Matematicheskaya fizika, analiz, geometriya
2001, v. 8, No. 1, p. 215–227

Determining functionals for nonlinear damped wave equations

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Received January 22, 2001

We prove the existence of a wide collection of finite sets of functionals that completely determine the long-time behaviour of solutions to nonlinearly damped wave equations. This collection contains finite sets of determining modes, nodes and local volume averages.

Introduction

In a smooth bounded domain $\Omega \subset \mathbf{R}^d$ we consider the following nonlinear wave equation:

$$u_{tt} + g(u_t) - \nu\Delta u + f(u) = h(t,x),\ x \in \Omega \subset \mathbf{R}^d,\ t > 0, \tag{1}$$

$$u|_{\partial\Omega} = 0, \tag{2}$$

$$u|_{t=0} = u_0(x),\ u_t|_{t=0} = u_1(x). \tag{3}$$

Under some natural conditions on the continuous functions $g(u)$ and $f(u)$ and on the right-hand side $h(t,x)$ we prove the existence of a wide collection of finite sets of functionals that completely determine, in the sense that we explain below, the long-time behaviour of solutions to the equations (1)–(3). The approach

Mathematics Subject Classification 2000: 35B40; 35L70, 35L15.

presented relies on the concept of a completeness defect (see [3] and [4]) for a set of linear functionals and it involves some ideas from abstract approximation theory. A similar problem for the second order in time evolution equations was studied under the hypothesis that the damping term $g(u_t)$ is linear function of u_t (see [2] and also the survey [3]). However the presence of the nonlinear damping term requires completely different approach to the problem.

The question of the number of functionals that are necessary for the description of the long-time behaviour of solutions to nonlinear partial differential equations was first discussed by Foias and Prodi [6] and by Ladyzhenskaya [10] for the 2D Navier–Stokes equations. They proved that the asymptotic behaviour of the solutions is completely determined by dynamics of the first N Fourier modes, if N is sufficiently large. After [6] and [10] similar results were obtained for other functionals and other evolutionary equations and a general approach to the problem of the existence of a finite number of determining parameters was developed (see the survey [2] and the literature quoted there).

Long time behaviour of solutions of wave equations with nonlinear damping is studied in [9], [7], [8],[12],[11], [5], [13] (see also the references therein). This paper can be considered as a development of [9],where the rate of decay of the difference of two solutions to the problem (1) and (2) with $f(u) \equiv 0$ is studied.

1. Preliminaries

We assume that $g(u)$ and $f(u)$ are continuous functions with the properties:

$$g(0) = 0, \ (g(u) - g(v))(u - v) \geq a_0(u - v)^2 + a_1|u - v|^{m+2}, \quad u, v \in \mathbf{R}; \quad (1.1)$$

$$|g(u) - g(v)| \leq a_2(1 + |u|^m + |v|^m) \cdot |u - v|, \quad u, v \in \mathbf{R}; \quad (1.2)$$

$$(f(u) - f(v))(u - v) + b_0(u - v)^2 \geq b_1|u - v|^{p+2}, \quad u, v \in \mathbf{R}; \quad (1.3)$$

$$|f(u) - f(v)| \leq b_2(1 + |u|^p + |v|^p) \cdot |u - v|, \quad u, v \in \mathbf{R}; \quad (1.4)$$

$$F(u) = \int_0^u f(s)\, ds \geq -C, \quad u \in \mathbf{R}. \quad (1.5)$$

Here a_0, b_0, C are nonnegative and a_1, a_2, b_1, b_2, p are positive numbers. We also assume that if $d \geq 3$ and $p < m$ then $m \leq \frac{4}{d-2}$. In the case $a_0 > 0$ we can also admit $a_1 = 0$. As a simple example of functions $g(u)$ and $f(u)$ with these properties we can consider

$$g(u) = a_0 u + a_1|u|^m u \quad \text{and} \quad f(u) = -b_0 u + b_1|u|^p u. \quad (1.6)$$

We suppose that $h(t, x) \in L_\infty(\mathbf{R}_+; L_2(\Omega))$ and we assume that for every $u_0 \in H_0^1(\Omega) \cap L_{p+2}(\Omega)$ and $u_1 \in L_2(\Omega)$ the problem (1)–(3) has a unique global solution such that

$$u(t) \in C(0, T; H_0^1(\Omega) \cap L_{p+2}(\Omega)) \quad (1.7)$$

and

$$u_t(t) \equiv u_t \in C(0,T; L_2(\Omega)) \cap L_{m+2}((0,T) \times \Omega) \tag{1.8}$$

for each $T > 0$. Here and below $H_0^1(\Omega)$ is the first order Sobolev space with the zero boundary conditions and equipped by the norm $\|\nabla \cdot\|$, where $\|\cdot\|$ is the norm in space $L_2(\Omega)$. We also use the notation $\|\cdot\|_q$ for the norm in $L_q(\Omega)$. We note that in the example (1.6) the global existence theorem in the class of functions possessing the properties (1.7) and (1.8) is well-known (see, e.g., [11]).

As in [2, 3, 4] we involve the concept of the completeness defect for a description of sets of determining functionals. Assume that X and Y are Banach spaces and X continuously and densely embedded into Y. Let $\mathcal{L} = \{l_j : j = 1, ..., N\}$ be a finite set of linearly independent continuous functionals on X. We define the completeness defect $\epsilon_{\mathcal{L}}(X,Y) \equiv \epsilon_{\mathcal{L}}$ of the set $\mathcal{L}$ with respect to the pair of the spaces X and Y by the formula

$$\epsilon_{\mathcal{L}} = \sup\{\|w\|_Y \ : w \in X,\ l_j(w) = 0,\ l_j \in \mathcal{L},\ \|w\|_X \le 1\}. \tag{1.9}$$

The value $\epsilon_{\mathcal{L}}$ is proved to be very useful for characterisation of sets of determining functionals (see, e.g., [2, 3, 4] and the references therein). One can show that the completeness defect $\epsilon_{\mathcal{L}}(X,Y)$ is the best possible global error of approximation in Y of elements $u \in X$ by elements of the form $u_{\mathcal{L}} = \sum_{j=1}^{N} l_j(u)\phi_j$, where $\{\phi_j : j = 1, \dots, N\}$ is an arbitrary set in X. The smallness of $\epsilon_{\mathcal{L}}(X,Y)$ is the main condition (see the results presented below) that guarantee the property of a set of functionals to be asymptotically determining. The so-called modes, nodes and local volume averages (the description of these functionals can be found in [3, 4], for instance) are the main examples of sets of functionals with a small completeness defect. For further discussions and for other properties of the completeness defect we refer to [3] and [4]. Here we only point out the following estimate

$$\|u\|_Y \le C_{\mathcal{L}} \cdot \eta_{\mathcal{L}}(u) + \epsilon_{\mathcal{L}} \cdot \|u\|_X, \quad u \in X, \tag{1.10}$$

where $C_{\mathcal{L}} > 0$ is a constant depending on $\mathcal{L}$ and

$$\eta_{\mathcal{L}}(u) = \max\{|l_j(u)| \ : j = 1, \dots, N\}. \tag{1.11}$$

2. Determining functionals

Our first result is the following assertion.

Theorem 2.1. *Assume that $u(t)$ is a solution to the problem (1)–(3) with $h(t,x) \equiv h(x) \in L_2(\Omega)$ possessing the properties (1.7) and (1.8). Let $w(x) \in H_0^1(\Omega) \cap L_{p+2}(\Omega)$ be a stationary solution to the problem (1) and (2), i.e. to the problem*

$$-\nu\Delta u + f(u) = h(x),\ x \in \Omega, \quad u|_{\partial\Omega} = 0. \tag{2.1}$$

Assume that $\mathcal{L} = \{l_j : j = 1, \dots, N\}$ *is a set of functionals on* $H_0^1(\Omega)$ *and* $\epsilon_{\mathcal{L}} \equiv \epsilon_{\mathcal{L}}(H_0^1(\Omega), L_2(\Omega))$ *is the corresponding completeness defect. Then the condition*

$$\lim_{t\to+\infty} l_j(u(t)) = l_j(w) \quad \text{for all} \quad j = 1, \dots, N \tag{2.2}$$

implies that

$$\lim_{t\to+\infty} \left(\|u_t(t)\|^2 + \|\nabla(u(t) - w)\|^2 + \|u(t) - w\|_{p+2}^{p+2} \right) = 0, \tag{2.3}$$

provided $\epsilon_{\mathcal{L}} < \sqrt{\nu b_0^{-1}}$.

P r o o f. We rely here on some ideas developed in [11]. Below we assume that $a_1 > 0$ (for the case $a_1 = 0$ and $a_0 > 0$ arguments are similar). Let $v(t) = u(t) - w$. Then for $v(t)$ we have the following problem

$$v_{tt} + g(v_t) - \nu\Delta v + f(w + v(t)) - f(w) = 0, \ x \in \Omega \subset \mathbf{R}^d, \ t > 0, \tag{2.4}$$

$$u|_{\partial\Omega} = 0, \ u|_{t=0} = u_0(x), \ u_t|_{t=0} = u_1(x). \tag{2.5}$$

Multiplying the equation (2.4) in $L_2(\Omega)$ by v_t we obtain:

$$\frac{1}{2} \cdot \frac{d}{dt} \left(\|v_t(t)\|^2 + \nu\|\nabla v(t)\|^2 \right) + (f(u) - f(w), v_t) + (g(v_t), v_t) = 0. \tag{2.6}$$

It is not difficult to see that

$$(f(u) - f(w), v_t) = (f(u), u_t) - (f(w), v_t) = \frac{d}{dt}\Phi(v(t)),$$

where

$$\Phi(v) = (F(u), 1) - (F(w), 1) - (f(w)v) \equiv \int_0^1 (f(w + zv) - f(w), v) dz.$$

Consequently from (2.6) we obtain the equality:

$$\frac{d}{dt}E(t) + (g(v_t), v_t) = 0, \tag{2.8}$$

where

$$E(t) = \frac{1}{2}\|v_t(t)\|^2 + \frac{\nu}{2}\|\nabla v(t)\|^2 + \Phi(v(t)). \tag{2.9}$$

We use the condition (1.3) to get the following inequalities:

$$\Phi(v) \geq -\frac{b_0}{2}\|v\|^2 + \frac{b_1}{p+2}\|v\|_{p+2}^{p+2}, \tag{2.10}$$

$$\Phi(v) - (f(w+v) - f(w), v) = \int_0^1 (f(w+zv) - f(w+v)), v)dz$$

$$= -\int_0^1 (f(w+v) - f(w+zv)), v)dz \le \frac{b_0}{2}\|v\|^2 - \frac{b_1}{p+2}\|v\|_{p+2}^{p+2}. \qquad (2.11)$$

In what follows we will use the following inequality which follows from (1.10) with $X = H_0^1(\Omega)$ and $Y = L_2(\Omega)$:

$$\|v\|^2 \le (1+\delta)\epsilon_{\mathcal{L}}^2 \|\nabla v\|^2 + C_{\mathcal{L},\delta} \max_{j=1,\dots,N} |l_j(u)|^2, \qquad (2.12)$$

for each $\delta > 0$. Using the inequalities (2.10) and (2.12) we get the following estimate of $E(t)$ from below:

$$E(t) \ge \frac{1}{2}\|v_t\|^2 + \left(\frac{\nu}{2} - \frac{b_0}{2}\epsilon_{\mathcal{L}}^2(1+\delta)\right)\|\nabla v\|^2$$

$$+\frac{b_1}{p+2}\|v\|_{p+2}^{p+2} - C_{\mathcal{L},\delta}\max_j |l_j(v)|^2. \qquad (2.13)$$

Let us note that it follows from (2.8) and the condition (1.1) that the function $E(t)$ is monotony decreasing. The inequality (2.13) also implies that if $\epsilon_{\mathcal{L}}^2 < \nu b_0^{-1}$ and $\lim_{t\to+\infty} l_j(v(t)) = 0$ then $\lim_{t\to+\infty} E(t) \ge 0$. Thus $E(t) \ge 0$ for each $t > 0$.

Integrating the equality (2.8) with respect to t and using the condition (1.1) we also obtain:

$$E(0) - E(t) \ge a_1 \int_0^t \|v_t(s)\|_{m+2}^{m+2} ds. \qquad (2.14)$$

Multiplying the equation (2.4) by v we find

$$\frac{d}{dt}(v, v_t) = \|v_t\|^2 - \nu\|\nabla v\|^2 - (f(v+w) - f(w), v) - (g(v_t), v).$$

Since $-\frac{\nu}{2}\|\nabla v\|^2 = -E(t) + \frac{1}{2}\|v_t\|^2 + \Phi(v)$ we have:

$$\frac{d}{dt}(v, v_t) = \frac{3}{2}\|v_t\|^2 - \frac{\nu}{2}\|\nabla v\|^2$$

$$+\{\Phi(v) - (f(w+v) - f(w), v)\} - (g(v_t), v) - E(t).$$

By using the inequalities (2.11) and (2.12) we obtain from the last relation the following inequality:

$$\frac{d}{dt}(v, v_t) \le -E(t) - \frac{1}{2}\left(\nu - \epsilon_{\mathcal{L}}^2(1+\delta)b_0\right)\|\nabla v\|^2$$

$$+C_{\mathcal{L},\delta}\max_j |l_j(v)|^2 - \frac{b_1}{p+2}\|v\|_{p+2}^{p+2} + \frac{3}{2}\|v_t\|^2 + |(g(v_t), v)|.$$

Integrating the last inequality with respect to t we obtain:

$$\int_0^t E(s)ds \leq -(v(t), v_t(t)) + (v(0), v_t(0)$$
$$+\frac{3}{2}\int_0^t \|v_s(s)\|^2 ds + C_{\mathcal{L},\delta}\int_0^t \max_j |l_j(v(s))|^2 ds$$
$$+a_2\int_0^t \|v_s(s)\|\|v(s)\|ds + a_2\int_0^t\int_\Omega |v_s(s,x)|^{m+1}|v(s,x)|dxds. \tag{2.15}$$

By using (2.14) and (2.10) one can easily get:

$$-\frac{b_0}{2}\|v\|^2 + \frac{b_1}{p+2}\|v\|_{p+2}^{p+2} + \frac{1}{2}\|v_t(t)\|^2 + \frac{\nu}{2}\|\nabla v\|^2 \leq E(t) \leq E(0) \tag{2.16}$$

The following estimate is an easy consequence of the last inequality

$$|(v(0), v_t(0)) - (v(t), v_t(t))| \leq C_1. \tag{2.17}$$

By using the Hölder inequality and the inequality (2.14) we obtain:

$$\int_0^t \|v_t(s)\|^2 ds = \int_0^t\int_\Omega |v_t(s,x)|^2 ds$$
$$\leq \left(\int_0^t \|v_t(s)\|_{m+2}^{m+2} ds\right)^{\frac{2}{m+2}} \cdot (\mathrm{Vol}\,(\Omega)\cdot t)^{\frac{m}{m+2}} \leq C_2 \cdot t^{\frac{m}{m+2}}. \tag{2.18}$$

Let us estimate the last two terms in the right hand side of the inequality (2.15). By using the inequalities (2.16) and (2.18) we obtain:

$$\int_0^t \|v_t(s)\|\|v(s)\|ds \leq \left(\int_0^t \|v_t(s)\|^2 ds\right)^{\frac{1}{2}} \cdot \left(\int_0^t \|v_(s)\|^2 ds\right)^{\frac{1}{2}}$$
$$\leq C \cdot \sup_{0\leq s\leq t}\|\nabla v(s)\| \cdot \left(\int_0^t \|v_t(s)\|^2 ds\right)^{\frac{1}{2}} \cdot t^{\frac{1}{2}} \leq C_3 \cdot t^{\frac{m+1}{m+2}}. \tag{2.19}$$

Due to the Hölder inequality and the inequality (2.16) under the condition $m \leq p$ we have:

$$\int_0^t\int_\Omega |v_t(s,x)|^{m+1}|v(s,x)|dxds$$
$$\leq \left(\int_0^t\int_\Omega |v_t(s,x)|^{m+2}dxds\right)^{\frac{m+1}{m+2}} \left(\int_0^t\int_\Omega |v(s,x)|^{m+2}dxds\right)^{\frac{1}{m+2}} \leq C_4 \cdot t^{\frac{1}{m+2}}. \tag{2.20}$$

If $m > p$ and if either $d \leq 2$ or $m \leq 4(d-2)^{-1}$, then we have $H^1(\Omega) \subset L_{m+2}(\Omega)$. Therefore using (2.16) we also have (2.20) for this case.

So due to the inequalities (2.17)-(2.20) and the inequality $tE(t) \leq \int_0^t E(t)dt$ ($E(t)$ is a decreasing function) we obtain from (2.15) the following inequality:

$$tE(t) \leq C_1 + C_2 \cdot t^{\frac{m}{m+2}} + C_3 \cdot t^{\frac{m+1}{m+2}} + C_4 \cdot t^{\frac{1}{m+2}}.$$

The last inequality implies that $E(t) \leq C_5 \cdot t^{-1/(m+2)}$. Therefore from (2.13) under the condition $\epsilon_{\mathcal{L}} < \sqrt{\nu b_0^{-1}}$ we have

$$\|v_t\|^2 + \|\nabla v\|^2 + \|v\|_{p+2}^{p+2} \leq C_1(\mathcal{L}) \cdot t^{-1/(m+2)} + C_2(\mathcal{L}) \cdot \max_j |l_j(v(t))|^2. \quad (2.21)$$

This implies the assertion of Theorem 2.1.

R e m a r k 2.1. Under the conditions of Theorem 2.1 the relation (2.21) provides us also an estimate of the decay rate of the value $\|v_t\|^2 + \|\nabla v\|^2 + \|v\|_{p+2}^{p+2}$ when $t \to \infty$.

R e m a r k 2.2. Theorem 2.1 remains true, if instead of stationarity of the right hand side $h(t,x)$ in the equation (1) we assume that $h(t,x) \to h(x)$ in the sense that

$$\int_0^\infty \tau \|h(\tau) - h\|^2 \, d\tau < \infty.$$

The proof follows step by step to the argument given above with minor modifications in the relations (2.8), (2.14) and (2.15). Instead of monotonicity of $E(t)$ we make use of monotonicity of the function $E(t) - \int_0^t (h(\tau) - h, v_t(\tau)\, d\tau$.

Theorem 2.1 implies immediately the following assertion.

Corollary 2.1. *Let $w_1(x), w_2(x) \in H_0^1(\Omega) \cap L_{p+2}(\Omega)$ be two stationary solutions to the (2.1) with $h(x) \in L_2(\Omega)$. Let $\mathcal{L} = \{l_j : j = 1, \ldots, N\}$ be a set of functionals on $H_0^1(\Omega)$ and $\epsilon_{\mathcal{L}}(H_0^1(\Omega), L_2(\Omega)) < \sqrt{\nu b_0^{-1}}$. Then the condition*

$$l_j(w_1) = l_j(w_2) \quad \text{for all} \quad j = 1, \ldots, N$$

implies that $w_1(x) \equiv w_2(x)$.

Two following corollaries deal with the case when the problem possesses precompact trajectories. For conditions that guarantee the precompactness of trajectories see [7] and [5], for example.

Let $h(t,x) \equiv h(x) \in L_2(\Omega)$ and let $u(t)$ be a solution to the problem (1)–(3). We recall that the set

$$\gamma_+(u_0, u_1) = \cup \{(u(t); u_t(t)) \; : \; t \geq 0\}$$

in the space $\mathcal{E} = H_0^1(\Omega) \cap L_{p+2}(\Omega) \times L_2(\Omega)$ is said to be the *semi-trajectory* emanating from $(u_0; u_1)$ of the dynamical system generated by (1) and (2) in $\mathcal{E}$. We also define the ω-limit set of the semi-trajectory $\gamma_+(u_0, u_1)$ by the formula

$$\omega(\gamma_+) \equiv \omega(u_0, u_1) = \cap_{\tau>0} \left[\cup \{(u(t); u_t(t)) \ : \ t \geq \tau\}\right]_{\mathcal{E}} \tag{2.22}$$

where $[A]_{\mathcal{E}}$ is the closure of the set A in $\mathcal{E}$.

Corollary 2.2. *Let $h(t,x) \equiv h(x) \in L_2(\Omega)$ and $\mathcal{L} = \{l_j : j = 1, \dots, N\}$ be a set of functionals on $H_0^1(\Omega))$ such that $\epsilon_{\mathcal{L}} \equiv \epsilon_{\mathcal{L}}(H_0^1(\Omega), L_2(\Omega)) < \sqrt{\nu b_0^{-1}}$. Assume that $u(t)$ is a solution to the problem (1)–(3) with precompact semi-trajectory $\gamma_+ = \gamma_+(u_0, u_1)$ and there exists the finite limits*

$$\lim_{t \to +\infty} l_j(u(t)) \equiv l_j \quad j = 1, \dots, N. \tag{2.23}$$

Then there exists a stationary solution $w(x) \in H_0^1(\Omega) \cap L_{p+2}(\Omega)$ such that (2.3) holds.

P r o o f. Precompactness of γ_+ implies that ω-limit set $\omega(\gamma_+)$ is non-empty compact set in $\mathcal{E}$. As in the proof of Theorem 2.1 it is easy to see that the functional

$$E_0(u(t), u_t(t)) = \frac{1}{2}\|u_t(t)\|^2 + \frac{\nu}{2}\|\nabla u(t)\|^2 + (F(u(t)), 1) \tag{2.24}$$

possesses the property

$$\frac{d}{dt} E_0(u(t), u_t(t)) + (g(u_t(t)), u_t(t)) = 0,$$

This implies that $E_0(u, u_t)$ is the Lyapunov function (see the books [1] and [7]) and therefore the ω-limit set $\omega(\gamma_+)$ lies in the set $\mathcal{N}$ of equilibrium points to the problem (1) and (2), i.e.

$$\omega(\gamma_+) \subset \mathcal{N} \equiv \{(w; 0) \in \mathcal{E} \ : w \ \text{ is a solution to (2.1)} \}.$$

From (2.23) we have that $l_j(w) = l_j$ for all $(w; 0) \in \omega(\gamma_+) \subset \mathcal{N}$. Consequently Corollary 2.1 implies that $\omega(\gamma_+)$ consists of a single point $(w; 0)$ and therefore (2.3) holds.

Corollary 2.3. *Let the assumptions of Corollary 2.2 concerning the function $h(t,x)$ and the set $\mathcal{L}$ of functionals on $H_0^1(\Omega))$ be valid. Assume that $u^{(1)}(t)$ and*

$u^{(2)}(t)$ are solution to the problem (1) and (2) with precompact semi-trajectories $\gamma_+^{(1)}$ and $\gamma_+^{(2)}$ and

$$\lim_{t\to+\infty}\Big(l_j(u^{(1)}(t))-l_j(u^{(2)}(t))\Big)=0 \quad j=1,\dots,N. \tag{2.25}$$

Then we have $\omega(\gamma_+^{(1)})\equiv\omega(\gamma_+^{(2)})$. If the set $\mathcal{N}$ of equilibrium points is finite, then there exists a stationary solution $w(x)\in H_0^1(\Omega)\cap L_{p+2}(\Omega)$ such that

$$\lim_{t\to+\infty}\Big(\|u_t^{(i)}(t)\|^2+\|\nabla(u^{(i)}(t)-w)\|^2+\|u^{(i)}(t)-w\|_{p+2}^{p+2}\Big)=0,\quad i=1,2.$$

P r o o f. Let $(z_1,0)\in\omega(\gamma_+^{(1)})\subset\mathcal{N}$. Then there exists a sequence $\{t_m\}$ such that $t_m\to+\infty$ and $u^{(1)}(t_m)\to z_1$ in the space $H_0^1(\Omega)\cap L_{p+2}(\Omega)$ when $m\to\infty$. Since $\gamma_+^{(2)}$ is precompact set, we can choose a subsequence $\{t_{m_k}\}\subset\{t_m\}$ such that $u^{(2)}(t_{m_k})\to z_2$, where $(z_2,0)\in\omega(\gamma_+^{(2)})\subset\mathcal{N}$. The property (2.25) gives that $l_j(z_1)=l_j(z_2)$ for all $j=1,2,\dots,N$. Consequently Corollary 2.1 implies that $z_1=z_2$ and therefore we have $(z_1,0)\in\omega(\gamma_+^{(2)})$. This implies that $\omega(\gamma_+^{(1)})=\omega(\gamma_+^{(2)})$. If $\mathcal{N}$ is finite, then it is easy to see that $\omega(\gamma_+^{(1)})=\omega(\gamma_+^{(2)})$ consists of a single equilibrium point. This implies the assertion of the corollary.

The following assertion gives some generalisation of Corollary 2.1 allowing one of the solutions to be non-stationary.

Theorem 2.2. *Assume that $u(t)$ is a solution to the problem (1) and (2) with $h(t,x)\equiv h(x)\in L_2(\Omega)$ defined on $\mathbf{R}$ and possessing the properties like (1.7) and (1.8) for every interval of $\mathbf{R}$. Assume also that*

$$\limsup_{t\to-\infty}\Big(\|u_t(t)\|^2+\|\nabla u(t)\|^2+\|u(t)\|_{p+2}^{p+2}\Big)<\infty, \tag{2.26}$$

Let $\mathcal{L}=\{l_j : j=1,\dots,N\}$ be a set of functionals on $H_0^1(\Omega))$ and $\epsilon_{\mathcal{L}}\equiv\epsilon_{\mathcal{L}}(H_0^1(\Omega),L_2(\Omega))<\sqrt{\nu b_0^{-1}}$. If for some solution $w(x)\in H_0^1(\Omega)\cap L_{p+2}(\Omega)$ to the problem (2.1) we have

$$l_j(u(t))=l_j(w) \quad \textit{for all} \quad t\in\mathbf{R} \textit{ and } \quad j=1,\dots,N, \tag{2.27}$$

then $u(t)\equiv w$.

P r o o f. We apply the same idea as in the proof of Theorem 2.1 but with reversed time. Let us consider the function $v(t)=u(-t)-w$ as a solution to the problem

$$v_{tt}-\nu\Delta v+f(w+v(t))-f(w)=-g(-v_t),\ x\in\Omega\subset\mathbf{R}^d,\ t>0, \tag{2.28}$$

$$u|_{\partial\Omega} = 0, \; v|_{t=0} = u_0(x) - w(x), \; v_t|_{t=0} = -u_1(x).$$

As in the proof of Theorem 2.1 it follows from (2.28) that the function $E(t)$ defined by (2.9) satisfies:

$$\frac{d}{dt}E(t) = -(g(-v_t), v_t) \geq a_1 \|v_t(t)\|_{m+2}^{m+2} \tag{2.29}$$

Integration of (2.29) over $(0, t)$ gives

$$E(t) \geq E(0) + a_1 \int_0^t \|v_t(s)\|_{m+2}^{m+2} ds, \quad t \geq 0. \tag{2.30}$$

Due to (2.27) instead of (2.12) we have

$$\|v\|^2 \leq \epsilon_{\mathcal{L}}^2 \|\nabla v\|^2. \tag{2.31}$$

Thanks to the the last inequality and also to the conditions (1.4) and (2.26) we have

$$\frac{1}{2}\|v_t\|^2 + \frac{\nu_0}{2}\|\nabla v\|^2 + \frac{b_1}{p+2}\|v\|_{p+2}^{p+2} \leq E(t) \leq C \tag{2.32}$$

for all $t \geq 0$, where $\nu_0 = \nu - b_0\epsilon_{\mathcal{L}}^2 > 0$ and C is a constant depending on the solutions $u(t)$ and w.

By using (2.27) and (2.11) as in the proof of Theorem 2.1 we obtain from the relation

$$\frac{d}{dt}(v, v_t) = \|v_t\|^2 - \nu\|\nabla v\|^2 - (f(v+w) - f(w), v) - (g(-v_t), v)$$

the following inequality

$$\int_0^t E(s)ds \leq -(v(t), v_t(t)) + (v(0), v_t(0) + \frac{3}{2}\int_0^t \|v_s(s)\|^2 ds$$

$$+a_2 \int_0^t \|v_s(s)\|\|v(s)\|ds + a_2 \int_0^t \int_\Omega |v_s(s,x)|^{m+1}|v(s,x)|dxds. \tag{2.33}$$

The condition (2.26) implies that

$$|(v(0), v_t(0)) - (v(t), v_t(t))| \leq D_1. \tag{2.34}$$

It follows from the Hölder inequality, (2.30) and (2.32) that

$$\int_0^t \|v_t(s)\|^2 \leq \left(\int_0^t \|v_t(s)\|_{m+2}^{m+2} ds\right)^{\frac{2}{m+2}} (\text{Vol }(\Omega) \cdot t)^{\frac{m}{m+2}}$$

$$\leq D_2 E(t)^{\frac{2}{m+2}} t^{\frac{m}{m+2}} \leq D_3 t^{\frac{m}{m+2}}. \tag{2.35}$$

By using the first two inequalities in (2.19) and also (2.35) and (2.32) we obtain

$$\int_0^t \|v_t(s)\| \|v(s)\| ds \le D_4 t^{\frac{m+1}{m+2}}. \tag{2.36}$$

The first inequality in (2.20) and also (2.30) and (2.32) imply the following inequality

$$\int_0^t \int_\Omega |v_s(s,x)|^{m+1} |v(s,x)| dx ds \le D_5 t^{\frac{1}{m+2}}. \tag{2.37}$$

Since $E(t)$ is nondecreasing function (see (2.25)) by using (2.34)–(2.37) we obtain from (2.33) the inequality

$$E(0)t \le \int_0^t E(s)ds \le D\left[1 + t^{\frac{m}{m+2}} + t^{\frac{m+1}{m+2}} + t^{\frac{1}{m+2}}\right].$$

The last inequality implies that $E(0)$ must be equal to 0. Thus we get $u_0 = w$ and $u_1 = 0$. Therefore $u(t) \equiv w$ because of the uniqueness of the solution.

Corollary 2.4. *Let the assumptions of Theorem 2.2 concerning the function $h(t,x)$ and the set $\mathcal{L}$ of functionals on $H_0^1(\Omega))$ be valid. Assume that $u^{(1)}(t)$ and $u^{(2)}(t)$ are solution to the problem (1) and (2) defined on $\mathbf{R}$ and possessing the properties like (1.7) and (1.8) for every interval of $\mathbf{R}$. Assume also that these solutions have precompact trajectories $\gamma^{(i)} \equiv \cup\{(u(t); u_t(t)) \ : \ t \in \mathbf{R}\}$, $i = 1, 2$, and*

$$l_j(u^{(1)}(t)) = l_j((u^{(2)}(t)) \quad \textit{for all} \quad t \in \mathbf{R} \textit{ and } \quad j = 1, \dots, N. \tag{2.38}$$

Then there exists a stationary solution $w(x) \in H_0^1(\Omega) \cap L_{p+2}(\Omega)$ such that $u^{(1)}(t) \equiv u^{(2)}(t) \equiv w$.

P r o o f. As in the proof of Corollary 2.3 it is not difficult to find that

$$\omega(\gamma^{(1)}) = \omega(\gamma^{(2)}) = \alpha(\gamma^{(1)}) = \alpha(\gamma^{(2)}), \tag{2.39}$$

where $\omega(\gamma^{(i)})$ is ω-limit set of the trajectory $\gamma^{(i)}$ defined by (2.22) and $\alpha(\gamma^{(i)})$ is α-limit set of the trajectory $\gamma^{(i)}$ defined by the equality

$$\alpha(\gamma^{(i)}) = \cap_{\tau<0}\left[\cup\left\{(u^{(i)}(t); u_t^{(i)}(t)) \ : \ t \le \tau\right\}\right]_{\mathcal{E}}, \quad i = 1, 2.$$

The property (2.39) implies that the functional $E_0(u, u_t)$ defined by (2.24) is constant on the both solutions $u^{(1)}(t)$ and $u^{(2)}(t)$. Therefore $u^{(1)}(t)$ and $u^{(2)}(t)$ are stationary solutions. Consequently (2.38) and Theorem 2.2 give that $u^{(1)}(t) \equiv u^{(2)}(t) \equiv w$, where w is a solution to the problem (2.1).

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Определяющие функционалы для волновых уравнений с нелинейным демпфированием

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Доказано существование широкого семейства конечных множеств функционалов, которые полностью определяют асимптотическое поведение решений волновых уравнений с нелинейным демпфированием. Это семейство содержит конечные множества определяющих мод, узлов и локальных объемных средних.

Визначаючі функціонали для хвильових рівнянь з нелінійним демпфуванням

І.Д. Чуєшов, В.К. Калантаров

Доведено існування широкого класу скінченних множин функціоналів, які повністю визначають асимптотичну поведінку розв'язків хвильових рівнянь з нелінійним демпфуванням. Цей клас містить скінченні множини визначаючих мод, вузлів та локальних об'ємних усереднень.