Matematicheskaya fizika, analiz, geometriya
2000, v. 7, No. 3, p. 331–344

Irregular subsets of the Grassmannian manifolds and their maps

M.A. Pankov

Institute of Mathematics, National Academy of Sciences of Ukraine
3 Tereshchenkovskaya Str., Kiev, 01601, Ukraine
E-mail:pankov@imath.kiev.ua

Received October 24, 1996
Communicated by A.A. Borisenko

The maps of the Grassmannian manifold $\mathbb{G}_k^n$ which preserve the class of irregular subsets are studied. It is shown that in the case $n \neq 2k$ any map of this class is induced by a linear automorphism of $\mathbb{R}^n$.

Irregular subsets of the Grassmannian manifolds were introduced in author's paper [4] (see also [5]), in which it was considered the connection between irregular subsets of the Grassmannian manifold $\mathbb{G}_k^n$ and projections of k-dimensional subsets of $\mathbb{R}^n$ onto k-dimensional planes (we recall the results of paper [4, 5] in Subsection 1.2). Here we study bijective maps of $\mathbb{G}_k^n$ into $\mathbb{G}_k^n$ preserving the class of irregular sets. We do not restrict ourself to the case of continuous maps.

1. Some properties of irregular subsets

1.1. Definition of regular and irregular sets. A set $R \subset \mathbb{G}_k^n$ is called *regular* ($R \in \mathfrak{R}_k^n$) if there exists a coordinate system in $\mathbb{R}^n$ such that any plane belonging to R is a coordinate plane for this system. Any coordinate system in $\mathbb{R}^n$ has

$$c_k^n = \frac{n!}{k!(n-k)!}$$

distinct k-dimensional coordinate planes. Therefore, if $R \in \mathfrak{R}_k^n$, then $|R| \leq c_k^n$. A regular set $R \subset \mathbb{G}_k^n$ is called *maximal* ($R \in \mathfrak{M}\mathfrak{R}_k^n$) if $|R| = c_k^n$. It is easy to see that for any $R \in \mathfrak{R}_k^n$ there exists $\hat{R} \in \mathfrak{M}\mathfrak{R}_k^n$ such that $R \subset \hat{R}$.

Let $R \in \mathfrak{MR}_k^n$. Consider the coordinate system in $\mathbb{R}^n$ such that any plane belonging to R is a coordinate plane for this system. Denote by $r_m(R)$ the set of all m-dimensional coordinate planes. Then $r_m(R) \in \mathfrak{MR}_m^n$ and the map

$$r_m : \mathfrak{MR}_k^n \to \mathfrak{MR}_m^n ,$$

$$R \to r_m(R) \quad \forall R \in \mathfrak{MR}_k^n$$

defines an one-to-one correspondence between $\mathfrak{MR}_k^n$ and $\mathfrak{MR}_m^n$.

A set $V \subset \mathbb{G}_k^n$ is called *irregular* ($V \in \mathfrak{I}_k^n$) if there exists no $R \in \mathfrak{MR}_k^n$ such that $R \subset V$. An irregular set $V \subset \mathbb{G}_k^n$ is said to be *maximal* ($V \in \mathfrak{MI}_k^n$) if for any $W \in \mathfrak{I}_k^n$ such that $V \subset W$ we have $V = W$.

Proposition 1.1 [5]. *For any $V \in \mathfrak{I}_k^n$ there exists $W \in \mathfrak{MI}_k^n$ such that $V \subset W$.*

Let $s \in \mathbb{G}_m^n$ and

$$\mathbb{G}_k^n(s) = \begin{cases} \{\, l \in \mathbb{G}_k^n \mid l \subset s \,\} & m \geq k , \\ \{\, l \in \mathbb{G}_k^n \mid s \subset l \,\} & m \leq k . \end{cases}$$

Then $\mathbb{G}_k^n(s) \in \mathfrak{I}_k^n$ and we have the following

Proposition 1.2 [4, 5]. *Let $V \in \mathfrak{MI}_k^n$ and $k = 1, n-1$. Then there exists $s \in \mathbb{G}_{n-k}^n$ such that $V = \mathbb{G}_k^n(s)$.*

1.2. Irregular subsets of $\mathbb{G}_k^n$ and projections of k-dimensional subsets of $\mathbb{R}^n$ onto k-dimensional planes. Let $X \subset \mathbb{R}^n$, l and s be k-dimensional and $(n-k)$-dimensional planes, respectively. Denote by $p_l^s(X)$ the projection of the set X into the plane l along the plane s. The projection $p_l^s(X)$ is well-defined if and only if the planes l and s are transversal.

It was proved by G. Nobeling [3] that for any k-dimensional F_σ-subset of $\mathbb{R}^n$ there exists a k-dimensional plane l such that $\dim p_l^{l^\perp}(X) = k$, where $l^\perp$ is the orthogonal complement to l.

The projection $p_l^s(X)$ is called *regular* if it is a set of second category in l. Consider

$$V_k^n(X) = \{\, l \in \mathbb{G}_k^n \mid \forall s \in \mathbb{G}_{n-k}^n \text{ the projection } p_l^s(X) \text{ is not regular } \} ,$$

$$W_{n-k}^n(X) = \{\, s \in \mathbb{G}_{n-k}^n \mid \forall l \in \mathbb{G}_k^n \text{ the projection } p_l^s(X) \text{ is not regular } \} .$$

We have the following

Theorem 1.1 [4, 5]. *If $\dim X \geq k$, then $V_k^n(X) \in \mathfrak{I}_k^n$ and $W_{n-k}^n(X) \in \mathfrak{I}_{n-k}^n$.*

Theorem 1.1 shows us that for any coordinate system in $\mathbb{R}^n$ there are k-dimensional and $(n-k)$-dimensional coordinate planes l and s such that the projection $p_l^s(X)$ is regular. It is not difficult to see that if X is an F_σ-subset of $\mathbb{R}^n$ and the projection $p_l^s(X)$ is regular, then $\dim p_l^s(X) = k$ and the Nobeling Theorem [3] is a consequence of Theorem 1.1.

We want to prove that for any k-dimensional set X the sets $V_k^n(X)$ and $W_{n-k}^n(X)$ are nowhere dense in $\mathbb{G}_k^n$ an= d $\mathbb{G}_{n-k}^n$, respectively. It seems to be natural to ask how large may be an irregular set. Propositions 1.1 and 1.2 imply that *in the cases* $k = 1, n-1$ *any irregular set is nowhere dense in* $\mathbb{G}_k^n$. For the case $1 < k < n-1$ this statement is not proved. However, we have the following result supporting our conjecture

Theorem 1.2 [5]. *If* $V \in \mathfrak{I}_k^n$, *then* $Int(V) = \emptyset$.

Therefore, *for any* k*-dimensional subset* X *of* $\mathbb{R}^n$ *the sets* $\mathbb{G}_k^n \setminus V_k^n(X)$ *and* $\mathbb{G}_{n-k}^n \setminus W_{n-k}^n(X)$ *are everywhere dense in* $\mathbb{G}_k^n$ *and* $\mathbb{G}_{n-k}^n$*; moreover, in the case* $k = 1, n-1$ *their complements are nowhere dense.*

2. Regular maps of the Grassmannian manifolds

2.1. Definition and elementary properties. A bijective map f of $\mathbb{G}_k^n$ into $\mathbb{G}_k^n$ is called *regular* ($f \in \mathfrak{R}(\mathbb{G}_k^n)$) if it preserves the class $\mathfrak{R}_k^n$; i.e., for any $R \subset \mathbb{G}_k^n$ we have $f(R) \in \mathfrak{R}_k^n$ if and only if $R \in \mathfrak{R}_k^n$. An immediate verification shows us that the following lemma holds true.

Lemma 2.1. *The following statements are equivalent:*

(i) $f \in \mathfrak{R}(\mathbb{G}_k^n)$;

(ii) f *preserves the class* $\mathfrak{M}\mathfrak{R}_k^n$;

(iii) f *preserves the class* $\mathfrak{I}_k^n$;

(iv) f *preserves the class* $\mathfrak{M}\mathfrak{I}_k^n$.

Let $\varphi_k^n : \mathbb{G}_k^n \to \mathbb{G}_{n-k}^n$ be the canonical homeomorphism; i.e., $\varphi_k^n(l) = l^\perp$ for any $l \in \mathbb{G}_k^n$. Then

$$\mathbb{G}_k^n(s) = \varphi_{n-k}^n(\mathbb{G}_{n-k}^n(s^\perp)) \tag{2.1}$$

and the following lemma holds.

Lemma 2.2 [4, 5]. *The canonical homeomorphism* φ_k^n *maps the classes* $\mathfrak{R}_k^n$, $\mathfrak{M}\mathfrak{R}_k^n$, $\mathfrak{I}_k^n$, $\mathfrak{M}\mathfrak{I}_k^n$ *into the classes* $\mathfrak{R}_{n-k}^n$, $\mathfrak{M}\mathfrak{R}_{n-k}^n$, $\mathfrak{I}_{n-k}^n$, $\mathfrak{M}\mathfrak{I}_{n-k}^n$, *respectively.*

All regular maps of $\mathbb{G}_k^n$ constitute the group $\mathfrak{R}(\mathbb{G}_k^n)$. Lemma 2.2 shows us that $\varphi_k^{2k} \in \mathfrak{R}(\mathbb{G}_k^{2k})$.

Let $s \in \mathbb{G}_m^n$. If $m \geq k$, then there exists the natural homeomorphism φ_s between $\mathbb{G}_k^n(s)$ and $\mathbb{G}_k^m$. In the case $m < k$ consider the map

$$\varphi_s = \varphi_{s^\perp}\varphi_k^n : \mathbb{G}_k^n(s) \to \mathbb{G}_{k-m}^{n-m}\ .$$

This is a homeomorphism of $\mathbb{G}_k^n(s)$ onto $\mathbb{G}_{k-m}^{n-m}$. Moreover, we have $R \in \mathfrak{R}_k^n$ if and only if $\varphi_s(R) \in \mathfrak{R}_k^m$ or $\varphi_s(R) \in \mathfrak{R}_{k-m}^{n-m}$ for any subset $R \subset \mathbb{G}_k^n(s)$.

2.2. Linear automorphisms. Any linear automorphism of $\mathbb{R}^n$ induces a regular map of $\mathbb{G}_k^n$. A map of $\mathbb{G}_k^n$ induced by a linear automorphism of $\mathbb{R}^n$ is called *linear*. All linear maps of $\mathbb{G}_k^n$ form a group. Denote it by $\mathfrak{L}(\mathbb{G}_k^n)$. Let $\mathfrak{GL}(n)$ be the group of all linear automorphisms of $\mathbb{R}^n$. Then we have the following

Proposition 2.1.

$$\mathfrak{L}(\mathbb{G}_k^n) \approx \begin{cases} \{\, l \in \mathfrak{GL}(n) \mid \det l = 1 \,\}, & n \neq 2i \ , \\ \{\, l \in \mathfrak{GL}(n) \mid |\det l| = 1 \,\}, & n = 2i \ . \end{cases}$$

P r o o f. Consider the homomorphism

$$\pi_k^n : \mathfrak{GL}(n) \to \mathfrak{L}(\mathbb{G}_k^n) \ ,$$

where $\pi_k^n(l)$ is the automorphism induced by $l \in \mathfrak{GL}(n)$. Then

$$Ker\pi_k^n = \{\, \alpha Id \mid \alpha \in \mathbb{R}\, , \alpha \neq 0 \,\}$$

and

$$\mathfrak{L}(\mathbb{G}_k^n) \approx \mathfrak{GL}(n)/Ker\pi_k^n \ .$$

Two last equations imply our statement (it must be pointed out that in the case $n = 2i$ we have $\det l > 0$ for any $l \in Ker\pi\underline{\overset{n}{\ }}k$).

Let $f \in \mathfrak{L}(\mathbb{G}_k^n)$ and $f' \in (\pi_k^n)^{-1}(f)$. For any $i = 1, \dots, n-1$ define

$$L_{k,i}^n(f) = \pi_i^n(f') \ .$$

If $f'' \in (\pi_k^n)^{-1}(f)$ and $f' \neq f''$, then

$$f'(f'')^{-1} \in Ker\pi_k^n = Ker\pi_i^n \ .$$

This implies that $\pi_i^n(f') = \pi_i^n(f'')$ and $L_{k,i}^n(f)$ is well-defined for any $f \in \mathfrak{L}(\mathbb{G}_k^n)$ and $i = 1, ..., n-1$. The similar arguments shows us that

$$L_{k,i}^n : \mathfrak{L}(\mathbb{G}_k^n) \to \mathfrak{L}(\mathbb{G}_i^n)$$

is an isomorphism between $\mathfrak{L}(\mathbb{G}_k^n)$ and $\mathfrak{L}(\mathbb{G}_i^n)$ for any $i = 1, \dots, n-1$. Moreover, for any $f \in \mathfrak{L}(\mathbb{G}_k^n)$ we have

$$f(\mathbb{G}_k^n(s)) = \mathbb{G}_k^n(f_i(s)), \quad \forall s \in \mathbb{G}_i^n \ , \tag{2.2}$$

where $f_i = L_{k,i}^n(f)$.

2.3. In the next section we prove the following

Theorem 2.1. *If $n \geq 3$ and $n \neq 2k$, then $\mathfrak{R}(\mathbb{G}_k^n) = \mathfrak{L}(\mathbb{G}_k^n)$. Moreover, the group $\mathfrak{R}(\mathbb{G}_k^{2k})$ $(k \geq 2)$ is generated by $\mathfrak{L}(\mathbb{G}_k^{2k})$ and φ_k^{2k}.*

For any $l_i \in \mathbb{G}_1^2$ $(i = 1, 2)$ such that $l_1 \neq l_2$ we have $\{l_1, l_2\} \in \mathfrak{M}\mathfrak{R}_1^2$. Therefore, any bijective map of $\mathbb{G}_1^2$ into $\mathbb{G}_1^2$ is regular.

The group $\mathfrak{L}(\mathbb{G}_k^{2k})$ is normal subgroup of $\mathfrak{R}(\mathbb{G}_k^{2k})$ and

$$\mathfrak{R}(\mathbb{G}_k^{2k})/\mathfrak{L}(\mathbb{G}_k^{2k}) \approx \mathbb{Z}_2$$

(this statement is a simple consequence of Theorem 2.1, and we do not prove it here). The subgroup $\{Id,\ \varphi_k^{2k}\}$ is not normal and the group $\mathfrak{R}(\mathbb{G}_k^{2k})$ is not isomorphic to the group $\mathfrak{L}(\mathbb{G}_k^{2k}) \times \mathbb{Z}_2$.

3. The Chow Theorem. Proof of Theorem 2.1 in the cases $k = 1, n-1$

3.1. Proof of Theorem 2.1 in the cases $k = 1, n-1$. In the case $k = 1$ our statement is a trivial consequence of the Main Theorem of Projection Geometry (see [2]). Consider the case $k = n - 1$.

Let $f \in \mathfrak{R}(\mathbb{G}_{n-1}^n)$. Then Lemma 2.1 and Proposition 1.2 imply the existence of bijective map f_1 of $\mathbb{G}_1^n$ into $\mathbb{G}_1^n$ such that equation (2.2) holds for $i = 1, k = n-1$. Moreover, for any $R \in \mathfrak{M}\mathfrak{R}_1^n$ we have

$$f_1(R) = r_1(f(r_{n-1}(R)))\ ; \tag{3.1}$$

i.e., $f_1 \in \mathfrak{R}(\mathbb{G}_1^n)$. Consider the homeomorphism

$$F : \mathfrak{R}(\mathbb{G}_{n-1}^n) \to \mathfrak{R}(\mathbb{G}_1^n) = \mathfrak{L}(\mathbb{G}_1^n)\ ,$$

$$F(f) = f_1\ .$$

If $F(f) = Id$, then equation (3.1) shows that $f = Id$ and F is a monomorphism. It is easy to see that

$$F|_{\mathfrak{L}(\mathbb{G}_{n-1}^n)} = L_{n-1,1}^n$$

and $F(\mathfrak{R}(\mathbb{G}_{n-1}^n)) = \mathfrak{L}(\mathbb{G}_1^n)$. Therefore, $\mathfrak{R}(\mathbb{G}_{n-1}^n) = \mathfrak{L}(\mathbb{G}_{n-1}^n)$.

3.2. The Chow Theorem. We say that $l \in \mathbb{G}_k^n$ and $s \in \mathbb{G}_k^n$ $(1 < k < n-1)$ *despoce in the neighbourhood* if there exists $p \in \mathbb{G}_{k+1}^n$ such that $l \in \mathbb{G}_k^n(p)$ and $s \in \mathbb{G}_k^n(p)$. This is equivalent the existence of $t \in \mathbb{G}_{k-1}^n$ such that $l \in \mathbb{G}_k^n(t)$ and $s \in \mathbb{G}_k^n(t)$.

We also say that a bijective map f of $\mathbb{G}_k^n$ into $\mathbb{G}_k^n$ *preserves the neighbourhood* if for any $l \in \mathbb{G}_k^n$, $s \in \mathbb{G}_k^n$ the planes $f(l)$, $f(s)$ despoce in the neighbourhood if and only if l and s are neighbouring. Denote by $\mathfrak{C}(\mathbb{G}_k^n)$ the class of all bijective maps of $\mathbb{G}_k^n$ into $\mathbb{G}_k^n$ preserving the neighbourhood. It is easy to see that $\mathfrak{C}(\mathbb{G}_k^n)$ is a group. To prove Theorem 2.1 we exploit the following

Theorem 3.1 [1] (see also [2]). *If $1 < k < n-1$ and $n \neq 2k$, then $\mathfrak{C}(\mathbb{G}_k^n) = \mathfrak{L}(\mathbb{G}_k^n)$. The group $\mathfrak{C}(\mathbb{G}_k^{2k})$ $(k \neq 1)$ is generated by φ_k^{2k} and $\mathfrak{L}(\mathbb{G}_k^{2k})$*

Theorem 3.1 implies that Theorem 2.1 will be proved if we show that $\mathfrak{R}(\mathbb{G}_k^n) = \mathfrak{C}(\mathbb{G}_k^n)$ for $1 < k < n-1$.

4. Proof of Theorem 2.1 in the case $1 < k < n-1$

4.1. Index of exactness of regular sets. In this subsection we introduce the index of exactness of regular sets and study its properties. We use it to prove Theorem 2.1.

A regular set $R \in \mathfrak{R}_k^n$ is said to be *exact* ($R \in \mathfrak{E}\mathfrak{R}_k^n$) if there exists a unique set $\hat{R} \in \mathfrak{M}\mathfrak{R}_k^n =$ such that $R \subset \hat{R}$. For any $R \in \mathfrak{R}_k^n$ consider

$$Ind(R) = \min_{\hat{R} \in \mathfrak{E}\mathfrak{R}_k^n,\ R \subset \hat{R}} \{|\hat{R}| - |R = |\} \ .$$

It is easy to see that $Ind(R) = 0$ if and only if $R \in \mathfrak{E}\mathfrak{R}_k^n$. The number $Ind(R)$ is called the *index of exactness* of the regular set R.

We have the following simple

Proposition 4.1. *The canonical homeomorphism φ_k^n and a map $f \in \mathfrak{R}(\mathbb{G}_k^n)$ preserve the index of exactness; i.e., for any $R \in \mathfrak{R}_k^n$ we have*

$$Ind(f(R)) = Ind(R) \ ,$$

$$Ind(\varphi_k^n(R)) = Ind(R) \ .$$

If $k = 1, n-1$, then $Ind(R) = n - |R|$ and $\mathfrak{E}\mathfrak{R}_k^n = \mathfrak{M}\mathfrak{R}_k^n$. In the case $1 < k < n-1$ the situation is more complicated.

Let $R' \in \mathfrak{M}\mathfrak{R}_k^n$. Consider the coordinate system in $\mathbb{R}^n$ such that any plane belonging to R' is a coordinate plane. Let s be an m-dimensional coordinate plane for that system, and

$$R(s) = R' \cap \mathbb{G}_k^n(s) \ .$$

Then

$$|R(s)| = \begin{cases} c_k^m \ , & m \geq k \ , \\ c_{k-m}^{n-m} \ , & m \leq k \ . \end{cases} \tag{4.1}$$

Proposition 4.2. *Let $R \in \mathfrak{R}_k^n$ and $R \subset R'$. Then the following statements holds true:*

(i) *if $n-k < k < n-1$ and $|R| = c_{k-1}^{n-1}$, then $Ind(R) \leq 2$ and $Ind(R) = 2$ if and only if there exists $s \in \mathbb{G}_1^n$ such that $R = R(s)$;*

(ii) *if* $n = 2k$ *and* $|R| = c_k^{n-1} = c_{k-1}^{n-1}$, *then* $Ind(R) \le 2$ *and* $Ind(R) = 2$ *if and only if there exists* $s \in \mathbb{G}_i^{n=}$ *(where* $i = 1, n-1$*) such that* $R = R(s)$*;*

(iii) *if* $1 < k < n-k$ *and* $|R| = c_k^{n-1}$, *then* $Ind(R) \le 2$ *and* $Ind(R) = 2$ *if and only if there exists* $s \in \mathbb{G}_{n-1}^n$ *such that* $R = R(s)$.

Statement (iii) of Proposition 4.2 is a consequence of statement (i), (2.1) and Proposition 4.1. In the next subsection we prove statements (i) and (ii).

4.2. P r o o f. Let

$$R_i = R \cap R(x_i) \ , \quad s_i = \bigcap_{l \in R_i} l$$

and

$$n_i = \begin{cases} \dim s_i \, , & R_i \neq \emptyset \, , \\ 0 \, , & R_i = \emptyset \, . \end{cases}$$

Then we have $R \in \mathfrak{E}\mathfrak{R}_k^n$ if and only if $n_i = 1$ for any $i = 1, \dots, n$. A set R_i is said to be *maximal* if for any j such that $R_i \subset R_j$ we have $R_i = R_j$.

Lemma 4.1. *If* R_i *is maximal, then* $n_i = 1$.

P r o o f. Let $l \in R$. Then we have the following two cases:

(i) $l \in R_i$;

(ii) $l \notin R_i$.

In the first case $l \in R(s_i)$. Consider the case (ii). Let s be the $(n-n_i)$-dimensional coordinate plane transverse to s_i. We prove that $l \in R(s)$. Assume that $l \notin R_i$ and $l \notin R(s)$. Then $\dim l \cap s_i \ge 1$. This implies the existence of j such that $i \neq j$ and $x_j \subset l \cap s_i$. It is easy to see that

$$R_i \cup \{l\} \subset R_j \text{ and } R_i \neq R_j \ ;$$

i.e., R_i is not maximal.

We have

$$R \subset R(s_i) \cup R(s) \ .$$

Then equation (4.1) implies that

$$c_{k-1}^{n-1} \le |R| \le |R(s_i)| + |R(s)| = c_{k-n_i}^{n-n_i} + c_k^{n-n_i} \ .$$

This inequality holds if and only if $n_i = 1$.

Consider the collection

$$\mathcal{R}_i = \{ \ R_j \mid R_j \subset R_i \text{ and } R_j \neq R_i \ \} \ .$$

A set $R_j \in \mathcal{R}_i$ is said to be *maximal in* $\mathcal{R}_{=i}$ if for any $R_p \in \mathcal{R}_i$ such that $R_j \subset R_p$ we have $R_j = R_p$.

Lemma 4.2. *If R_i is maximal and R_j is maximal in $\mathcal{R}_i$, then $n_j = 2$.*

P r o o f. Let $l \in \mathbb{G}_k^n$. Lemma 4.1 implies that $n_i = 1$, and we have the following three cases:

(i) $l \in R_j$;

(ii) $l \notin R_j$ and $l \in R_i$;

(iii) $l \notin R_i$.

In the first case $l \in R(s_j)$. Consider the cases (ii) and (iii). Let s be the $(n-n_j+1)$-dimensional plane generated by x_i and the $(n-n_j)$-dimensional coordinate plane transverse to s_j. Let s' be the $(n-2)$-dimensional coordinate plane transverse to the plane generated by x_i and x_j. We show that $l \in R(s) \cap R(s_i)$ in the case (ii) and $l \in R(s')$ in the case (iii). Let $l \in R_i$ (i.e., $l \in R(s_i)$) and $l \notin R_j$. Recall that R_j is maximal in $\mathcal{R}_i$. Therefore, $l \in R(s)$ (see the case (ii) in the proof of Lemma 4.1). Let $l \notin R_i$ (the case (iii)). Then $R_j \subset R_i$. This implies that $x_i \not\subset l$ and $x_j \not\subset l$; i.e., $l \in R(s')$.

We have

$$R \subset R(s_j) \cup (R(s) \cap R(s_i)) \cup R(s') \,.$$

Then equation (4.1) implies that

$$c_{k-1}^{n-1} \le |R| \le |R(s_j)| + |R(s) \cap R(s_i)| + |R(s')| = c_{k-n_j}^{n-n_j} + c_{k-1}^{n-n_j} + c_k^{n-2} \,.$$

This inequality holds if and only if $n_j \le 2$. Recall that $R_j \subset R_i$ and $R_j \ne R_i$. Therefore, $n_j = 2$.

Lemma 4.3. *In the case $n-k<k$ we have $n_i > 0$ for any $i = 1, \ldots, n$. If $n = 2k$ and there exists i such that $n_i = 0$, then $R = R(s)$, where s is the $(n-1)$-dimensional coordinate plane transverse to x_i.*

P r o o f. If $n_i = 0$ (i.e., $R_i = \emptyset$), then $R \subset R(s)$. Equation (2.2) shows that $|R| \le c_k^{n-1}$. It is easy to see that $c_k^{n-1} < c_{k-1}^{n-1}$ $(n-k<k)$ and $c_k^{n-1} = c_{k-1}^{n-1}$ $(n = 2k)$.

Lemma 4.4. *For any $i = 1, \ldots, n$ we have $n_i \le 2$.*

P r o o f. Let $n_p > 1$. Then Lemmas 4.1 and 4.2 imply the existence of i and j such that $n_i = 1, n_j = 2$ and $R_p \subset R_j \subset R_i$. Let $l \in R$. Then we have the following four cases:

(i) $l \in R_p$;

(ii) $l \notin R_i$;

(iii) $l \in R_i$ and $l \notin R_j$;

(iv) $l \in R_j$ and $l \notin R_p$.

In the case (i) $l \in R(s_p)$. Let s be the $(n-n_p)$-dimensional coordinate plane transverse to s_p. Then in the case (ii) $l \in R(s)$ (see the case (iii) in the proof of Lemma 4.2). Let s' be the $(n-n_p+1)$-dimensional plane generated by s and x_i, and s'' be the $(n-n_p+2)$-dimensional plane generated by s' and x_j. In the cases (iii) and (iv) $l \in R(s') \cap R(x_i)$ and $R(s'') \cap R(s_j)$, respectively (these cases are similar to the case (ii) in the proof of Lemma 4.2).

We have

$$R \subset R(s_p) \cup R(s) \cup (R(s') \cap R(x_i)) \cup (R(s'') \cap R(s_j)) \ .$$

Equation (4.1) implies that

$$\begin{aligned} c_{k-1}^{n-1} &\leq |R| \leq |R(s_p)| + |R(s)| + |R(s') \cap R(x_i)| + |R(s'') \cap R(s_j)| \\ &= c_{k-n_p}^{n-n_p} + c_k^{n-n_p} + c_{k-1}^{n-n_p} + c_{k-=2}^{n-n_p} = c_{k-n_p}^{n-n_p} + c_k^{n-n_p} + c_{k-1}^{n-n_p+1} \ . \end{aligned}$$

This inequality holds if and only if $n_p \leq 2$.

Denote by m_R (resp. n_R) the number of all R_i such that $n_i = 1$ (resp. $n_i = 2$). It is easy to see that $m_R + n_R = n$ if and only if $n_i > 0$ for any $i = 1, \dots, n$.

Lemma 4.5. *Suppose that $n-k \leq k \leq n-1$, $n_i > 0$ for any $i = 1, \dots, n$, and $n_R \geq n-k$. Then $n_R = n-1$.*

P r o o f. Let $p = n-k$. Consider the case $p = 1$ $(k = n-1)$. We have $|R| = c_{n-2}^{n-1} = n-1$ or $|R| = n$. It is easy to see that in the first case $n_R = n-1$ and in the second case $n_R = 0$. This implies the required.

Let $p > 1$. Consider i and j such that $R_j \subset R_i$, $n_j = 2$ and $n_i = 1$. Let s be the $(n-1)$-dimensional coordinate plane transverse to x_j. Then

$$R \subset R(s_j) \cup R(s) \ . \tag{4.2}$$

Consider

$$R' = \varphi_s(R \cap R(s)) \in \mathfrak{R}_k^{n-1} \ .$$

Equations (4.1) and (4.2) imply that

$$|R'| \geq |R| - |R(s_j)| \geq c_{k-1}^{n-1} - c_{k-2}^{n-2} = c_{k-1}^{n-2} \ .$$

The inductive hypothesis shows that we have the following two cases:

(i) $n_{R'} < n-k-1$ and $m_{R'} > k$;

(ii) $n_{R'} = n-2$.

Equation (4.2) shows that $n_R = n_{R'} + 1$. Therefore, in the case (i) $n_R < n-k$ and in the case (ii) $n_R = n-1$.
By Lemmas 4.3–4.5 we have the following four cases:

(i) $n-k \le k < n-1$ and $n_R = 0$, $m_R = n$;

(ii) $n-k \le k < n-1$ and $0 < n_R = n - m_R < n-k$;

(iii) $n-k \le k < n-1$ and $n_R = n-1$, $m_R = 1$;

(iv) $n = 2k$ and there exists i such that $n_i = 0$.

In the case (i) we have $R \in \mathfrak{E}\mathfrak{R}^n_k$ and $Ind(R) = 0$.

Consider the case (ii). Let

$$I_1 = \{\ i \mid n_i = 2\ \}\ ,$$

$$I_2 = \{\ i \mid n_i = 1 \text{ and there exists } j \in I_1 \text{ such that } R_j \subset R_i\ \}\ ,$$

$$I_3 = \{1, \dots, n\} \setminus (I_1 \cup I_2)\ .$$

It is easy to see that for any $i \in I_1$ there exists unique $j \in I_2$ such that $R_i \subset R_j$. Therefore,

$$|I_2| \le |I_1| < n-k \text{ and } |I_1 \cup I_3| > k\ .$$

This implies the existence of a set $I_4 \subset I_3$ such that $|I_4 \cup I_1| = k$. Let

$$I_4 \cup I_1 = \{i_1, \dots, i_k\}$$

and l be the plane generated by $x_{i_1}, \dots, x_{i_k}$. Then

$$R \cup \{l\} \in \mathfrak{E}\mathfrak{R}^n_k$$

and $Ind(R) = 1$.

In the case (iii) there exists a unique i such that $n_i = 1$ and $R_j \subset R_i$ for any $j = 1, \dots, n$. Therefore, $R = R(x_i)$. For any $l \in \hat{R} \setminus R$

$$R \cup \{l\} \notin \mathfrak{E}\mathfrak{R}^n_k\ ,$$

and for the set $R \cup \{l\}$ we have the case (ii). Therefore,

$$Ind(R \cup \{l\}) = 1$$

and $Ind(R) = 2$.

In the case (iv) there exists $s \in \mathbb{G}^n_{n-1}$ such that $R = R(s)$. Then for the set $\varphi^{2k}_k(R)$ we have the case (iii) and, by Proposition 4.1 and (2.1), $Ind(R) = 2$.

4.3. Now we prove Theorem 2.1. Let $f \in \mathfrak{R}(\mathbb{G}^n_k)$ and $R \in \mathfrak{M}\mathfrak{R}^n_k$. Then

$$R_f = f(R) \in \mathfrak{M}\mathfrak{R}^n_k \ .$$

Consider the coordinate systems $\{x_i\}^n_{i=1}$ and $\{y_i\}^n_{i=1}$ in $\mathbb{R}^n$ such that any plane belonging to R (resp. R_f) is a coordinate plane for the system $\{x_i\}^n_{i=1}$ (resp. $\{y_i\}^n_{i=1}$). Let s' be a coordinate plane for the system $\{y_i\}^n_{i=1}$. Set

$$R_f(s') = R_f \cap \mathbb{G}^n_k(s') \ .$$

Lemma 4.6. *Let s be an $(n-1)$-dimensional coordinate plane for the system $\{x_i\}^n_{i=1}$. Then the following statements hold true:*

(i) *if $1 < k < n-1$ and $n \neq 2k$, then in the system $\{y_i\}^n_{i=1}$ there exists an $(n-1)$-dimensional coordinate plane s' such that*

$$f(R(s)) = R_f(s') \qquad (4.3)$$

(recall that $R(s) = R \cap \mathbb{G}^n_k(s)$);

(ii) *if $n = 2k$, then in the system $\{y_i\}^n_{i=1}$ there exists an m-dimensional coordinate plane s' (where $m = 1, n-1$) such that equation (4.3) holds.*

P r o o f. (i). In the case $1 < k < n-k$ our statement is a consequence of Propositions 4.1, 4.2. Consider the case $n-k < k < n-1$.

In the system $\{x_i\}^n_{i=1}$ consider the axis x_i transverse to the plane s. Then

$$R(s) = R \setminus R(x_i) \ .$$

Proposition 4.1, 4.2 imply the existence of an axis y_j in the system $\{y_i\}^n_{i=1}$ such that

$$f(R(x_i)) = R_f(y_j) \ .$$

Denote by s' the $(n-1)$-dimensional coordinate plane in the system $\{y_i\}^n_{i=1}$ transverse to the axis y_j. Then we get the required.

(ii). In this case our statement is a consequence of Propositions 4.1 and 4.2.

Lemma 4.7. *Let s be a $(k+1)$-dimensional coordinate plane for the system $\{x_i\}^n_{i=1}$. Then the following statements hold true:*

(i) *if $1 < k < n-1$ and $n \neq 2k$, then in the system $\{y_i\}^n_{i=1}$ there exists a $(k+1)$-dimensional coordinate plane s' such that equation (4.3) holds;*

(ii) *if $n = 2k$, then in the system $\{y_i\}_{i=1}^n$ there exists an m-dimensional coordinate plane s' (where $m = k-1, k+1$) such that equation (4.3) holds.*

P r o o f. Denote by s_i (resp. s'_i) the $(n-1)$-dimensional coordinate plane in the system $\{x_i\}_{i=1}^n$ (resp. $\{y_i\}_{i=1}^n$) transverse to the axis x_i (resp. y_i).

(i) Lemma 4.6 implies that for any i there exists j_i such that

$$f(R(s_i)) = R_f(s'_{j_i}) \ . \tag{4.4}$$

Consider the subset $\{i_1, \ldots, i_{k+1}\}$ of $\{1, \ldots, n\}$ such that the axes $x_{i_1}, \ldots, x_{i_{k+1}}$ generate the plane s. Assume that

$$\{i_1, \ldots, i_{k+1}\} = \{1, \ldots, k+1\} \ .$$

Then

$$R(s) = \bigcap_{i=k+2}^{n} R(s_i)$$

and

$$f(R(s)) = \bigcap_{i=k+2}^{n} R_f(s'_{j_i}) = R_f(s') \, ,$$

where s' is the plane generated by the axes $y_{j_1}, \ldots, y_{j_{k+1}}$.

(ii) Lemma 4.6 shows that we have the following two cases:

(a) there exists j_1 such that equation (4.4) holds for $i = 1$;

(b) there exists j_1 such that $f(R(s_1)) = R_f(y_{j_1})$.

Consider the case (a). We show that for any i there exists i_j such that equation (4.4) holds. Then the proof of statement (ii) is similar to the proof of statement (i).

Assume that there exist i and j_i such that $f(R(s_i)) = R_f(y_{j_i})$. Let $\hat{s} = s_1 \cap s_i$. Then

$$R(\hat{s}) = R(s_1) \cap R(s_i)$$

and

$$f(R(\hat{s})) = R_f(s'_{j_1}) \cap R_f(y_{j_i}) \ .$$

Equation (5.1) imply that

$$|R(\hat{s})| = c_k^{2k-2}$$

and

$$|R_f(s'_{j_1}) \cap R_f(y_{j_i})| = c_{k-1}^{2k-2} \ .$$

An immediate verification shows us that $c_k^{2k-2} \neq c_{k-1}^{2k-2}$ and our hypothesis fails.

Now consider the case (b). It is not difficult to see that $\varphi_k^{2k} f$ satisfies the condition defining the case (a). Therefore, there exists $s'' \in \mathbb{G}_{k+1}^{2k}$ such that

$$(\varphi_k^{2k} f)(R(s)) = \varphi_k^{2k}(R_f) \cap \mathbb{G}_k^n(s'') \, .$$

This implies that equation (4.3) holds, if $s' = \varphi_{k+1}^{2k}(s'')$.

Lemma 4.7 implies that for any neighbouring $l \in \mathbb{G}_k^n$, $s \in \mathbb{G}_k^n$ the planes $f(l)$, $f(s)$ despoce in the neighbourhood. Consider the regular map f^{-1}. Lemma 4.7 shows that if $f(l)$, $f(s)$ despoce in the neighbourhood then l and s are neighbouring. Therefore, $\mathfrak{R}(\mathbb{G}_k^n) \subset \mathfrak{C}(\mathbb{G}_k^n)$. The inverse inclusion is a consequence of Theorem 3.1.

Acknowledgment. I wish to express our deep gratitude to S.I. Maximenko, E.A. Polulyakh, V.V. Sharko for their interest in my research and a number of valuable conments and the reviewer who acquainted me with Chow's results.

References

[1] *W.L. Chow*, On the geometry of algebraic homogeneous spaces. — Ann. Math. (1949), v. 50, p. 32–67.

[2] *J. Dieudonné*, La géométrie des groupes classiques. Springer–Verlag, Berlin, Heidelberg, New York (1971).

[3] *G. Nobeling*, Die Projektionen einer kompakten m-dimensionalen Menge in $\mathbb{R}_k$. — Ergebnisse Math. Koll. (1933), v. 4, p. 24–25.

[4] *M.A. Pankov*, Projections of k-dimensional subsets of $\mathbb{R}^n$ onto k-dimensional planes. — Mat. fiz., analiz, geom. (1998), v. 5, No. 1/2, p. 114–124.

[5] *M.A. Pankov*, Projections of k-dimensional subsets of $\mathbb{R}^n$ onto k-dimensional planes and irregular subsets of the Grassmannian manifolds. — Top. and App. (2000), v. 101, No. 2, p. 121–135.

Иррегулярные подмножества грассмановых многообразий и их отображения

М.А. Панков

Изучаются отображения грассманова многообразия $\mathbb{G}_k^n$ в себя, сохраняющие класс иррегулярных подмножеств. Доказано, что при $n \neq 2k$ отображения данного класса индуцированы линейными автоморфизмами $\mathbb{R}^n$.

Іррегулярні підмножини грассманових многовидів та їх відображення

М.О. Панков

Вивчаються відображення грассманова многовиду $\mathbb{G}_k^n$, які зберігають клас іррегулярних підмножин. Доведено, що у випадку $n \neq 2k$ ці відображення індуковані лінійними автоморфізмами $\mathbb{R}^n$.