2} \right|^2 + \sum_{k=-m+1}^{m-1} |b_{j,j+k}|^2 + \left| b_{j,j+m} - \frac{1}{2} \right|^2 \right) = 0,$$

where $b_{j,l} = 0$ *for* $l < 0$.

P r o o f. We only sketch out the proof. Take $B_0 = \{b^0_{k,j}\}_{k,j\ge 0}$ with $b^0_{k,j} = \delta_{k,j\pm m}$, where $\delta_{p,q}$ is the standard Kronecker symbol (B_0 plays the same role as J_0 above). It is clear again that for $\zeta = \cos m\theta$ and $\eta(p,q)$ (26)–(27) we have $\|\eta(p,q)\| \ge C(\zeta) > 0$ and $\lim_{q-p\to\infty} \|(B_0 - \zeta I)\eta(p,q)\| = 0$.

Next, for $V = B - B_0$

$$\|(B - \zeta I)\eta(p,q;\zeta)\| \le \|(B_0 - \zeta I)\eta(p,q;\zeta)\| + \|V\eta(p,q;\zeta)\|,$$

and the second term on the right-hand side goes to zero for a suitable choice of p, q by (28). ■

Going back to Jacobi matrices (22) put $\alpha_n = \gamma_n = 1/2$.* Theorem 5 states now that $[-1,1] \subset \sigma(J)$ as long as

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} |\beta_j|^2 = 0 \quad (29)$$

holds. The latter is known to be equivalent to $\lim_{n\in\Lambda} \beta_n = 0$ along some subsequence $\Lambda \subset \mathbb{Z}^+$ of density 1 (cf. e.g. [10, Theorem 13.7.2]). Recall that a sequence $\Lambda \subset \mathbb{Z}^+$ is said to have *density* $d(\Lambda)$ if

$$d(\Lambda) \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{|\Lambda \cap \{0,1,\ldots,n\}|}{n+1}$$

exists, where $|X|$ stands for the number of points in $X \subset \mathbb{Z}^+$.

It is worth pointing out that Theorem 5 is sharp with regard to the density. More precisely, given $0 < d < 1$ there is a discrete Schrödinger operator J such

*Such matrices are usually called the *discrete Schrödinger operators*.

that $\beta_n \to 0$ along a sequence Δ with $d < d(\Delta) < 1$ and $[-1,1]\backslash\sigma(J) \neq \emptyset$. To make up such example pick a positive integer m with $d < m/(m-1)$ and put

$$\beta_n = 0, \quad n \neq km, \qquad \beta_n = \beta > 0, \quad n = km$$

with an appropriate choice of β.

Let us go over to the portion of $\sigma(J)$ off $[-1,1]$.

We start out with the following

Lemma 8. *Let J be a discrete Schrödinger operator and let there exist a sequence of intervals $[p_k, q_k]$ with $q_k - p_k \to \infty$ as $k \to \infty$ such that $\beta_j = w$, $j \in [p_k, q_k]$. Then $w \in \sigma(J)$.*

P r o o f. The line of reasoning here is quite similar to that in Theorem 5. The operator $J - wI$ has now an increasing sequence of zero blocks along the main diagonal. Hence the test vectors

$$\eta_k = \frac{r_k}{\sqrt{q_k - p_k}}\,, \ r_k = \{0, \dots, 0, \cos(p_k+1)\frac{\pi}{2}, \dots, \cos q_k \frac{\pi}{2}, 0, \dots\}$$

do the job. ∎

It is well known that (23) yields quite regular behavior of $\sigma(J)$ off $[-1,1]$. In contrast to this under condition (25) the spectrum turns out to be "nearly arbitrary" on the complex plane. More precisely,

Theorem 9. *Given a compact $K \subset \mathbb{C}$ there exists the discrete Schrödinger operator J such that* (29) *holds and $K \subset \sigma(J)$.*

P r o o f. Pick any countable dense set $\{w_1, w_2, \dots\}$ in K. To define β_n we proceed as follows. Consider a partition of the set $\mathbb{Z}^+$

$$\mathbb{Z}^+ = \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{n} I_{nk}\,,$$

wherein $I_{n,p+1}$ follows I_{np} and $I_{n-1,p}$ precedes I_{nq} for all p, q. Let

$$|I_{n0}| = 2^n, \qquad |I_{nk}| = n \quad k = 1, 2, \dots, n. \tag{30}$$

Now put

$$\beta_j = 0, \quad j \in I_0 \stackrel{\text{def}}{=} \bigcup_{n=0}^{\infty} I_{n0}; \qquad \beta_j = w_k, \quad j \in I_k \stackrel{\text{def}}{=} \bigcup_{n=k}^{\infty} I_{nk}.$$

The sequence I_0 has density 1 thanks to (30) and hence (29) holds. By Lemma 8 each $w_k \in \sigma(J)$ and thus $K \subset \sigma(J)$, as claimed. ∎

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О спектрах бесконечных матриц Хессенберга и Якоби

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Изучаются вероятностные меры на единичной окружности в связи с операторами умножения на независимую переменную действующими в соответствующих пространствах Гильберта. Мера с постоянными коэффициентами отражения и соответствующий оператор умножения рассматриваются как невозмущенные обьекты. При определенных возмущениях показано, что носитель возмущенной меры содержит носитель исходной меры. Более общо, мы оцениваем размеры лакун в носителе возмущенной меры. Аналогичные результаты приведены для якобиевых и полосковых матриц.

Про спектри нескінченних матриць Хесенберга та Якобі

Л.Б. Голінський

Вивчаються ймовірносні міри на одиничному колі в зв'язку з операторами множення на незалежну змінну у відповідних просторах Гільберта. Міра з постійними коефіцієнтами відбиття та відповідний оператор множення вважаються незбуреними об'єктами. За певними збуреннями показано, що носій збуреної міри містить носій початкової міри. Більш загально, ми оцінюємо розміри лакун у носії збуреної міри. Відповідні результати наведені для якобієвих та смугастих матриць.