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Geometric realizations for some series of representations of the quantum group $SU_{2,2}$

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The paper solves the problem of analytic continuation for the holomorphic discrete series of representations for the quantum group $SU(2,2)$. In particular, a new realization of the ladder representation of this group is produced. Besides, q-analogues are constructed for the Shilov boundary of the unit ball in the space of complex 2×2 matrices and the principal degenerate series representations of $SU(2,2)$ associated to that boundary. A possibility is discussed of transferring some well known geometric constructions of the representation theory to the quantum case: the Penrose transform, the Beilinson–Bernstein approach to the construction of Harish–Chandra modules (for the case of the principal nondegenerate series).

1. Introduction

We consider some series of modules over the quantum universal enveloping Drinfeld–Jimbo algebra $U_q\mathfrak{g}$ in the special case $\dim\mathfrak{g} < \infty$, $0 < q < 1$. The finite dimensional $U_q\mathfrak{g}$-modules are closely related to compact quantum groups; those were investigated well enough [4, 13]. Infinite dimensional $U_q\mathfrak{g}$-modules we deal with in this work originate from our earlier paper [19], together with some applications therein to the theory of q-Cartan domains. To make the exposition more transparent, we restrict ourselves to a q-analogue of the ball in the space of

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all complex 2×2 matrices $\mathbb{U}=\{\boldsymbol{z}\in \mathrm{Mat}_2 | \boldsymbol{z}\boldsymbol{z}^*<1\}$, which is among the simplest Cartan domains.

The classes of infinite dimensional $U_q\mathfrak{g}$-modules in question differ from those considered by Letzter [11]. The problem of producing and investigating of the principal series of quantum Harish–Chandra modules in our case appears to be essentially more complicated.

It is worthwhile to note that some properties of the ladder representation of the quantum $SU_{2,2}$ described below are already well known [2].

Everywhere in the sequel $0<q<1$, the ground field is $\mathbb{C}$, and all the algebras are assumed to be unital, unless the contrary is stated explicitly.

Consider the Hopf algebra $U_q\mathfrak{g}=U_q\mathfrak{sl}_4$ determined by the standard lists of generators E_j, F_j, $K_j^{\pm 1}$, $j=1,2,3$, and relations [4, 13]. The coproduct $\triangle$, the counit ε, and the antipode S are given as follows:

$$\begin{aligned}
\triangle E_j &= E_j\otimes 1 + K_j\otimes E_j, & \varepsilon(E_j) &= 0, & S(E_j) &= -K_j^{-1}E_j,\\
\triangle F_j &= F_j\otimes K_j^{-1} + 1\otimes F_j, & \varepsilon(F_j) &= 0, & S(F_j) &= -F_jK_j,\\
\triangle K_j &= K_j\otimes K_j, & \varepsilon(K_j) &= 1, & S(K_j) &= K_j^{-1}.
\end{aligned}$$

We call a $U_q\mathfrak{g}$-module V $\mathbb{R}^3$-admissible if $V=\bigoplus_{\boldsymbol{\mu}} V_{\boldsymbol{\mu}}$ with $\boldsymbol{\mu}=(\mu_1,\mu_2,\mu_3)\in\mathbb{R}^3$, $V_{\boldsymbol{\mu}}=\{v\in V|\ K_j^{\pm1}v=q^{\pm\mu_j}v,\ j=1,2,3\}$. Let $U_q\mathfrak{k}\subset U_q\mathfrak{g}$ be the Hopf subalgebra generated by $K_2^{\pm1}$, E_j, F_j, $K_j^{\pm1}$, $j=1,3$. Every $U_q\mathfrak{g}$-module inherits a structure of $U_q\mathfrak{k}$-module. We are interested in quantum $(\mathfrak{g},\mathfrak{k})$-modules, i.e., $\mathbb{R}^3$-admissible $U_q\mathfrak{g}$-modules which are direct sums of finite dimensional $U_q\mathfrak{k}$-modules.

Equip the Hopf algebra $U_q\mathfrak{g}$ with an involution:

$$E_2^*=-K_2F_2,\qquad F_2^*=-E_2K_2^{-1},\qquad K_2^*=K_2,$$

$$E_j^*=K_jF_j,\qquad F_j^*=E_jK_j^{-1},\qquad K_j^*=K_j,\qquad j=1,3.$$

We thus get a $*$-Hopf algebra $(U_q\mathfrak{g},*)$ which is a q-analogue of $U\mathfrak{su}_{2,2}$ and its subalgebra $(U_q\mathfrak{k},*)$ is a q-analogue of $U\mathfrak{s}(\mathfrak{u}_2\times\mathfrak{u}_2)$.

A quantum $(\mathfrak{g},\mathfrak{k})$-module V is said to be unitarizable if $(\xi v_1,v_2)=(v_1,\xi^*v_2)$ for some Hermitian scalar product in V and all $v_1,v_2\in V$, $\xi\in U_q\mathfrak{g}$. Our purpose here is to produce some series of unitarizable quantum $(\mathfrak{g},\mathfrak{k})$-modules by means of non-commutative geometry and non-commutative function theory in q-Cartan domains [19, 15–18].

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2. The $U_q\mathfrak{su}_{2,2}$-module algebra $\mathrm{Pol}(\mathrm{Pl}_{2,4})_{q,x}$

Let e_1, e_2, e_3, e_4 be the standard basis in $\mathbb{C}^4$. Associate to every linear operator in $\mathbb{C}^2$ its graph, a two-dimensional subspace in $\mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2$, which has trivial intersection with the linear span of e_1, e_2. We are interested in the pairs (L, ω), with L a subspace as above and ω its non-zero volume form (an skew-symmetric bilinear form) in L. We need a q-analogue of this algebraic variety which we call the Plücker manifold $\mathrm{Pl}_{2,4}$. The matrix elements $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ of the linear operator L, together with $t^{\pm 1}$ related to the volume element ω, work as 'coordinates' on $\mathrm{Pl}_{2,4}$.

An algebra F is called a $U_q\mathfrak{g}$-module algebra if the multiplication $m : F \otimes F \to F$ is a morphism of $U_q\mathfrak{g}$-modules, and the unit $1 \in F$ is a $U_q\mathfrak{g}$-invariant. To rephrase, one can say that for all $f_1, f_2 \in F$, $j = 1, 2, 3$,

$$\begin{aligned} E_j(f_1 f_2) &= E_j(f_1) f_2 + (K_j f_1)(E_j f_2), & E_j 1 &= 0, \\ F_j(f_1 f_2) &= (F_j f_1)(K_j^{-1} f_2) + f_1(F_j f_2), & F_j 1 &= 0, \\ K_j^{\pm 1}(f_1 f_2) &= (K_j^{\pm 1} f_1)(K_j^{\pm 1} f_2), & K_j^{\pm 1} 1 &= 1. \end{aligned}$$

In the case of a $*$-algebra F one should impose an additional compatibility requirement for involutions:

$$(\xi f)^* = (S(\xi))^* f^*, \quad \xi \in U_q\mathfrak{g},\ f \in F.$$

Once the $*$-algebra F is given by the list of its generators and relations, the $U_q\mathfrak{g}$-module structure in F is determined unambiguously by the action of the generators E_j, F_j, $K_j^{\pm 1}$, $j = 1, 2, 3$, on the generators of F.

Consider the $*$-algebra $\mathrm{Pol}(\mathrm{Mat}_2)_q$ given by its generators α, β, γ, δ and the following commutation relations (the initial six of those are well known and the rest was obtained in [16]):

$$\left\{ \begin{array}{lcl} \alpha\beta &=& q\beta\alpha \\ \gamma\delta &=& q\delta\gamma \end{array} \right., \qquad \left\{ \begin{array}{lcl} \alpha\gamma &=& q\gamma\alpha \\ \beta\delta &=& q\delta\beta \end{array} \right., \qquad \left\{ \begin{array}{lcl} \beta\gamma &=& \gamma\beta \\ \alpha\delta &=& \delta\alpha + (q - q^{-1})\beta\gamma \end{array} \right.,$$

$$\left\{ \begin{array}{lcl} \delta^*\alpha &=& \alpha\delta^* \\ \delta^*\beta &=& q\beta\delta^* \\ \delta^*\gamma &=& q\gamma\delta^* \\ \delta^*\delta &=& q^2\delta\delta^* + 1 - q^2 \end{array} \right., \qquad \left\{ \begin{array}{lcl} \gamma^*\alpha &=& q\alpha\gamma^* - (q^{-1} - q)\beta\delta^* \\ \gamma^*\beta &=& \beta\gamma^* \\ \gamma^*\gamma &=& q^2\gamma\gamma^* - (1 - q^2)\delta\delta^* + 1 - q^2 \end{array} \right.,$$

$$\left\{ \begin{array}{lcl} \beta^*\alpha &=& q\alpha\beta^* - (q^{-1} - q)\gamma\delta^* \\ \beta^*\beta &=& q^2\beta\beta^* - (1 - q^2)\delta\delta^* + 1 - q^2 \end{array} \right.,$$

$$\alpha^*\alpha = q^2\alpha\alpha^* - (1 - q^2)(\beta\beta^* + \gamma\gamma^*) + (q^{-1} - q)^2\delta\delta^* + 1 - q^2.$$

The $*$-algebra $\mathrm{Pol}(\mathrm{Pl}_{2,4})_{q,x}$ is given by the generators α, β, γ, δ, t, t^{-1}, the commutation relations as in the above definition of $\mathrm{Pol}(\mathrm{Mat}_2)_q$, and the additional relations $tt^{-1}=t^{-1}t=1$, $tt^*=t^*t$, $zt=qtz$, $zt^*=qt^*z$, with $z\in\{\alpha,\beta,\gamma,\delta\}^\star$.

An application of a q-analogue for the above geometric interpretation of the Plücker manifold allows one to prove

Proposition 2.1. *i) There exists a unique structure of $U_q\mathfrak{su}_{2,2}$-module algebra in* $\mathrm{Pol}(\mathrm{Mat}_2)_q$ *such that*

$$\begin{pmatrix}E_1\alpha & E_1\beta\\ E_1\gamma & E_1\delta\end{pmatrix}=q^{-1/2}\begin{pmatrix}0&\alpha\\0&\gamma\end{pmatrix},\qquad \begin{pmatrix}E_3\alpha & E_3\beta\\ E_3\gamma & E_3\delta\end{pmatrix}=q^{-1/2}\begin{pmatrix}0&0\\ \alpha&\beta\end{pmatrix},$$

$$\begin{pmatrix}F_1\alpha & F_1\beta\\ F_1\gamma & F_1\delta\end{pmatrix}=q^{1/2}\begin{pmatrix}\beta&0\\ \delta&0\end{pmatrix},\qquad \begin{pmatrix}F_3\alpha & F_3\beta\\ F_3\gamma & F_3\delta\end{pmatrix}=q^{1/2}\begin{pmatrix}\gamma&\delta\\ 0&0\end{pmatrix},$$

$$\begin{pmatrix}K_1\alpha & K_1\beta\\ K_1\gamma & K_1\delta\end{pmatrix}=\begin{pmatrix}q\alpha&q^{-1}\beta\\ q\gamma&q^{-1}\delta\end{pmatrix},\qquad \begin{pmatrix}K_3\alpha & K_3\beta\\ K_3\gamma & K_3\delta\end{pmatrix}=\begin{pmatrix}q\alpha&q\beta\\ q^{-1}\gamma&q^{-1}\delta\end{pmatrix},$$

$$\begin{pmatrix}E_2\alpha & E_2\beta\\ E_2\gamma & E_2\delta\end{pmatrix}=-q^{1/2}\begin{pmatrix}q^{-1}\beta\gamma&\delta\beta\\ \delta\gamma&\delta^2\end{pmatrix},\qquad \begin{pmatrix}F_2\alpha & F_2\beta\\ F_2\gamma & F_2\delta\end{pmatrix}=q^{1/2}\begin{pmatrix}0&0\\ 0&1\end{pmatrix},$$

$$\begin{pmatrix}K_2\alpha & K_2\beta\\ K_2\gamma & K_2\delta\end{pmatrix}=\begin{pmatrix}\alpha&q\beta\\ q\gamma&q^2\delta\end{pmatrix}.$$

ii) There exists a unique structure of $U_q\mathfrak{su}_{2,2}$-module algebra in $\mathrm{Pol}(\mathrm{Pl}_{2,4})_{q,x}$ *such that the action of E_j, F_j, $K_j^{\pm 1}$ on α, β, γ, δ is given by the above equations and*

$$\left\{\begin{array}{lcl}E_jt&=&0\\ F_jt&=&0\\ K_jt&=&t\end{array}\right.,\quad j=1,3;\qquad \left\{\begin{array}{lcl}E_2t&=&q^{-1/2}t\delta\\ F_2t&=&0\\ K_2t&=&q^{-1}t\end{array}\right..$$

Note that a much more general result is obtained in [16].

To produce the series of quantum $(\mathfrak{g},\mathfrak{k})$-modules considered in the sequel we use essentially the specific dependencies of the elements E_2t^λ, F_2t^λ, $K_2^{\pm1}t^\lambda$, $E_2((\alpha\delta-q\beta\gamma)^\lambda)$, $F_2((\alpha\delta-q\beta\gamma)^\lambda)$, $K_2^{\pm1}((\alpha\delta-q\beta\gamma)^\lambda)$ on q^λ. It is easily deducible from the definitions that for all $\lambda\in\mathbb{Z}_+$

$$E_2t^\lambda=q^{-3/2}\frac{q^{-2\lambda}-1}{q^{-2}-1}\delta t^\lambda,\qquad F_2t^\lambda=0,\qquad K_2^{\pm1}t^\lambda=q^{\mp\lambda}t^\lambda,$$

$$E_2((\alpha\delta-q\beta\gamma)^\lambda)=-q^{1/2}\frac{1-q^{2\lambda}}{1-q^2}\delta(\alpha\delta-q\beta\gamma)^\lambda,$$

*The notation $x=tt^*$ and $\mathrm{Pol}(\mathrm{Pl}_{2,4})_{q,x}$ are justified by the fact that the algebra $\mathrm{Pol}(\mathrm{Pl}_{2,4})_{q,x}$ in question can be derived as a localization of another useful algebra $\mathrm{Pol}(\mathrm{Pl}_{2,4})_q$ with respect to the multiplicative system $x^{\mathbb{N}}$.

$$F_2((\alpha\delta - q\beta\gamma)^\lambda) = q^{1/2}\frac{q^{-2\lambda}-1}{q^{-2}-1}\alpha(\alpha\delta - q\beta\gamma)^{\lambda-1}, \qquad \lambda \neq 0,$$
$$K_2^{\pm 1}((\alpha\delta - q\beta\gamma)^\lambda) = q^{\pm 2\lambda}(\alpha\delta - q\beta\gamma)^\lambda.$$

For instance, the first relation is obvious for $\lambda = 0$, and the general case is accessible via an induction argument:

$$\begin{aligned} E_2(t^{\lambda+1}) &= (E_2t)t^\lambda + (K_2t)(E_2t^\lambda) = q^{-1/2}t\delta t^\lambda + q^{-1}tq^{-3/2}\frac{q^{-2\lambda}-1}{q^{-2}-1}\delta t^\lambda \\ &= \left(q^{-1/2} + q^{-5/2}\frac{q^{-2\lambda}-1}{q^{-2}-1}\right)q^{-1}\delta t^{\lambda+1} = q^{-3/2}\frac{q^{-2(\lambda+1)}-1}{q^{-2}-1}\delta t^{\lambda+1}. \end{aligned}$$

3. The analytic continuation of the holomorphic discrete series: step one

Consider the subalgebra $\mathbb{C}[\mathrm{Pl}_{2,4}]_{q,t} \subset \mathrm{Pol}(\mathrm{Pl}_{2,4})_{q,x}$ generated by α, β, γ, δ, t, t^{-1}. Equip it with a $\mathbb{Z}$-grading: $\deg\alpha = \deg\beta = \deg\gamma = \deg\delta = 0$, $\deg(t^{\pm 1}) = \pm 1$. The homogeneous components of this algebra are quantum $(\mathfrak{g}, \mathfrak{k})$-modules*.

Consider the subalgebra $\mathbb{C}[\mathrm{Mat}_2]_q \subset \mathrm{Pol}(\mathrm{Mat}_2)_q$ generated by α, β, γ, δ. This algebra constitutes a famous subject of a research in the quantum group theory. Associate to each $\lambda \in \mathbb{Z}$ a linear operator $i_\lambda : \mathbb{C}[\mathrm{Mat}_2]_q \to \mathbb{C}[\mathrm{Pl}_{2,4}]_{q,t}$, $i_\lambda : f \mapsto ft^{-\lambda}$. This isomorphism between the vector space $\mathbb{C}[\mathrm{Mat}_2]_q$ and a homogeneous component of $\mathbb{C}[\mathrm{Pl}_{2,4}]_{q,t}$ allows one to transfer the structure of $U_q\mathfrak{sl}_4$-module from $\mathbb{C}[\mathrm{Pl}_{2,4}]_{q,t}$ to $\mathbb{C}[\mathrm{Mat}_2]_q$. Thus we obtain a representation of $U_q\mathfrak{sl}_4$ in $\mathbb{C}[\mathrm{Mat}_2]_q$, to be denoted by π_{q^λ}. For all $\xi \in U_q\mathfrak{sl}_4$, $f \in \mathbb{C}[\mathrm{Mat}_2]_q$, the vector valued function $\pi_{q^\lambda}(\xi)f$ appears to be a Laurent polynomial of an indeterminate $\zeta = q^\lambda$. This leads to the canonical analytic continuation of the operator valued function π_{q^λ}. The term 'analytic continuation of the holomorphic discrete series' stands for the above family π_{q^λ} of representations of $U_q\mathfrak{sl}_4$.

The results of the work by H.P. Jakobsen [5] imply that the quantum $(\mathfrak{g}, \mathfrak{k})$-modules π_{q^λ} are unitarizable for all $\lambda > 1$. We follow [17] in finding an explicit form for the related scalar product.

Consider the $\mathrm{Pol}(\mathrm{Mat}_2)_q$-module given by a single generator v and the relations $\alpha^*v = \beta^*v = \gamma^*v = \delta^*v = 0$. The associated representation T of $\mathrm{Pol}(\mathrm{Mat}_2)_q$ in the vector space $H = \mathbb{C}[\mathrm{Mat}_2]_qv$ is faithful; it is called the vacuum representation.

*The notation $\mathbb{C}[\mathrm{Pl}_{2,4}]_{q,t}$ can be justified in the same way as the notation $\mathrm{Pol}(\mathrm{Pl}_{2,4})_{q,x}$ introduced in the previous section.

Let $\check{\rho}$ be the linear operator in H that realizes the action of the 'half-sum of positive coroots':

$$\check{\rho}(\alpha^a\beta^b\gamma^c\delta^d v)=\frac{1}{2}(3a+2b+2c+d)\alpha^a\beta^b\gamma^c\delta^d v,$$

with $a,b,c,d\in\mathbb{Z}_+$. We need also the element

$$y=1-(\alpha\alpha^*+\beta\beta^*+\gamma\gamma^*+\delta\delta^*)+(\alpha\delta-q\beta\gamma)(\alpha\delta-q\beta\gamma)^*,$$

which is a q-analogue of the determinant $\det(1-\mathbf{z}\mathbf{z}^*)$, with $\mathbf{z}=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$.

As a consequence of the results of [17] we have

Proposition 3.1. *i) For $\lambda>3$ the linear functional*

$$\int\limits_{\mathbb{U}_q} f d\nu_\lambda \stackrel{\text{def}}{=} \frac{\operatorname{tr}(T(fy^\lambda)q^{-2\check{\rho}})}{\operatorname{tr}(T(y^\lambda)q^{-2\check{\rho}})}$$

is well defined and positive on $\operatorname{Pol}(\operatorname{Mat}_2)_q$.

ii) For $\lambda>3$ the scalar product $(f_1,f_2)_{q^{2\lambda}}=\int\limits_{\mathbb{U}_q} f_2^* f_1 d\nu_\lambda$ in $\mathbb{C}[\operatorname{Mat}_2]_q$ is well defined, positive, and

$$(\pi_{q^\lambda}(\xi)f_1,f_2)_{q^{2\lambda}}=(f_1,\pi_{q^\lambda}(\xi^*)f_2)_{q^{2\lambda}},\qquad \xi\in U_q\mathfrak{g},\ f_1,f_2\in\mathbb{C}[\operatorname{Mat}_2]_q.$$

The representations π_{q^λ}, $\lambda=3,4,5,\ldots$, are q-analogues of the holomorphic discrete series representations, and the completions of $\mathbb{C}[\operatorname{Mat}_2]_q$ with respect to the norms $\|f\|_{q^{2\lambda}}=(f,f)_{q^{2\lambda}}^{1/2}$ are q-analogues of the weighted Bergman spaces. Our intention in what follows is to present explicit formulae for the analytic continuation of the scalar product $(f_1,f_2)_{q^{2\lambda}}$ with respect to the parameter $q^{2\lambda}$ and to prove the positivity of this scalar product for $\lambda>1$.

To conclude, consider the $U_q\mathfrak{k}$-invariants

$$y_1=\alpha\alpha^*+\beta\beta^*+\gamma\gamma^*+\delta\delta^*,\qquad y_2=(\alpha\delta-q\beta\gamma)(\alpha\delta-q\beta\gamma)^*.$$

Prove that $T(y_1)T(y_2)=T(y_2)T(y_1)$, or equivalently, $y_1y_2=y_2y_1$. In fact, observe that H admits a structure of $U_q\mathfrak{k}$-module ($\xi(fv)=(\xi f)v$, $\xi\in U_q\mathfrak{k}$, $f\in\mathbb{C}[\operatorname{Mat}_2]_q$) and splits into a sum of pairwise non-isomorphic simple $U_q\mathfrak{k}$-modules $H=\bigoplus\limits_{k_1\geq k_2\geq 0} H^{(k_1,k_2)}$, $H^{(k_1,k_2)}=U_q\mathfrak{k}\delta^{k_1-k_2}(\alpha\delta-q\beta\gamma)^{k_2}v$. What remains is to take into account that the restrictions of $T(y_1)$, $T(y_2)$ onto $H^{(k_1,k_2)}$ are scalar operators by the 'Schur lemma'. Those scalars are easily deducible:

$$T(y_1)|_{H^{(k_1,k_2)}}=1-q^{2k_1}+q^{-2}(1-q^{2k_2}),$$

$$T(y_2)|_{H^{(k_1,k_2)}} = q^{-2}(1-q^{2k_2})(1-q^{2(k_1+1)}).$$

Just as one could expect, the joint spectrum of the operators $T(y_1)$, $T(y_2)$ tends to

$$\{(\mathrm{tr}(\boldsymbol{z}\boldsymbol{z}^*), \det(\boldsymbol{z}\boldsymbol{z}^*))|\, \boldsymbol{z} \in \mathbb{U}\} = \{(y_1, y_2)|\, 0 \le y_1 \le 2 \quad \& \quad 0 \le y_2 \le y_1^2/4\}$$

as q goes to 1.

4. An invariant integral on the Shilov boundary

Let $c = \alpha\delta - q\beta\gamma$ and $\mathbb{C}[GL_2]_q$ be the localization of $\mathbb{C}[\mathrm{Mat}_2]_q$ with respect to the multiplicative system $c^{\mathbb{N}}$. It is easy to prove the existence and uniqueness of an extension of the $U_q\mathfrak{g}$-module structure from $\mathbb{C}[\mathrm{Mat}_2]_q$ onto $\mathbb{C}[GL_2]_q$. Equip the $U_q\mathfrak{g}$-module algebra $\mathbb{C}[GL_2]_q$ with an involution:

$$\alpha^* = q^{-2}(\alpha\delta - q\beta\gamma)^{-1}\delta, \qquad \beta^* = -q^{-1}(\alpha\delta - q\beta\gamma)^{-1}\gamma,$$
$$\gamma^* = -q^{-1}(\alpha\delta - q\beta\gamma)^{-1}\beta, \qquad \delta^* = (\alpha\delta - q\beta\gamma)^{-1}\alpha,$$

and introduce the notation $\mathrm{Pol}(S(\mathbb{U}))_q = (\mathbb{C}[GL_2]_q, *)$.

The following propositions justify our choice of the involution.

Proposition 4.1. *For all $f \in \mathrm{Pol}(S(\mathbb{U}))_q$, $\xi \in U_q\mathfrak{g}$ one has*

$$(\xi f)^* = (S(\xi))^* f^*.$$

Proposition 4.2. *There exists a unique homomorphism of $U_q\mathfrak{g}$-module $*$-algebras $j:\ \mathrm{Pol}(\mathrm{Mat}_2)_q \to \mathrm{Pol}(S(\mathbb{U}))_q$ such that $j(\alpha) = \alpha$, $j(\beta) = \beta$, $j(\gamma) = \gamma$, $j(\delta) = \delta$.*

These statements are proved in an essentially more general form in [18]. It also follows from the results of the work that the $U_q\mathfrak{g}$-module $*$-algebra $\mathrm{Pol}(S(\mathbb{U}))_q$ is a q-analogue of the polynomial algebra on the Shilov boundary $S(\mathbb{U})$ of the unit ball $\mathbb{U}$ in the space Mat_2 of complex 2×2 matrices.

The $U_q\mathfrak{k}$-module $\mathrm{Pol}(S(\mathbb{U}))_q$ splits into a sum of pairwise non-isomorphic simple finite dimensional submodules. In particular, the trivial $U_q\mathfrak{k}$-module appears in $\mathrm{Pol}(S(\mathbb{U}))_q$ with multiplicity 1 and there exists a unique $U_q\mathfrak{k}$-invariant integral $\mu:\ \mathrm{Pol}(S(\mathbb{U}))_q \to \mathbb{C}$, $f \mapsto \int\limits_{S(\mathbb{U})_q} f d\mu$, with $\int\limits_{S(\mathbb{U})_q} 1 d\mu = 1$.

Proposition 4.3. *The above $U_q\mathfrak{k}$-invariant integral is positive definite.*

P r o o f. Consider the $*$-algebra $\mathrm{Pol}(U_2)_q$ of regular functions on the quantum U_2 [9], together with the $*$-homomorphism of algebras $i : \mathrm{Pol}(S(\mathbb{U}))_q \to \mathrm{Pol}(U_2)_q$ given by

$$i(\alpha) = q^{-1}\alpha, \qquad i(\beta) = q^{-1}\beta,$$
$$i(\gamma) = \gamma, \qquad i(\delta) = \delta.$$

The positivity of an invariant integral on the quantum group U_2 constitutes a well known fact. So, what remains is to prove the invariance of the integral

$$\mathrm{Pol}(U_2)_q \to \mathbb{C}, \qquad f \mapsto \int\limits_{S(\mathbb{U})_q} i^{-1}(f) d\mu$$

with respect to the action of $U_q\mathfrak{u}_2$ by right translations on the quantum U_2. This is a consequence of the invariance of μ with respect to the action of the subalgebra in $U_q\mathfrak{k}$ generated by E_1, F_1, $K_1^{\pm 1}$, $(K_1K_2^2K_3)^{\pm 1}$. ■

Now introduce an auxiliary $U_q\mathfrak{g}$-module $*$-algebra $\mathrm{Pol}(\widehat{S}(\mathbb{U}))_q$, to be used in a construction of the principal degenerate series of quantum $(\mathfrak{g}, \mathfrak{k})$-modules.

The $*$-algebra $\mathrm{Pol}(\widehat{S}(\mathbb{U}))_q$ is defined by adding t, t^{-1} to the list α, β, γ, δ, c^{-1} of generators of $\mathrm{Pol}(S(\mathbb{U}))_q$ and

$$tt^{-1} = t^{-1}t = 1, \qquad tt^* = t^*t,$$

$$zt = qtz, \qquad zt^* = qt^*z, \qquad \text{with} \quad z \in \{\alpha, \beta, \gamma, \delta\}$$

to the list of relations.

The next two statements follow from the results of [18].

Proposition 4.4. *i) There exists a unique extension of the structure of $U_q\mathfrak{g}$-module $*$-algebra from $\mathrm{Pol}(S(\mathbb{U}))_q$ onto $\mathrm{Pol}(\widehat{S}(\mathbb{U}))_q$ such that*

$$\begin{cases} E_jt & = & 0 \\ F_jt & = & 0 \\ K_jt & = & 0 \end{cases}, \quad j = 1,3, \qquad \begin{cases} E_2t & = & q^{-1/2}t\delta \\ F_2t & = & 0 \\ K_2t & = & q^{-1}t \end{cases}.$$

ii) There exists a unique homomorphism $\widehat{j} : \mathrm{Pol}(\mathrm{Pl}_{2,4})_{q,x} \to \mathrm{Pol}(\widehat{S}(\mathbb{U}))_q$ of $U_q\mathfrak{g}$-module $*$-algebras such that

$$\widehat{j}(\alpha) = \alpha, \qquad \widehat{j}(\beta) = \beta, \qquad \widehat{j}(\gamma) = \gamma, \qquad \widehat{j}(\delta) = \delta, \qquad \widehat{j}(t^{\pm 1}) = t^{\pm 1}.$$

Proposition 4.5. *The subspace* $t^{*-2}\mathrm{Pol}(S(\mathbb{U}))_q t^{-2}$ *is a submodule of the* $U_q\mathfrak{g}$*-module* $\mathrm{Pol}(\widehat{S}(\mathbb{U}))_q$*, and the linear functional*

$$t^{*-2}\mathrm{Pol}(S(\mathbb{U}))_q t^{-2} \to \mathbb{C}, \qquad t^{*-2} f t^{-2} \mapsto \int\limits_{S(\mathbb{U})_q} f d\mu$$

is an invariant integral (i.e., a morphism of $U_q\mathfrak{g}$*-modules).*

5. An analytic continuation of the holomorphic discrete series: step two

Just as in the classical case $q = 1$, one has

$$\mathbb{C}[\mathrm{Mat}_2]_q = \bigoplus_{k_1 \geq k_2 \geq 0} \mathbb{C}[\mathrm{Mat}_2]_q^{(k_1,k_2)} = U_q\mathfrak{k}\delta^{k_1-k_2}(\alpha\delta - q\beta\gamma)^{k_2},$$

with $\mathbb{C}[\mathrm{Mat}_2]_q^{(k_1,k_2)}$ being simple pairwise non-isomorphic $U_q\mathfrak{k}$-submodules of the $U_q\mathfrak{k}$-module $\mathbb{C}[\mathrm{Mat}_2]_q$. Introduce the notation $f^{(k_1,k_2)}$ for a projection of f onto the $U_q\mathfrak{k}$-isotypic component $\mathbb{C}[\mathrm{Mat}_2]_q^{(k_1,k_2)}$ parallel to the sum of all other $U_q\mathfrak{k}$-isotypic components.

By the 'Schur lemma', every $U_q\mathfrak{k}$-invariant Hermitian form (f_1, f_2) on $\mathbb{C}[\mathrm{Mat}_2]_q$ is given by

$$(f_1, f_2) = \sum_{k_1 \geq k_2 \geq 0} c(k_1, k_2) \int\limits_{S(\mathbb{U})_q} (f_2^{(k_1,k_2)})^* f_1^{(k_1,k_2)} d\mu.$$

We are going to obtain this decomposition for $(f_1, f_2)_{q^{2\lambda}}$, $\lambda > 3$. Recall the notation $(a; q^2)_m = \prod\limits_{j=0}^{m-1}(1 - aq^{2j})$.

Proposition 5.1. *For all* $\lambda > 3$, $f_1, f_2 \in \mathbb{C}[\mathrm{Mat}_2]_q$,

$$\int\limits_{\mathbb{U}_q} f_2^* f_1 d\nu_\lambda = \sum_{k_1 \geq k_2 \geq 0} c(k_1, k_2, q^{2\lambda}) \int\limits_{S(\mathbb{U})_q} (f_2^{(k_1,k_2)})^* f_1^{(k_1,k_2)} d\mu,$$

with

$$c(k_1, k_2, q^{2\lambda}) = \frac{(q^4; q^2)_{k_1}(q^2; q^2)_{k_2}}{(q^{2\lambda}; q^2)_{k_1}(q^{2(\lambda-1)}; q^2)_{k_2}}. \tag{5.1}$$

P r o o f. In the case $q=1$ a similar result was obtained by Faraut and Koranyi [3] in a very big generality. Our proof here imitates that of [3].

First introduce the subalgebra $\mathbb{C}[\overline{\mathrm{Mat}}_2]_q \subset \mathrm{Pol}(\mathrm{Mat}_2)_q$ generated by α^*, β^*, γ^*, δ^*, and the algebra $\mathbb{C}[\overline{\mathrm{Mat}}_2]_q^{\mathrm{op}}$ which differs from $\mathbb{C}[\overline{\mathrm{Mat}}_2]_q$ by replacement of the multiplication law with the opposite one. We use the algebra $\mathbb{C}[\mathrm{Mat}_2 \times \overline{\mathrm{Mat}}_2]_q \stackrel{\mathrm{def}}{=} \mathbb{C}[\mathrm{Mat}_2]_q \otimes \mathbb{C}[\overline{\mathrm{Mat}}_2]_q^{\mathrm{op}}$ as a q-analogue for the algebra of (degenerate) kernels of integral operators.

Equip $\mathbb{C}[\mathrm{Mat}_2 \times \overline{\mathrm{Mat}}_2]_q$ with a bigrading

$$\deg(\alpha \otimes 1) = \deg(\beta \otimes 1) = \deg(\gamma \otimes 1) = \deg(\delta \otimes 1) = (1,0),$$

$$\deg(1 \otimes \alpha^*) = \deg(1 \otimes \beta^*) = \deg(1 \otimes \gamma^*) = \deg(1 \otimes \delta^*) = (0,1)$$

and the associated topology. The completed algebra $\mathbb{C}[[\mathrm{Mat}_2 \times \overline{\mathrm{Mat}}_2]]_q$ will work as the algebra of generalized kernels of integral operators [17].

Just as in the case $q=1$ one can deduce Proposition 5.1. from the following three lemmas.

Lemma 5.2. *Given $k_1, k_2 \in \mathbb{Z}$, $k_1 \geq k_2 \geq 0$, denote by P_{k_1,k_2} the projection in $\mathbb{C}[\mathrm{Mat}_2]_q$ onto the component $\mathbb{C}[\mathrm{Mat}_2]_q^{(k_1,k_2)}$ parallel to the sum of all other $U_q\mathfrak{k}$-isotypic components. There exists a unique element $p_{k_1,k_2} \in \mathbb{C}[\mathrm{Mat}_2 \times \overline{\mathrm{Mat}}_2]_q$ such that*

$$P_{k_1,k_2} f(\boldsymbol{z}) = \int\limits_{S(\mathbb{U})_q} p_{k_1,k_2}(\boldsymbol{z},\boldsymbol{\zeta}) f(\boldsymbol{\zeta}) d\mu(\boldsymbol{\zeta})$$

for all $f \in \mathbb{C}[\mathrm{Mat}_2]_q$.

Introduce the notation $L^2(d\nu_\lambda)_q$, $L^2_a(d\nu_\lambda)_q$ for completions of vector spaces $\mathrm{Pol}(\mathrm{Mat}_2)_q$, $\mathbb{C}[\mathrm{Mat}_2]_q$ respectively, with respect to the norm $\|f\|_{q^{2\lambda}} = \left(\int\limits_{\mathbb{U}_q} f^* f d\nu_\lambda\right)^{1/2}$. These are well defined for $\lambda > 3$, and certainly $L^2_a(d\nu_\lambda)_q \subset L^2(d\nu_\lambda)_q$.

Lemma 5.3. *Given $\lambda > 3$, denote by P_λ the orthogonal projection in $L^2(d\nu_\lambda)_q$ onto $L^2_a(d\nu_\lambda)_q$. There exists a unique $K_\lambda \in \mathbb{C}[[\mathrm{Mat}_2 \times \overline{\mathrm{Mat}}_2]]_q$ such that*

$$P_\lambda f(\boldsymbol{z}) = \int\limits_{\mathbb{U}_q} K_\lambda(\boldsymbol{z},\boldsymbol{\zeta}) f(\boldsymbol{\zeta}) d\nu_\lambda(\boldsymbol{\zeta}).$$

for all $f \in \mathrm{Pol}(\mathrm{Mat}_2)_q$.

Lemma 5.4. *In* $\mathbb{C}[[\mathrm{Mat}_2 \times \overline{\mathrm{Mat}_2}]]_q$ *one has*

$$K_\lambda = \sum_{k_1 \geq k_2 \geq 0} \frac{1}{c(k_1, k_2, \lambda)} p_{k_1,k_2},$$

with $c(k_1, k_2, \lambda)$ *being given by (5.1).*

Lemmas 5.2, 5.3 can be proved in the same way as in the case $q = 1$. Turn to the proof of Lemma 5.4.

We are going to use the Schur polynomials

$$s_{k_1 k_2}(x_1, x_2) = (x_1 x_2)^{k_2} \cdot \frac{x_1^{k_1-k_2+1} - x_2^{k_1-k_2+1}}{x_1 - x_2}.$$

These are expressible in terms of elementary symmetric polynomials:

$$s_{k_1 k_2}(x_1, x_2) = u_{k_1 k_2}(x_1 + x_2, x_1 x_2).$$

(The polynomials $u_{k_1 k_2}$ are closely related to the well known Chebyshev polynomials of the second kind $U_{k_1-k_2}(x)$).

Recall the notation $[j]_q = \dfrac{q^j - q^{-j}}{q - q^{-1}}$, $(a; q^2)_\infty = \prod\limits_{j=0}^{\infty}(1 - aq^{2j})$ and consider the kernels $\chi_1 = \alpha \otimes \alpha^* + \beta \otimes \beta^* + \gamma \otimes \gamma^* + \delta \otimes \delta^*$, $\chi_2 = c \otimes c^*$ with $c = (\alpha\delta - q\beta\gamma) \in \mathrm{Pol}(\mathrm{Mat}_2)_q$.

Lemma 5.5. *i)* $p_{k_1,k_2} = q^{k_1+k_2} \cdot [k_1 - k_2 + 1]_q \cdot u_{k_1 k_2}(\chi_1, \chi_2)$,

ii) $K_\lambda = \prod\limits_{j=0}^{\infty} \left(1 - q^{2(\lambda+j)}\chi_1 + q^{4(\lambda+j)}\chi_2\right) \left(\prod\limits_{j=0}^{\infty} \left(1 - q^{2j}\chi_1 + q^{4j}\chi_2\right)\right)^{-1}.$

The first statement of Lemma 5.5 are easily deducible from the orthogonality relations for matrix elements of representations of the quantum group U_2. The second statement follows from the results of [17].

Lemma 5.4 is a consequence of Lemma 5.5 and the following well known relation in the theory of Schur polynomials [12]:

$$\frac{(q^{2\lambda}x_1; q^2)_\infty}{(x_1; q^2)_\infty} \cdot \frac{(q^{2\lambda}x_2; q^2)_\infty}{(x_2; q^2)_\infty}$$

$$= \sum_{k_1 \geq k_2 \geq 0} \frac{(q^{2\lambda}; q^2)_{k_1} (q^{2(\lambda-1)}; q^2)_{k_2}}{(q^4; q^2)_{k_1} (q^2; q^2)_{k_2}} [k_1 - k_2 + 1]_q \cdot q^{(k_1+k_2)} s_{k_1 k_2}(x_1, x_2).$$

The above proof of Proposition 5.1 is transferable quite literally onto the case of quantum $SU_{n,n}$ and a q-analogue of the unit ball in the space of $n \times n$ matrices.

6. Analytic continuation of the holomorphic discrete series: ladder representation of the quantum group $\mathbf{SU_{2,2}}$

It is explained in [3] that the results like our Proposition 5.1 allow one to solve the problems of irreducibility, unitarizability, and composition series of the representations π_{q^λ}. We restrict ourselves to some simplest corollaries from Proposition 5.1.

Proposition 6.1. *Suppose that either* $\lambda > 1$ *or* $\operatorname{Im}\lambda \in \frac{\pi}{\lg q}\mathbb{Z}$. *Then the sesquilinear form* $(f_1, f_2)_{q^{2\lambda}}$ *is positive definite, and for all* $f_1, f_2 \in \mathbb{C}[\mathrm{Mat}_2]_q$, $\xi \in U_q\mathfrak{g}$ *one has*

$$(\pi_{q^\lambda}(\xi) f_1, f_2)_{q^{2\lambda}} = (f_1, \pi_{q^\lambda}(\xi^*) f_2)_{q^{2\lambda}}. \tag{6.2}$$

P r o o f. The positivity follows from Proposition 5.1. Let $\zeta = q^\lambda$. If $\operatorname{Im}\zeta = 0$, both sides of (6.2) are rational functions of ζ. So, what remains is to use the fact that this equality is true for $0 < \zeta < q^3$. ∎

Turn to the case $\lambda = 1$. It follows from Proposition 6.1 that the kernel of the sesquilinear form $\langle f_1, f_2\rangle = \lim\limits_{\lambda\to 1+0}(1 - q^{2\lambda-2})(f_1, f_2)_{q^{2\lambda}}$ is a common invariant subspace for all the operators $\pi_q(\xi)$, $\xi \in U_q\mathfrak{g}$. Explicitly, this kernel is

$$L = \bigoplus_{k=0}^{\infty} \mathbb{C}[\mathrm{Mat}_2]_q^{(k,0)}.$$

On L one has a well defined Hermitian form $(f_1, f_2) = \lim\limits_{\lambda\to 1+0}(f_1, f_2)_{q^{2\lambda}}$, and hence the quantum $(\mathfrak{g}, \mathfrak{k})$-module associated to the restriction $\pi_q|_L$ is unitarizable. The representation $\pi_q|_L$ is a q-analogue of the well known *ladder representation.*

In the case $q = 1$ the subspace $\bigoplus_{k=0}^{\infty} \mathbb{C}[\mathrm{Mat}_2]^{(k,0)}$ coincides with the kernel of the covariant differential operator $\Box = \dfrac{\partial}{\partial\alpha}\dfrac{\partial}{\partial\delta} - \dfrac{\partial}{\partial\beta}\dfrac{\partial}{\partial\gamma}$. Our intention is to obtain a q-analogue of this result*.

We use a notion of the first order differential calculus over an algebra A and a covariant first order differential calculus as in [10].

Among well known $U_q\mathfrak{k}$-invariant first order differential calculi over $\mathbb{C}[\mathrm{Mat}_2]_q$ one has to distinguish a unique $U_q\mathfrak{g}$-invariant calculus. A general method of producing such differential calculi (with hidden symmetry) for q-Cartan domains is described in [19].

*A similar result was obtained by V. Dobrev [2] and H.P. Jakobsen [6] in a different context.

The first order differential calculus we need is determined by the following 'commutation relations between coordinates and differentials' (these are written in R-matrix form in [16]):

$$d\alpha \cdot \alpha = q^2 \alpha d\alpha, \qquad d\alpha \cdot \beta = q\beta d\alpha - (1-q^2)\alpha d\beta,$$
$$d\alpha \cdot \gamma = q\gamma d\alpha - (1-q^2)\alpha d\gamma, \qquad d\alpha \cdot \delta = \delta d\alpha - (q^{-1}-q)(\gamma d\beta + \beta d\gamma) + (q^{-1}-q)^2 \alpha d\delta,$$
$$d\beta \cdot \alpha = q\alpha \cdot d\beta, \qquad d\beta \cdot \beta = q^2 \beta d\beta,$$
$$d\beta \cdot \gamma = \gamma d\beta - (q^{-1}-q)\alpha d\delta, \qquad d\beta \cdot \delta = q\delta d\beta - (1-q^2)\beta d\delta,$$
$$d\gamma \cdot \alpha = q\alpha d\gamma, \qquad d\gamma \cdot \gamma = q^2 \gamma d\gamma,$$
$$d\gamma \cdot \beta = \beta d\gamma - (q^{-1}-q)\alpha d\delta, \qquad d\gamma \cdot \delta = q\delta d\gamma - (1-q^2)\gamma d\delta,$$
$$d\delta \cdot \alpha = \alpha d\delta, \qquad d\delta \cdot \gamma = q\gamma d\delta,$$
$$d\delta \cdot \beta = q\beta d\delta, \qquad d\delta \cdot \delta = q^2 \delta d\delta.$$

It is worthwhile to note that it admits an extension up to a $U_q\mathfrak{g}$-module first order differential calculus over $U_q\mathfrak{g}$-module algebra $\mathbb{C}[\mathrm{Pl}_{2,4}]_{q,t}$: $dt \cdot t = q^{-2} t dt$,

$$dz \cdot t = q^{-1} t dz, \qquad dt \cdot z = q^{-1} z dt + (q^{-2}-1) t dz \qquad \text{for all} \quad z \in \{\alpha, \beta, \gamma, \delta\}.$$

Turn back to $\mathbb{C}[\mathrm{Mat}_2]_q$. The operator d is given on the generators of this algebra in an obvious way and is extended onto the entire algebra via the Leibnitz rule. The operators $\frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\beta}, \frac{\partial}{\partial\gamma}, \frac{\partial}{\partial\delta}$ in $\mathbb{C}[\mathrm{Mat}_2]_q$ are imposed in a standard way:

$$df = \frac{\partial f}{\partial\alpha} d\alpha + \frac{\partial f}{\partial\beta} d\beta + \frac{\partial f}{\partial\gamma} d\gamma + \frac{\partial f}{\partial\delta} d\delta.$$

As an easy consequence of the definitions one has

Proposition 6.2. *Let* $\Box_q = \frac{\partial}{\partial\alpha}\frac{\partial}{\partial\delta} - q\frac{\partial}{\partial\beta}\frac{\partial}{\partial\gamma}$.

i) $\Box_q$ *intertwines* π_q *and* π_{q^3}*:*

$$\pi_{q^3}(\xi)\Box_q = \Box_q \pi_q(\xi), \qquad \xi \in U_q\mathfrak{g}.$$

ii) $L = \mathrm{Ker}\,\Box_q$.

iii) $(\alpha\delta - q\beta\gamma)\Box_q|_{\mathbb{C}[\mathrm{Mat}_2]_q^{(k_1,k_2)}} = q^{-2} \cdot \frac{1-q^{2k_2}}{1-q^2} \cdot \frac{1-q^{2(k_1+1)}}{1-q^2}$.

Corollary 6.3. *For all $s \in \mathbb{N}$*

$$\Box_q(\alpha\delta - q\beta\gamma)^s = b_q(s)(\alpha\delta - q\beta\gamma)^{s-1},$$

$$b_q(s) = q^{-2} \cdot \frac{1-q^{2s}}{1-q^2} \cdot \frac{1-q^{2(s+1)}}{1-q^2}.$$

$b_q(s)$ is a q-analogue of the Sato–Bernstein polynomial $b(s) = s(s+1)$ for the prehomogeneous vector space Mat_2. In a recent preprint [7] and the works cited therein, another approach to q-analogues for algebras of differential operators was used to produce q-analogues of the Bernstein polynomials.

Consider the vector space $\mathbb{C}^4$ (with its standard coordinate system t_1, t_2, t_3, t_4), together with the associated projective space $\mathbb{CP}^3$. Let $\mathbb{L} \subset \mathbb{CP}^3$ be a projectivization of the plane $t_3 = t_4 = 0$. It is well known that in the case $q = 1$ the ladder representation is isomorphic to the natural representation of $U\mathfrak{g}$ in the cohomologies $H^1(\mathbb{CP}^3 \setminus \mathbb{L}, \mathcal{O}(-2))$. A computation of these cohomologies by the Čech method leads to the Laurent polynomials:

$$H^1(\mathbb{CP}^3 \setminus \mathbb{L}, \mathcal{O}(-2)) = \left\{ \sum_{(j_1,j_2,j_3,j_4)\in J} c_{j_1,j_2,j_3,j_4} t_1^{j_1} t_2^{j_2} t_3^{j_3} t_4^{j_4} \right\},$$

with $J = \{(j_1, j_2, j_3, j_4) \in \mathbb{Z}^4 | j_1 \geq 0,\ j_2 \geq 0,\ j_3 < 0,\ j_4 < 0,\ j_1+j_2+j_3+j_4 = -2\}$.

So, one has two geometric realizations of the ladder representation of $SU_{2,2}$ (those in $H^1(\mathbb{CP}^3 \setminus \mathbb{L}, \mathcal{O}(-2))$ and in $\mathrm{Ker}\,\Box$).

The lowest weight subspace in $H^1(\mathbb{CP}^3 \setminus \mathbb{L}, \mathcal{O}(-2))$ is generated by the Laurent polynomial $t_3^{-1} t_4^{-1}$, and in the kernel of $\Box = \frac{\partial}{\partial\alpha}\frac{\partial}{\partial\delta} - \frac{\partial}{\partial\beta}\frac{\partial}{\partial\gamma}$ by the constant function 1. There exists a unique isomorphism between the two realizations of the ladder representation which takes $t_3^{-1}t_4^{-1}$ to 1. This operator is very essential in the mathematical physics and is called the Penrose transform [1]. A replacement of the commutation relation $t_i t_j = t_j t_i$ by $t_i t_j = q t_j t_i$, $i < j$, allows one to transfer easily the above observations onto the case $0 < q < 1$ (more precisely, everything but the notion of cohomologies for quasi-coherent sheaves). It is just the way of on which another realization of the ladder representation and the quantum Penrose transform appear.

7. The principal degenerate series of quantum Harish–Chandra modules

In the classical theory the principal series of Harish–Chandra modules are associated to parabolic subgroups P. Our purpose is to produce a q-analogue of

the principal series of Harish–Chandra modules associated to a stability group P for a point of the Shilov boundary $p \in S(\mathbb{U})$.
We call a $U_q\mathfrak{g}$-module V $\mathbb{Z}^3$-admissible if $V = \bigoplus\limits_{\boldsymbol{\mu}} V_{\boldsymbol{\mu}}$ with $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3) \in \mathbb{Z}^3$, $V_{\boldsymbol{\mu}} = \{v \in V \,|\, K_j^{\pm 1} v = q^{\pm \mu_j} v,\ j = 1, 2, 3\}$.

A quantum Harish–Chandra module is a finitely generated $\mathbb{Z}^3$-admissible $U_q\mathfrak{g}$-module V such that

i) V is a sum of finite dimensional simple $U_q\mathfrak{k}$-modules,

ii) each simple finite dimensional $U_q\mathfrak{k}$-module W occurs in V with finite multiplicity $(\dim \mathrm{Hom}_{U_q\mathfrak{k}}(W, V) < \infty)$.

Quantum Harish–Chandra modules are quantum $(\mathfrak{g}, \mathfrak{k})$-modules, and the notion of unitarizability is applicable here. The rest of this section is devoted to producing the principal degenerate series of the unitarizable quantum Harish–Chandra modules. Note that producing and classification of simple unitarizable quantum Harish–Chandra modules still constitute an open problem even in our special case of quantum $SU_{2,2}$.

In the case $\lambda \in -2\mathbb{Z}_+$ one has a well defined linear operator $\mathrm{Pol}(S(\mathbb{U}))_q \to \mathrm{Pol}(\widehat{S}(\mathbb{U}))_q$, $f \mapsto f \cdot (\alpha\delta - q\beta\gamma)^{-\lambda/2} t^{-\lambda}$. The same argument as that applied in Section 3 to produce π_{q^λ}, yields

Proposition 7.1. *There exists a unique one-parameter family τ_{q^λ} of representations of $U_q\mathfrak{g}$ in the space $\mathrm{Pol}(S(\mathbb{U}))_q$ of polynomials on the Shilov boundary of the quantum matrix ball such that*

i) for all $\lambda \in -2\mathbb{Z}_+$, $\xi \in U_q\mathfrak{g}$, $f \in \mathrm{Pol}(S(\mathbb{U}))_q$ one has

$$(\tau_{q^\lambda}(\xi) f)(\alpha\delta - q\beta\gamma)^{-\lambda/2} t^{-\lambda} = \xi(f(\alpha\delta - q\beta\gamma)^{-\lambda/2} t^{-\lambda});$$

ii) for all $\xi \in U_q\mathfrak{g}$, $f \in \mathrm{Pol}(S(\mathbb{U}))_q$, the vector function $\tau_{q^\lambda}(\xi) f$ is a Laurent polynomial of the indeterminate $\zeta = q^\lambda$.

Note that the multiple $(\alpha\delta - q\beta\gamma)^{-\lambda/2}$ provides the integral nature for weight of τ_{q^λ}. We are to produce a q-analogue of the principal degenerate series of Harish–Chandra modules associated to the Shilov boundary $S(\mathbb{U})$.

Turn to a construction of the corresponding principal unitary series.

Proposition 7.2. *In the case $\mathrm{Re}\,\lambda = 2$ the quantum Harish–Chandra module associated to τ_{q^λ} is unitarizable:*

$$\int\limits_{S(\mathbb{U})_q} f_2^*(\tau_{q^\lambda}(\xi) f_1) d\mu = \int\limits_{S(\mathbb{U})_q} (\tau_{q^\lambda}(\xi^*) f_2)^* f_1 d\mu \tag{7.3}$$

for all $f_1, f_2 \in \mathrm{Pol}(S(\mathbb{U}))_q$, $\xi \in U_q\mathfrak{g}$.

P r o o f. The representation τ_{q^λ} can be defined in a different way, as one can extend the $U_q\mathfrak{g}$-module algebra $\mathrm{Pol}(\widehat{S}(\mathbb{U}))_q$ via adding to the list of generators the powers t^λ, $(t^*)^\lambda$, $(\alpha\delta-q\beta\gamma)^\lambda$ for any $\lambda\in\mathbb{C}$. The relations between the generators of the extended algebra as well as the action of E_j, F_j, $K_j^{\pm 1}$, $j=1,2,3$, on them are derived from the corresponding formulae for integral powers of t, t^*, and $\alpha\delta-q\beta\gamma$ via the analytic continuation which uses Laurent polynomials of the indeterminate $\zeta=q^\lambda$. Moreover, this new algebra may be endowed with an involution as follows

$$(t^\lambda)^*=(t^*)^{\overline{\lambda}},\qquad ((\alpha\delta-q\beta\gamma)^\lambda)^*=q^{-2\overline{\lambda}}\cdot(\alpha\delta-q\beta\gamma)^{-\overline{\lambda}}$$

(where bar denotes the complex conjugation), and thus it is made a $U_q\mathfrak{g}$-module $*$-algebra.

Now the relation (7.3) follows from

Lemma 7.3. *Let* $\mathrm{Re}\,\lambda=2$. *The linear subspace*

$$((\alpha\delta-q\beta\gamma)^{-\lambda/2}\cdot t^{-\lambda})^*\cdot\mathrm{Pol}(S(\mathbb{U}))_q\cdot(\alpha\delta-q\beta\gamma)^{-\lambda/2}\cdot t^{-\lambda}$$

is a $U_q\mathfrak{g}$*-module, and the linear functional*

$$((\alpha\delta-q\beta\gamma)^{-\lambda/2}\cdot t^{-\lambda})^*\cdot f\cdot(\alpha\delta-q\beta\gamma)^{-\lambda/2}\cdot t^{-\lambda}\mapsto\int\limits_{S(\mathbb{U})_q}f\,d\mu$$

is a $U_q\mathfrak{g}$*-invariant integral.*

P r o o f o f L e m m a 7.3. Suppose that $\lambda=2+i\rho$ with $\rho\in\mathbb{R}$. Then, by definitions,

$$\begin{aligned}&((\alpha\delta-q\beta\gamma)^{-\lambda/2}\cdot t^{-\lambda})^*\cdot f\cdot(\alpha\delta-q\beta\gamma)^{-\lambda/2}\cdot t^{-\lambda}\\&=(t^*)^{-\overline{\lambda}}\cdot((\alpha\delta-q\beta\gamma)^*)^{-\overline{\lambda}/2}\cdot f\cdot(\alpha\delta-q\beta\gamma)^{-\lambda/2}\cdot t^{-\lambda}\\&=\mathrm{const}(\rho)\cdot(t^*)^{-2}\cdot(t^*)^{i\rho}\cdot(\alpha\delta-q\beta\gamma)^{-i\rho}\cdot f\cdot t^{-i\rho}\cdot t^{-2}.\end{aligned}$$

Now it suffices to apply Proposition 4.5, the equality

$$\int\limits_{S(\mathbb{U})_q}f\,d\mu=\int\limits_{S(\mathbb{U})_q}f^{(0,0)}d\mu,$$

and the observation that the element $t^*t^{-1}(\alpha\delta-q\beta\gamma)^{-1}\in\mathrm{Pol}(\widehat{S}(\mathbb{U}))_q$ commutes with any element of the subalgebra $\mathrm{Pol}(S(\mathbb{U}))_q$ and is a $U_q\mathfrak{g}$-invariant.

A construction of the second part τ'_{q^λ} of the principal degenerate series of quantum Harish–Chandra modules we are interested in is described in the following proposition. Its proof is just the same as that of Proposition 7.1.

Proposition 7.4. *There exists a unique one-parameter family τ'_{q^λ} of representations of $U_q\mathfrak{g}$ in the space $\mathrm{Pol}(S(\mathbb{U}))_q$ such that*

i) for all $\lambda \in -2\mathbb{Z}_+$, $\xi \in U_q\mathfrak{g}$, $f \in \mathrm{Pol}(S(\mathbb{U}))_q$, one has

$$(\tau'_{q^\lambda}(\xi)f)(\alpha\delta - q\beta\gamma)^{-\lambda/2}t^{-\lambda-1} = \xi(f(\alpha\delta - q\beta\gamma)^{-\lambda/2}t^{-\lambda-1});$$

ii) for all $\xi \in U_q\mathfrak{g}$, $f \in \mathrm{Pol}(S(\mathbb{U}))_q$, the vector function $\tau'_{q^\lambda}(\xi)f$ is a Laurent polynomial of the indeterminate $\zeta = q^\lambda$.

R e m a r k. Both parts τ_{q^λ}, τ'_{q^λ} of the series of quantum Harish–Chandra modules in question could be also derived via embeddings of vector spaces $\mathrm{Pol}(S(\mathbb{U}))_q \to \mathrm{Pol}(\widehat{S}(\mathbb{U}))_q$, $f \mapsto ft^{l_1}t^{*l_2}$. For that, with $l_1 - l_2 \in \mathbb{Z}$ being fixed, one should arrange 'an analytic continuation in $\zeta = q^{l_1+l_2}$'. An equivalence of the two above approaches to producing the principal degenerate series follows from properties of the element $t^*t^{-1}(\alpha\delta - q\beta\gamma)^{-1}$ (see proof of Lemma 7.3).

8. The principal non-degenerate series of unitarizable quantum Harish–Chandra modules

The finite dimensional simple admissible $U_q\mathfrak{g}$-modules allow a plausible description in terms of generators and relations when the highest weight vectors are chosen as generators. In the infinite dimensional case the capability of this approach is much lower. The well known method of inducing from a parabolic subgroup in our case is also inapplicable due to the absence of a valuable q-analogue of the Iwasawa decomposition.

Fortunately, there exists one more approach to a description of Harish–Chandra modules, that of Beilinson and Bernstein [14]. Within the framework of this approach simple Harish–Chandra modules are produced in cohomologies with supports on K-orbits in the space of full flags $X = G/B$ (in our case $G = SL_4$, $K = S(GL_2 \times GL_2)$, and B a standard Borel subgroup). The principal non-degenerate series is related to an open orbit, which is an affine algebraic variety. This fact sharply simplifies the problem of producing the principal non-degenerate series, and makes it possible to solve the problem for $0 < q < 1$.

An application of the results of Kostant [8] allows one to obtain an analogue of Proposition 4.5 for full flags and to distinguish the principal non-degenerate series of unitarizable quantum Harish–Chandra modules.

The previous section contains an exposition of the principal degenerate series of Harish–Chandra modules. Its geometric realization could be produced in the same way as that of the principal non-degenerate series in this section. For that, it suffices to replace the space of full flags G/B with $G/P \simeq \mathrm{Gr}_2(\mathbb{C}^4)$. The open K-orbit in G/P is isomorphic to $\{\boldsymbol{z} \in \mathrm{Mat}_2 |\ \det \boldsymbol{z} \neq 0\}$.

Appendix 1. A complete list of irreducible $*$-representations of $\mathrm{Pol}(\mathrm{Mat}_2)_q$

This appendix presents an outline of the results of L. Turowska [20] on classification of irreducible $*$-representations of $\mathrm{Pol}(\mathrm{Mat}_2)_q$.

To forestall the exposition, note that every irreducible representation from the list of L. Turowska possesses a distinguished vector v (determined up to a scalar multiple) and is a completion of the $\mathrm{Pol}(\mathrm{Mat}_2)_q$-module $V = \mathrm{Pol}(\mathrm{Mat}_2)_q v$ with respect to a suitable topology. Our intention is to produce the list of relations which determine the above $\mathrm{Pol}(\mathrm{Mat}_2)_q$-modules. As one can observe from the results of L. Turowska, the non-negative linear functionals

$$l_q : \mathrm{Pol}(\mathrm{Mat}_2)_q \to \mathbb{C}, \qquad l_q : f \mapsto (fv, v)$$

lead in the classical limit $q \to 1$ to non-negative linear functionals on the polynomial algebra $\mathrm{Pol}(\mathrm{Mat}_2)$. The limit functionals are just the delta-functions in some points of the closure of the unit ball $\mathbb{U}$.

We list below those points, together with the lists of determining relations for the associated $\mathrm{Pol}(\mathrm{Mat}_2)_q$-modules*.

0-dimensional leaves

$$\begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{i\varphi_2} \end{pmatrix} \qquad \begin{array}{llll} \alpha v = e^{i\varphi_1} v, & \beta v = 0, & \gamma v = 0, & \delta v = e^{i\varphi_2} v, \\ \alpha^* v = e^{-i\varphi_1} v, & \beta^* v = 0, & \gamma^* v = 0, & \delta^* v = e^{-i\varphi_2} v, \\ \varphi_1, \varphi_2 \in \mathbb{R}/2\pi\mathbb{Z}. & & & \end{array}$$

2-dimensional leaves

$$\begin{pmatrix} 0 & 0 \\ 0 & e^{i\varphi} \end{pmatrix} \qquad \begin{array}{lllll} & \beta v = 0, & \gamma v = 0, & \delta v = e^{i\varphi} v, & \\ \alpha^* v = 0, & \beta^* v = 0, & \gamma^* v = 0, & \delta^* v = e^{-i\varphi} v, & \varphi \in \mathbb{R}/2\pi\mathbb{Z}. \end{array}$$

$$\begin{pmatrix} 0 & e^{i\varphi_1} \\ e^{i\varphi_2} & 0 \end{pmatrix} \qquad \begin{array}{lll} \alpha v = 0, & \beta v = e^{i\varphi_1} v, & \gamma v = e^{i\varphi_2} v, \\ \alpha^* v = -q^{-1} e^{-i(\varphi_1 + \varphi_2)} \delta v, & \beta^* v = e^{-i\varphi_1} v, & \gamma^* v = e^{-i\varphi_2} v, \\ \delta^* v = 0, \quad \varphi_1, \varphi_2 \in \mathbb{R}/2\pi\mathbb{Z}. & & \end{array}$$

*Consider the Poisson bracket $\{f_1, f_2\} = \lim\limits_{h \to 0} \frac{f_1 f_2 - f_2 f_1}{ih}$, $h = 2\log(q^{-1})$, and associate to each of those points a bounded symplectic leaf containing this point. An important invariant of the irreducible $*$-representation is the dimension of the associated symplectic leaf.

4-dimensional leaves

$$\begin{pmatrix} 0 & 0 \\ e^{i\varphi} & 0 \end{pmatrix} \qquad \begin{matrix} & \beta v = 0, & \gamma v = e^{i\varphi} v, & & \\ \alpha^* v = 0, & \beta^* v = 0, & \gamma^* v = e^{-i\varphi} v, & \delta^* v = 0, & \varphi \in \mathbb{R}/2\pi\mathbb{Z}. \end{matrix}$$

$$\begin{pmatrix} 0 & e^{i\varphi_1} \\ 0 & 0 \end{pmatrix} \qquad \begin{matrix} & \beta v = e^{i\varphi} v, & \gamma v = 0, & & \\ \alpha^* v = 0, & \beta^* v = e^{-i\varphi} v, & \gamma^* v = 0, & \delta^* v = 0, & \varphi \in \mathbb{R}/2\pi\mathbb{Z}. \end{matrix}$$

6-dimensional leaves

$$\begin{pmatrix} e^{i\varphi} & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{matrix} \alpha v = e^{i\varphi} v, & & \\ \alpha^* v = e^{-i\varphi} v, & \beta^* v = \gamma^* v = \delta^* v = 0, & \varphi \in \mathbb{R}/2\pi\mathbb{Z}. \end{matrix}$$

8-dimensional leaf

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \alpha^* v = \beta^* v = \gamma^* v = \delta^* v = 0.$$

It follows from the results of L. Turowska that every of the above $\mathrm{Pol}(\mathrm{Mat}_2)_q$-modules can be equipped with a structure of pre-Hilbert space in such a way that the $\mathrm{Pol}(\mathrm{Mat}_2)_q$-action is extendable onto the associated Hilbert space, and this procedure provides a complete list of irreducible $*$-representations of $\mathrm{Pol}(\mathrm{Mat}_2)_q$*. Note that the $*$-representation associated to the 8-dimensional symplectic leaf is faithful; it is unique (up to a unitary equivalence) faithful irreducible $*$-representation. The uniqueness is easily deducible from the commutation relations between α, β, γ, δ, α^*, β^*, γ^*, δ^*, y (the later element is defined in Section 3).

Another two series of $*$-representations are related to the leaves that contain unitary matrices

$$\begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{i\varphi_2} \end{pmatrix}, \qquad \begin{pmatrix} 0 & e^{i\varphi_1} \\ e^{i\varphi_2} & 0 \end{pmatrix}, \qquad \varphi_1, \varphi_2 \in \mathbb{R}/2\pi\mathbb{Z}.$$

These two series are due to the $*$-homomorphism $\mathrm{Pol}(\mathrm{Mat}_2)_q \to \mathbb{C}[U_2]_q$ described in the main sections of this work. They could be obtained within the theory of $*$-representations of the algebra $\mathbb{C}[U_2]_q$ of regular functions on the quantum U_2.

*More precisely, the work by L. Turowska [20] presents explicit formulae that describe the action of the operators $\alpha, \beta, \gamma, \delta$ in the Hilbert space $l^2(\mathbb{Z}_+)^{\otimes d/2}$, with d being the dimension of the corresponding symplectic leaf.

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Геометрические реализации некоторых серий представлений квантовой группы $SU(2,2)$

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Решена задача об аналитическом продолжении голоморфной дискретной серии представлений для квантовой группы $SU(2,2)$. В частности, получена новая реализация лестничного представления этой группы. Кроме того, построены q-аналоги границы Шилова единичного шара в пространстве комплексных матриц второго порядка и отвечающих ей представлений основной вырожденной серии группы $SU(2,2)$. Обсуждается возможность обобщения на квантовый случай некоторых хорошо известных в теории представлений геометрических конструкций: преобразования Пенроуза, подхода Бейлинсона– Бернштейна к построению модулей Хариш–Чандры (для случая основной невырожденной серии).

Геометричні реалізації деяких серій представлень квантової групи $SU(2,2)$

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Вирішено задачу про аналітичне продовження голоморфної дискретної серії представлень для квантової групи $SU(2,2)$. Зокрема, отримано нову реалізацію сходового представлення цієї групи. Крім того, побудовано q-аналоги межі Шилова одиничної кулі в просторі комплексних матриць другого порядку та представлень основної виродженої серії групи $SU(2,2)$, що їй відповідають. Обговорюється можливість узагальнення на квантовий випадок деяких добре відомих у теорії представлень геометричних конструкцій: перетворення Пенроуза, підходу Бейлінсона–Бернштейна до побудови модулей Харіш–Чандри (для випадку основної невиродженої серії).