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Knotted totally geodesic submanifolds in positively curved spheres

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A diffeomorphic to sphere three-manifold of positive sectional curvature with geodesics of given torus type is constructed.

Theorem 0.1. *Given a pair of coprime integers* $(m, n), |m|, |n| > 2$, *there exists a metric of positive sectional curvature in a compact three-manifold* M, *with the following properties:*

(a) M *is diffeomorphic to* S^3;

(b) there exist geodesic embeddings of S^1 *in* M, *isotopic to torus knots* (m, n) *and* $(m - n)$.

In connection to the Theorem 0.1 above, we would like to propose the following problem.

0.2. Is it true, that a positively curved metric in S^3 admits only finitely many different geodesic knot types?

The problem 0.2 seems to be conceptually related to a theorem of Choi-Schoen on compactness of the space of embedded minimal surfaces of a given genius in a Ricci-positive three-manifold (see 3.4).

1. Topology of cyclic quotients

1.1. Here we collect the facts we need on the topology of cyclic quotients. Let N be a smooth manifold with a smooth action of a cyclic group $\mathbb{Z}_n$. Assume that

(a) all stationary subgroups of points in N are either trivial or $\mathbb{Z}_n$ itself (this is automatically so, if n is prime);

(b) all components of the fixed point set $\mathrm{Fix}(N)$ are of codimension 2.

Then the quotient $N/\mathbb{Z}_n$ has a canonical structure of a smooth manifold. Indeed, let $Q \subset \mathrm{Fix}(N)$ be a connected component. Fix an invariant Riemannian metric in N. Let η be the rank two normal bundle to Q and let εD be the disc bundle η of radius ε. For ε small the exponential map establishes a diffeomorphism of εD onto a tubular neighbourhood of Q. We may assume η to be orientable (see below). Consider η as a complex line bundle. Then cut $Exp(\varepsilon D)$ off and glue the unit disc bundle of $\eta^{\otimes n}$ to $(N \backslash Exp(\varepsilon D))/\mathbb{Z}_n$. Doing this simultaneously for all Q, we get a new manifold, homeomorphic to $N/\mathbb{Z}_n$. If η is not orientable, consider the double covering $\pi : \tilde{Q} \to Q$ such that $\tilde{\eta} = \pi^*\eta$ is orientable. Denote τ the involution of $\tilde{Q}$ and $\tilde{\eta}$ such that $\tilde{Q}/\tau = Q$ and $\tilde{\eta}/\tau = \eta$, Observe that τ is an orthogonal and antilinear automorphism of $\tilde{\eta}$, so it induces an orthogonal and antilinear automorphism of $\tilde{\eta}^{\otimes n}$ which we denote again by τ. Now, glue $\tilde{\eta}^{\otimes n}/\tau$ to $(N \backslash Exp(\varepsilon D))/\mathbb{Z}_n$.

1.2. Example. Let n be a smooth quasiprojective variety over $\mathbb{R}$. Let τ be the canonical involution of $N(\mathbb{C})$, coming from $Gal(\mathbb{C}/\mathbb{R})$. Then $N(\mathbb{C})/\tau$ is a (real) smooth manifold. In particular, $\mathbb{C}P^2/\tau$ is a four-sphere, [5, 6].

1.3. Cyclic quotients of the three-sphere. Consider S^3 as a unit sphere in the Hermitian space $\mathbb{C}^2$ with the coordinates (z_1, z_2). Denote by K and L the geodesic circles $z_2 = 0$ and $z_1 = 0$. For $a, b > 0$ with $a^2 + b^2 = 1$ consider a torus $T_{a,b} : |z_1| = a, |z_2| = b$. The family $T_{a,b}$ form a fibration of $S^3 \backslash (K \cup L)$, "converging" to K and L. Now, any Hopf circle, i.e. an intersection of S^3 with a complex line, lies in one of $T_{a,b}$, namely, $\{z_2 = \lambda z_1\} \cap S^3$ lies on $T_{a,b}$ with $a = \dfrac{1}{\sqrt{1+|\lambda|^2}}$ and $b = \dfrac{\lambda}{\sqrt{1+|\lambda|^2}}$. Observe that all tori $T_{a,b}$ are equidistant from K and L in spherical metric. Any such torus is a Heegard surface of the decomposition $S^3 = D^2 \times S^1 \cup S^1 \times D^2$.

Now, consider a $\mathbb{Z}_m$-action $(z_1, z_2) \to (z_1, e^{\frac{2\pi k}{m}} z_2)$. It has K as a fixed point set and acts free in $S^3 \backslash K$. According to 1.1, $S^3/\mathbb{Z}_m$ is a manifold. We claim $S^3/\mathbb{Z}_m \approx S^3$. Indeed, the action in the handle, which contains L, is free and the quotient is obviously a handle again. The action in the other handle, which contains K is fiber like, as in 1.1, and since the normal bundle to K is trivial, the quotient is again a handle, which proves the statement.

Let n be an integer, coprime to m. Consider action of $\mathbb{Z}_m \times \mathbb{Z}_n$ by $(z_1, z_2) \to (e^{\frac{2\pi ik}{m}} z_1, e^{\frac{2\pi k}{m}} z_2)$. Applying the diffeomorphism above twice, we get a following lemmas.

Lemma 1.4. *The quotient $S_3/\mathbb{Z}_m \times \mathbb{Z}_n$ is diffeomorphic to S_3.*

2. Constructing a metric in the cyclic quotient

The construction below uses the computations of Gromov and Thurston [4]. In that paper, Gromov and Thurston introduced negatively curved metrics on ramified coverings of hyperbolic manifolds. Our situation is "dual" to that considered in [4], in particular, lifting to a ramified covering is replaced by descending to a cyclic quotient.

Lemma 2.1. *(Comp [4], p. 4). Given $n \in \mathbb{N}$ and $p > 0$, there exists a smooth function $\sigma(r)$ with the following properties:*

(i) $\sigma(r) = \sin r$ for small r,

(ii) $\sigma'(r) > 0$ and $\sigma''(r) < 0$,

(iii) $\sigma(r) = \dfrac{\sin r}{n}$ for $r \geq p$.

The proof is immediate.

Now, the metric of $S^3 \backslash L$ can be written as

$$g = dr^2 + \sin^2 r d\Theta^2 + \cos^2 r dt^2.$$

Here t is the length parameter along K and (r, Θ) are polar coordinates in geodesic two-spheres, orthogonal to K.

Consider the cyclic quotient $S^3/\mathbb{Z}_n$ and equip it with the metric

$$\tilde{g} = dr^2 + \sigma^2(r) d\Theta^2 + \cos^2 r dt^2,$$

where Θ here is the <u>new</u> angle parameter. Outside the ρ-neighbourhood of $K/\mathbb{Z}_n$ this is just a descend of the spherical metric by the (free) action of $\mathbb{Z}_n$. The crucial fact is the following.

Lemma 2.2. *The metric $\tilde{g}$ is a well-defined smooth metric on $S^3/\mathbb{Z}_n$, of strictly positive curvature, which is invariant under the (descend of) the $\mathbb{Z}_m$-action. Out of the small neighbourhood of K, the curvature of $\tilde{g}$ is constant.*

P r o o f. It is elementary to check that $\tilde{g}$ is smooth with respect to the manifold structure of $S^3/\mathbb{Z}_n$. The positivity of curvature follows form computations of [4], p. 4-5, with obvious changes (cosh $\to$ cos etc.). The invariance under the $\mathbb{Z}_n$-action is obvious from the construction.

2.3. Taking ρ small and repeating the construction with respect to the $\mathbb{Z}_m$-action, we come to the following lemma.

Lemma 2.3. *The quotient $S^3/\mathbb{Z}_m \times \mathbb{Z}_n$ can be equipped with the metric $\tilde{\tilde{g}}$ with following properties:*

(a) the curvature of $(S^3/\mathbb{Z}_m \times \mathbb{Z}_n, \tilde{\tilde{g}})$ is strictly positive

(b) outside arbitrary small neighbourhood of $K/\mathbb{Z}_m$ and $L/\mathbb{Z}_n$ the metric $\tilde{\tilde{g}}$ is a descend of the spherical metric of S^3.

3. Knotted geodesics

Lemma 3.1. *The image of any Hopf geodesic circle in S^3 outside the p-neighbourhoods of K, L is a torus knot (m, n) in $S^3/\mathbb{Z}_m \times \mathbb{Z}_n \approx S^3$.*

P r o o f. Let $\gamma = (z_2 = \lambda z_1) \cap S^3$, $\lambda \in \mathbb{C}$, be a Hopf circle. According to 1.3, $\gamma \subset T_{a,b}$ with $a = \dfrac{1}{\sqrt{1+|\lambda|^2}}$. In angle coordinates $(e^{i\Theta}, e^{i\tau})$ on $T_{a,b}$, γ may be written in parametric form as $t \to (e^{it}, e^{it})$. The quotient map $T_{a,b} \to T_{a,b}/\mathbb{Z}_m \times \mathbb{Z}_n \approx T^2$ can be written as $(e^{i\Theta}, e^{i\Theta}) \to (e^{mi\Theta}, e^{ni\tau})$. So the image of γ is $t \to (e^{mit}, e^{nit})$ which is (m, n) torus knot.

3.2. Geodesics of different knot type. Consider a new complex structure in $\mathbb{C}^2$ defined by the matrix $\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$. Observe that K and L are still Hopf circles which respect to this new complex structure. The tori $T_{a,b}$ have the same equation $|w_1| = a, |w_2| = b$ with respect to new coordinates $w_1 = z_1, w_2 = \overline{z}_2$. Hence they contain a Hopf circle $w_2 = \lambda w_1$, which descends to a geodesic in M, which is isotopic to the torus knot $(m, -n)$. Since torus knots are invertible, but not amphicherical ([3]), we get two different knot types among geodesics in M. This concludes the proof of the Theorem 0.1.

3.3. Questions and remarks

3.3.1. May knots other than torus knots be realized as geodesics of positively curved metric in S^3?

3.3.2. Fix a pinching constant δ. Can a three-sphere with a δ-pinched metric have geodesics of arbitrary torus knot type?

3.4. Suppose $S^g \subset (S^3, \mathrm{can})$ be an immersed minimal surface of genus g whose image do not touch $K \cup L$. Applying the construction above, we come to a minimal surface in M with "a lot of" selfintersections. In particular, one may start with a Clifford torus close to $T_{1/\sqrt{2},1/\sqrt{2}}$. This situation contrasts the compactness theorem of embedded minimal surface of a given genus [7]. For $g > 2$, are may therefore ask a following question:

May a compact minimal surface in S^3 of genus $g > 2$ avoid a geodesic circle?

3.5. We sketch a different type of examples which lead to Theorem 0.1. in case when both m, n are odd. Start with a positively curved metric in S^2. Let $\overline{M} = US^2$, a unit tangent bundle with the Sasaki metric (making $US^2 \to S^2$ be a Riemannian submersion). If the curvature of S^2 is less than $1/\sqrt{3}$, then US^2 is positively curved. This follows from O'Neil formula for Riemannian submersion with totally geodesic fibers ([2, Ch. 9]). Since $US^2 \simeq \mathbb{R}P^2$, the double cover of US^2 is the three-sphere.

Now, we may take the metric of S^2 to be rotationally invariant. Then the computations of [1] show that we have the full control on closed geodesics of S^2.

In particular, there are closed trajectories of the geodesic flow in US^2, whose lift to S^3 will be a torus knot of a given type (m,n) if both m,n are odd. Unfortunately, already the trefoil knot may not be realized in this way.

3.6. Concluding remarks. It looks like that there exists a positively curved metric on S^4 with a totally geodesic $\mathbb{R}P^2$ having the normal Euler number four. Indeed, look at the standard Kahler metric in $\mathbb{C}P^2$. The canonical autoholomorphic involution $\tau : \mathbb{C}P^2 \to \mathbb{C}P^2$ is an isometry and $\mathbb{C}P^2/\tau \simeq S^4$. The fixed point set $\mathrm{Fix}(\tau)$ is a totally geodesic $\mathbb{R}P^2 \subset \mathbb{C}P^2$. It is possible to mimic the construction of 2.2. and find a perturbation of the quotient metric in $\mathbb{C}P^2/\tau$ in directions orthogonal to $\mathbb{R}P^2$ (recall that there exists exactly one totally geodesic surface, also isometric to $\mathbb{R}P^2$, meeting $\mathrm{Fix}(\tau)$ orthogonally at a given point). The curvature tensor, however, is no more diagonal, and it is a nontrivial problem to check if the curvature is positive. Observe that the resulting metric admit an isometric $SO(3)$-action, and the equidistant from $\mathbb{R}P^2$ manifolds are lens spaces $SO(3)/\mathbb{Z}_2 \simeq S^3/\mathbb{Z}_4$ with a homogeneous metric, which is different from Berger's metrics.

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Заузленные вполне геодезические подмногообразия на сферах положительной кривизны

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