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Construction of the maximal subharmonic minorant for functions of a special form

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For a function continuous on a compact set of the complex plane and satisfying some additional conditions, an algorithm for construction of the maximal subharmonic minorant is found.

1. Introduction and main result

Let $D \subset \mathbb{C}$ be a bounded domain and $m(z)$ be a continuous function in $\overline{D}$.

Denote by $SH(m, D)$ the set of functions subharmonic in D that satisfy the condition

$$u(z) \leq m(z), \quad z \in \overline{D}.$$

The function

$$u_{\max}(z, m) = \sup\{u(z) : u \in SH(m, D)\}$$

is a subharmonic function and is called the *maximal subharmonic minorant* of $m(z)$ in D .

If m has a subharmonic minorant it has the maximal subharmonic minorant.

Construction of a subharmonic minorant is very often a stage when the wishes to find an entire functions with prescribed growth along the real axis or other curves (see, for example, [1, Ch. 5; 2]). And $m(z)$ often is harmonic outside the curves.

This is a ground for the next supposition.

Let Δ be the Laplace operator. We suppose that

1) Δm is a *charge* (signed measure),

i.e., it can be represented by the Jordan decomposition in the form

$$\Delta m = \mu_s - \mu_e,$$

where $\mu_s,\ \mu_e$ are measures concentrated on non-intersecting sets E_s and E_e.

It is not a difficult requiering to $m(z)$ because every function that is sufficiently smooth has this property.

Suppose also that

2) $\overline{E}_s$, the support of μ_s, is a set of positive capacity which has no inner points.

It may be, for example, a curve in D.

Let μ_K denote the balayage of the measure μ having its support in D on a compact $K \subset D$ (see, for example, [2]).

For a subharmonic function $v(z,\mu)$ in D, which has μ as its mass distribution, this means that we replace $v(z,\mu)$ by a new subharmonic function v_K, which has the same values on K and ∂D (except perhaps for a set of capacity zero) and is harmonic in $D \setminus K$.

Now we consider the following sequence of the charges:

$$
\begin{aligned}
&\mu^0 = \mu_s - \mu_e\,,\\
&\mu^1 = \mu_s - (\mu_e)_{\overline{E}_s} = \mu_s^1 - \mu_e^1\,,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\mu^n = \mu_s^{n-1} - (\mu_e^{n-1})_{\overline{E}_s^{n-1}} = \mu_s^n - \mu_e^n\,,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
\tag{1}
$$

where $\overline{E}_s^n$ is the support of the measure μ_s^n. Note that the first equalities in (1) are not the Jordan decompositions: only the second ones. There corresponds to this sequence of measures a sequence of functions $m_n(z)$, which is obtained in the following way. The function $m_n(z)$ is obtained from $m_{n-1}(z)$ as the function which coincides with $m_{n-1}(z)$ on the set $E^n = (\overline{E}_s^{n-1}) \cup \partial D$ and is the solution of the Dirichlet problem in the domain $D \setminus E^n$ with boundary values $m_{n-1}(z)$.

We will explain this algorithm for the one-dimensional case, i.e., for construction of the maximal convex minorant of a function $m(z)$ on the interval (a,b), which is a broken line. An analogy of a solution of the Dirichlet problem is a line that connects two points. All the masses of $m(z)$ are concentrated in the abscisses of tops of the broken line angles. And a mass is negative if the corresponding angle points up and positive if it points down. The process is shown on the figure.

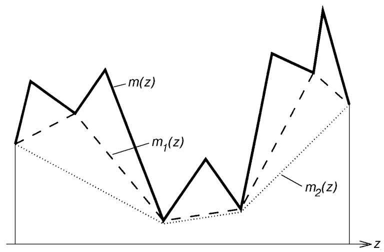

Figure.

The next function $m_1(z)$ also is a broken line. The process is completed in two steps. The function $m_2(z)$ is the maximal convex minorant of $m(z)$.

Theorem. *Let $m(z)$ be a function continuous on the $\overline{D}$ which satisfies the conditions 1) and 2), and let $m_n(z)$ be the sequence of the functions constructed above. Then $\lim\limits_{n\to\infty} m_n(z) = u_{\max}(z, m)$.*

2. Proof

We will represent the function $m(z)$ in the form

$$m(z) = H(z, m, D) - \int\limits_D G(z, \zeta)\mu_s(d\zeta) + \int\limits_D G(z, \zeta)\mu_e(d\zeta),$$

where $H(z, m, D)$ is the solution of the Dirichlet problem on D with the boundary conditions $H\mid_{\partial D}= m$, and $G(z, \zeta)$ is the Green function of D.

The function $H(z, m, D)$ is not changed in the process of applying the algorithm. So the function $m_n(z)$ can be represented in the form

$$m_n(z) = H(z, m, D) - \int G(z, \zeta)\mu_s^n(d\zeta) + \int G(z, \zeta)\mu_e^n(d\zeta).$$

Denote the last two terms on the RHS by $-\Pi(z, \mu_s^n)$ and $\Pi(z, \mu_e^n)$, respectively.

After each sweeping the function

$$w_n(z) = -\Pi(z, \mu_s^n) + \Pi(z, \mu_e^n) \qquad (2)$$

preserves its values on E_s^n and decreases off E_s^n.

So the sequence $m_n(z)$ decreases monotonically.

We have also

$$m_n(z) \geq \min\{m(z) : z \in \overline{E}_s^n \cup \partial D\} \geq \min\{m(z) : z \in \overline{D}\},$$

because of the minimum principle.

This means the sequence is bounded from below.

Let $m_\infty(z) = \lim_{n\to\infty} m_n(z)$. It is clear that $m_\infty(z) \leq m_0(z) = m(z)$. We will show that $m_\infty(z)$ is subharmonic. The sequence $\{m_n(z)\}$ converges monotonically. So it converges as a sequence of distributions. This means that the sequence of charges $\Delta m_n = \mu_s^n - \mu_e^n$ converges distributionally to a charge $\nu = \mu_s^\infty - \mu_e^\infty$, because $\mu_s^n + \mu_e^n \leq \mu_s^0 + \mu_e^0$ for all n.

From (2) we have $w_\infty(z) = -\Pi(z, \mu_s^\infty) + \Pi(z, \mu_e^\infty)$.

Suppose that $\mu_e^\infty \neq 0$. Then we can perform the next sweeping and decrease $w^\infty(z)$, which is impossible.

So, $\mu_e^\infty = 0$ and $m_\infty(z)$ is subharmonic. Now we show that it is the maximal subharmonic minorant.

Let $u(z)$ be any subharmonic minorant of $m(z)$. Then

$$u(z) \leq m(z), \quad z \in \overline{E}_s^\infty \cup \partial D.$$

By the maximum principle, $u(z) \leq m_\infty(z), \quad z \in D$, because $m_\infty(z)$ is harmonic in $D \setminus \overline{E}_s^\infty$.

This means that $m_\infty(z) = u_{\max}(z, m)$. The theorem is proved. ■

References

[1] *P. Koosis*, Lecons sur le Theoreme de Beurling and Malliavin. Les Publications CRM, Montreal (1996).

[2] *Y. Domar*, Entire functions of order ≤ 1 with bounds on both axes. — Ann. Acad. Sci. Fenn. Math. (1997), v. 22, No. 2, p. 339–348.

[3] *N.S. Landkoff*, Foundations of Modern Potential Theory. Springer–Verlag, Berlin, Heidelberg, New York (1972).

Нахождение максимальной субгармонической миноранты для функции специального вида

В. Азарин

Для функции, непрерывной на компактном множестве комплексной плоскости и удовлетворяющей некоторым дополнительным условиям, найден алгоритм для построения максимальной субгармонической миноранты.

Знаходження максимальної субгармонічної міноранти для функції спеціального вигляду

В. Азарін

Для функції, яка неперервна на компактній множині комплексної площини і така, що задовольняє певним додатковим умовам, знайдено алгоритм для побудування максимальної субгармонічної міноранти.