# A question by Alexei Aleksandrov and logarithmic determinants

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We construct an analytic function $f$ of Smirnov's class in the unit disk such that $\mathrm{Re}\,f$ vanishes almost everywhere on the unit circle and

$$\liminf_{t\to\infty} t\,\mathrm{meas}\{\zeta : |\zeta| = 1, |f(\zeta)| \geq t\} = 0\,.$$

This answers negatively to the question posed by A. Aleksandrov. We also find new sufficient conditions for representations of functions of Smirnov's class by the Schwarz and Cauchy integrals. These conditions extend previous results by Aleksandrov. ·

## § 1.  Analytic functions represented by Schwarz' integrals

This note is motivated by the following question: *when is an analytic function $f$ in the unit disc $\mathbb{D}$ represented by the Schwarz integral*

$$f(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta) \tag{1.1}$$

*of a real measure $\mu$ on the unit circle $\mathbb{T}$?* Two classical necessary conditions are:
**(i)** (Smirnov). $f \in \mathcal{N}_+$, that is,

$$\log|f(z)| \leq \int_{\mathbb{T}} \log|f(\zeta)|\,\mathrm{Re}\left(\frac{\zeta + z}{\zeta - z}\right)\,dm(\zeta)\,, \qquad z \in \mathbb{D}\,, \tag{1.2}$$

where $m$ is the normalized Lebesgue measure on $\mathbb{T}$, $m(\mathbb{T}) = 1$. (We use the same notation $f$ for an analytic function in the unit disc and for its non-tangential boundary values on the unit circle.)

In fact, Smirnov proved a stronger result: $f \in \bigcap_{p<1} H^p$.

**(ii)** (Kolmogorov). $f \in L^{1,\infty}(\mathbb{T})$, that is, $m_f(t) = O(t^{-1})$, $t \to \infty$, where $m_f(t) = m\{\zeta \in \mathbb{T} : |f(\zeta)| \geq t\}$. More precisely, if $f$ is represented in the form (1.1), then

$$m_f(t) \leq \frac{C\|\mu\|}{t}, \qquad 0 < t < \infty,$$

where $C$ is a numerical constant.

Another necessary condition is

**(iii)** (Hruschev and Vinogradov [12]) if $f$ is represented in the form (1.1), then there exists a limit

$$\lim_{t \to \infty} t m_f(t) = \frac{\|\mu_{\text{sing}}\|}{\pi}.$$

A weaker result that $\lim_{t \to \infty} t m_f(t) = 0$ if the measure $\mu$ in the representation (1.1) is absolutely continuous, is due to Titchmarsh [11].

Besides the above conditions, there are the trivial restrictions:

**(iv)**

$$\text{Re} f \left( = \frac{d\mu_{\text{a.c}}}{dm} \right) \in L^1(\mathbb{T}), \qquad \text{and} \qquad f(0) \left( = \int_{\mathbb{T}} d\mu \right) \in \mathbb{R}.$$

In [1] and [2] Aleksandrov proved

**Theorem A1.** *The set of conditions (i), (ii), and (iv) is sufficient for representation (1.1), with*

$$\|\mu\| \leq \|\text{Re} f\|_1 + C\|f\|_{1,\infty},$$

*where $\|.\|_1 = \|.\|_{L^1(\mathbb{T})}$, $\|f\|_{1,\infty} = \sup_{t>0} t m_f(t)$, and $C$ is a positive numerical constant.*

That is, an analytic function $f$ is represented by the Schwarz integral (1.1) iff conditions (i), (ii) and (iv) hold. Furthermore, Aleksandrov suggested that in the sufficiency part condition (ii) can be weakened and asked *whether conditions (i),*

$$\liminf_{t \to \infty} t m_f(t) < \infty,$$

*and (iv) already guarantee that $f$ is represented by the Schwarz integral (1.1).* Another form of this question is (cf. [3]): *whether there exists a non-constant analytic function $f$ of Smirnov's class $\mathcal{N}_+$ such that $\text{Re} f = 0$ a.e. on $\mathbb{T}$ and*

$$\liminf_{t \to \infty} t m_f(t) = 0. \qquad (1.4)$$

If such a function exists, then in view of the necessary conditions cited above, it cannot be represented by the Schwarz integral (1.1). Here, we answer this question.

**Theorem 1.** *There exists a non-constant analytic function $f \in \mathcal{N}_+$ satisfying (1.4) and such that $\operatorname{Re} f = 0$ a.e. on $\mathbb{T}$.*

On the other hand, condition (ii) can be really weakened:

**Theorem 2.** *Let $f$ be an analytic function in $\mathbb{D}$ satisfying conditions (i) and (iv), and let*

$$\liminf_{R \to \infty} R \int_R^\infty \frac{m_f(t)}{t}\, dt < \infty\,. \tag{1.5}$$

*Then $f$ is represented by the Schwarz integral (1.1) of a real measure $\mu$, and*

$$\|\mu\| \le \|\operatorname{Re} f\|_1 + C \liminf_{R \to \infty} R \int_R^\infty \frac{m_f(t)}{t}\, dt\,,$$

*where $C$ is a positive numerical constant.*

It is worth to note that after integration by parts condition (1.5) can be written as

$$\liminf_{R \to \infty} R \mathcal{N}\left(\frac{f}{R}\right) < +\infty\,,$$

where

$$\mathcal{N}(f) = \int_{\mathbb{T}} \log^+ |f|\, dm$$

is a "norm" in the Smirnov class $\mathcal{N}_+$.

## § 2.    Proof of Theorem 1

We construct $f$ as a universal covering of $\mathbb{C} \setminus E$, where $E$ is a closed subset of $i\mathbb{R}_+$, by the unit disc $\mathbb{D}$. We shall use some classical facts about universal coverings, harmonic measures and Green functions [9].

Let $E = \bigcup_{n \ge 1}[ir_n, 2ir_n]$, where $r_1 = 1$, $r_{n+1}/r_n \to \infty$, and let

$$\omega(t) = \omega(0, E \cap \{|z| \ge t\}, \mathbb{C} \setminus E)$$

be the harmonic measure of $E \cap \{z : |z| \ge t\}$ with respect to $\mathbb{C} \setminus E$ evaluated at the origin. We can choose the sequence $\{r_n\}$ increasing so fast that

$$\liminf_{t \to \infty} t\omega(t) = 0\,. \tag{2.1}$$

Indeed, let $h_n(z)$ be a bounded harmonic function in $\mathbb{C} \setminus ([i, 2i] \bigcup [ir_n, i\infty))$, vanishing on $[i, 2i]$, and having its boundary values equal identically to 1 on $[ir_n, i\infty)$. Evidently, $\lim_{n \to \infty} h_n(0) = 0$. Hence, on the $n + 1$-st step we can choose a value $r_{n+1}$ sufficiently large that $h_{n+1}(0) \leq r_n^{-2}$. Then by the maximum principle

$$
\begin{aligned}
\liminf_{t \to \infty} t\omega(t) & \leq \liminf_{n \to \infty} 2r_n \omega(2r_n) \\
& = 2 \liminf_{n \to \infty} r_n \omega(r_{n+1}) \leq 2 \liminf_{n \to \infty} r_n h_{n+1}(0) = 0 \,,
\end{aligned}
$$

proving (2.1).

Now, let $f : \mathbb{D} \to \mathbb{C} \setminus E$ be the universal covering map normalized by $f(0) = 0$. Since $E$ has a positive capacity, $f$ is of bounded type in $\mathbb{D}$, and therefore, has non-tangential boundary values a.e. on $\mathbb{T}$. By the invariance of the harmonic measure,

$$
\int_{\mathbb{T}} (\varphi \circ f)(\zeta) \operatorname{Re}\left(\frac{\zeta + z}{\zeta - z}\right) dm(\zeta) = \int_E \varphi(\eta)\, \omega(f(z), d\eta, \mathbb{C} \setminus E)\,, \qquad (2.2)
$$

where $\varphi$ is an arbitrary continuous function on $E$ (as the boundary of $\mathbb{C} \setminus E$). In particular,

$$
\int_T (\varphi \circ f)\, dm = \int_E \varphi(\eta)\, \omega(0, d\eta, \mathbb{C} \setminus E)\,. \qquad (2.2a)
$$

After a monotonic limit transitions, relations (2.2) and (2.2a) also hold for semi-continuous functions, and therefore

$$
m_f(t) = \omega(t)\,, \qquad 0 < t < \infty\,. \qquad (2.3)
$$

Now, the function $f$ has pure imaginary boundary values a.e. on $\mathbb{T}$, and

$$
\liminf_{t \to \infty} tm_f(t) \overset{(2.3)}{=} \liminf_{t \to \infty} t\omega(t) \overset{(2.1)}{=} 0\,.
$$

It remains to observe that $f$ is of Smirnov's class $\mathcal{N}_+$. Indeed, representing a subharmonic function $\log|w|$ in $\mathbb{C} \setminus E$ as a sum of the Green function and the Poisson integral, we have

$$
\log|w| = G_{\mathbb{C} \setminus E}(w, 0) + \int_E \log|\eta|\, \omega(w, d\eta, \mathbb{C} \setminus E) + \alpha K_E(w)\,, \qquad (2.4)
$$

where $G_{\mathbb{C} \setminus E}(w, 0)$ is Green's function for $\mathbb{C} \setminus E$ with pole at $w = 0$, $K_E(w)$ is a positive harmonic function in $\mathbb{C} \setminus E$ vanishing on $E$ (so-called Martin function), and $\alpha \in \mathbb{R}$.

First, we shall show that $\alpha \leq 0$ (in fact, a minor modification of the following argument shows that $\alpha = 0$, though it is not needed for our purposes). Indeed, if $\alpha > 0$, then as follows from (2.4), the function $K_E$ has a logarithmic growth at

infinity. Setting $K_E(z) = 0$ for $z \in E$ we obtain a subharmonic function in $\mathbb{C}$ of logarithmic growth, so that the set $E$ must be *thin* at infinity [4] and according to the Wiener criterion

$$\sum_{n=1}^{\infty} \frac{n}{\log \frac{2}{c(E_n)}} < \infty \,,$$

where $E_n = E \cap \{z : 2^n \le |z| < 2^{n+1}\}$ and $c(\,.\,)$ is the logarithmic capacity (cf. [5, Chapter 7] or [10, Chapter 5]). Since $E$ contains intervals $[ir_n, 2ir_n]$ and since the logarithmic capacity of the interval equals one quarter of its length, we see that the series must diverge and therefore $\alpha \le 0$.

Omitting the non-positive terms on the RHS of (2.4), we get

$$\log |w| \le \int_E \log |\eta| \, \omega(w, d\eta, \mathbb{C} \setminus E) \,.$$

Setting here $w = f(z)$, $\eta = f(\zeta)$, and making use of (2.2), we obtain

$$\log |f(z)| \le \int_{\mathbb{T}} \log |f(\zeta)| \operatorname{Re} \left( \frac{\zeta + z}{\zeta - z} \right) dm(\zeta) \,, \qquad z \in \mathbb{D},$$

which completes the proof. ∎

## §3. Proof of Theorem 2

We may assume that $\operatorname{Re} f = 0$ a.e. on $\mathbb{T}$. Otherwise, we decompose

$$f(z) = f_1(z) + \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \left( \operatorname{Re} f \right)(\zeta) \, dm(\zeta) = f_1 + f_2 \,,$$

where $\operatorname{Re} f_1 = 0$ a.e. on $\mathbb{T}$, and $\lim_{t \to \infty} t m_{f_2}(t) = 0$.

Following [6, 7, 8] we introduce a "logarithmic determinant"

$$u_f(w) := \int_{\mathbb{T}} \log |1 - w f(\zeta)| \, dm(\zeta) \,. \tag{3.1}$$

The function $u_f$ is subharmonic in $\mathbb{C}$, harmonic in the right and left half-planes $\Pi_{\pm}$, and $u(0) = 0$. Furthermore,

$$u_f(w) \le O(\log |w|) \,, \qquad w \to \infty \,,$$

so that, $u_f$ is represented by the Poisson integrals in $\Pi_{\pm}$. The function $u_f$ has a lower bound

$$u_f(it) \ge \log |1 - it f(0)| \ge 0 \,, \qquad t \in \mathbb{R}.$$

By the maximum principle applied to $u_f$ in $\Pi_{\pm}$, the function $u_f$ is non-negative everywhere in $\mathbb{C}$.

Our goal is to estimate from above the integral

$$\mathcal{I}_f := \frac{1}{\pi} \int_{\mathbb{R}} \frac{u_f(it)}{t^2} \, dt \, .$$

But first, we shall show that this integral controls the norm $||f||_{1,\infty}$ (cf. the proof of Theorem 2 in [7]). Indeed, estimating the Poisson integrals in $\Pi_{\pm}$ we have

$$u(re^{i\theta}) = \frac{r|\cos\theta|}{\pi} \int_{\mathbb{R}} \frac{u_f(it)}{t^2} \frac{t^2}{|it - re^{i\theta}|^2} \, dt \le \frac{r}{|\cos\theta|} \mathcal{I}_f \, , \qquad 0 < r < \infty \, .$$

In particular, $u_f(re^{i\theta}) \le r\sqrt{2}\,\mathcal{I}_f$ within the angles $\{|\theta| \le \pi/4\}$ and $\{|\theta| \ge 3\pi/4\}$. Applying the maximum principle to the harmonic function $u_f(z) - 2\mathcal{I}_f|\mathrm{Im}z|$ within the complementary angles $\{\pi/4 \le |\theta| \le 3\pi/4\}$, we get

$$M(r, u_f) := \max_{\theta \in [-\pi,\pi]} u_f(re^{i\theta}) \le 2\mathcal{I}_f r \, , \qquad 0 < r < \infty \, .$$

Let $\mu_f$ be the Riesz measure of the function $u_f$, and let $\mu_f(r) = \mu_f\{|w| \le r\}$ be its counting function. Then by the Jensen formula

$$\mu_f(r) \le M(er, u_f) \le 2e\mathcal{I}_f r \, , \qquad 0 < r < \infty \, .$$

It remains to observe that

$$m_f(\tau) \overset{(3.1)}{=} \mu_f(1/\tau) \, , \tag{3.2}$$

so that

$$||f||_{1,\infty} \le 2e\mathcal{I}_f \, . \tag{3.3}$$

Now, we estimate the integral $\mathcal{I}_f$ using an integral formula which has been used previously in a similar situation (cf. the proof of Theorem 3 in [6]). For $|\theta| < \pi/2$ we have

$$u_f(re^{i\theta}) = \frac{r\cos\theta}{\pi} \int_{\mathbb{R}} \frac{u_f(it)dt}{|re^{i\theta} - it|^2} \, .$$

Integrating this against $\cos\theta$, we get

$$\int_{-\pi/2}^{\pi/2} u_f(re^{i\theta})\cos\theta \, d\theta = \frac{r}{\pi} \int_{\mathbb{R}} u_f(it) \, dt \int_{-\pi/2}^{\pi/2} \frac{\cos^2\theta d\theta}{|re^{i\theta} - it|^2}$$

$$= \frac{r}{2} \int_{\mathbb{R}} u_f(it) \min\left(\frac{1}{t^2}, \frac{1}{r^2}\right) dt \, ,$$

since

$$\int_{-\pi/2}^{\pi/2} \frac{\cos^2\theta d\theta}{|re^{i\theta} - it|^2} = \frac{\pi}{2} \min\left(\frac{1}{t^2}, \frac{1}{r^2}\right) \, .$$

Using a similar relation in the left half-plane, we have

$$
\begin{aligned}
\int_{\mathbb{R}} u_f(it) \min\left(\frac{1}{t^2}, \frac{1}{r^2}\right) dt &= \frac{1}{r} \int_{-\pi}^{\pi} u_f(re^{i\theta})|\cos\theta|\, d\theta \\
&< \frac{1}{r} \int_{-\pi}^{\pi} u_f(re^{i\theta})\, d\theta \\
&= \frac{2\pi}{r} \int_0^r \frac{\mu_f(s)}{s}\, ds\,.
\end{aligned}
$$

At last, making the monotonic limit transition as $r \to 0$, and setting $R = 1/r$, we get

$$
\mathcal{I}_f \le 4\pi \liminf_{r\to 0} \frac{1}{r} \int_0^r \frac{\mu_f(s)}{s}\, ds \overset{(3.2)}{\le} 4\pi \liminf_{R\to\infty} R \int_R^\infty \frac{m_f(\tau)}{\tau}\, d\tau\,. \qquad (3.4)
$$

Combining inequalities (3.3) and (3.4) with Theorem A1, we complete the proof.

∎

## §4.  Concluding remarks

**4.1.** The technique of logarithmic determinants used in the proof of Theorem 2 also allows us to prove another theorem of Aleksandrov [1, 2]:

**Theorem A2.** *Let $f$ be an analytic function in the unit disc satisfying condition (iv), let $f \in \bigcap_{p<1} H^p$, and let*

$$
\liminf_{p\uparrow 1}(1-p)\|f\|_{H^p} < \infty\,.
$$

*Then $f$ is represented by the Schwarz integral (1.1) of a real measure $\mu$, and*

$$
\|\mu\| \le \|\mathrm{Re}f\|_1 + C \liminf_{p\uparrow 1}(1-p)\|f\|_{H^p}\,. \qquad (4.1)
$$

(In fact, Aleksandrov proved this estimate with $C = \frac{\pi}{2}$ on the RHS.)

Our proof follows the same lines as that of Theorem 2, only in the last step we use other integral formulas to estimate the integral $\mathcal{I}_f$:

$$
\int_{\mathbb{R}} \log|1 - it\lambda|\, \frac{dt}{|t|^{1+p}} = |\lambda|^p \frac{\pi}{p} \cot\frac{\pi p}{2}\,, \qquad \lambda \in i\mathbb{R}, \quad 0 < p < 1\,,
$$

and

$$
\int_{\mathbb{R}} \frac{u_f(it)}{|t|^{1+p}}\, dt = \frac{\pi}{p} \cot\frac{\pi p}{2} \int_{\mathbb{T}} |f(\zeta)|^p\, dm(\zeta)\,, \qquad 0 < p < 1\,.
$$

On the other hand, Theorem A2 follows directly from our Theorem 2 since the $p$-th power of the $L^p$ norm can be written as the integral

$$p \int_0^\infty t^p \, \frac{m_f(t)}{t} \, dt = p^2 \int_0^\infty t^{p-1} \left( \int_t^\infty \frac{m_f(s)}{s} \, ds \right) dt \,.$$

This yields that if

$$\lim_{t \to \infty} t \int_t^\infty \frac{m_f(s)}{s} \, ds = \infty \,,$$

then $\lim_{p \uparrow 1} (1-p) \|f\|_p = \infty$.

We also note that it is not difficult to construct a function $h$ on $[-\pi, \pi]$ such that

$$\liminf_{t \to \infty} t \int_t^\infty \frac{m_h(s)}{s} \, ds = 0$$

while $h$ does not belong to any of the $L^p$ spaces for $p > 0$. This shows that the assumptions of Theorem 2 are really weaker than those of Theorem A2.

**4.2.** The counter-example provided by Theorem 1 is related to the logarithmic determinants by the following formula (in the notation of §2):

$$\int_{\mathbb{T}} \log |1 - wf(\zeta)| \, dm(\zeta) \overset{(2.2a)}{=} G_{\mathbb{C} \backslash E^*}(w, \infty) \,,$$

where $E^* = \{\lambda : 1/\lambda \in E\}$.

**4.3.** Theorems 1 and 2 can be easily reformulated for analytic functions represented by Cauchy-type integrals

$$f(z) = \int_{\mathbb{T}} \frac{d\mu(\zeta)}{\zeta - z}$$

of complex-valued measures $\mu$ of finite variations. In this case, condition (iv) must be replaced by

**(iv')** $Tf \in L^1(\mathbb{T})$, where $Tf(\zeta) = \lim_{r \uparrow 1} \left( f(r\zeta) - f(r^{-1}\zeta) \right)$.

We leave the details to the reader (cf. [1, 2]).

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## Вопрос Алексея Александрова и логарифмические детерминанты

М. Содин

Построена аналитическая функция $f$, принадлежащая классу Смирнова в единичном круге, такая что $\operatorname{Re} f = 0$ почти всюду на единичной окружности и

$$\liminf_{t \to \infty} t \operatorname{meas}\{\zeta : |\zeta| = 1, |f(\zeta)| \geq t\} = 0\,.$$

Этот пример дает отрицательный ответ на вопроос А. Александрова. Также найдены новые достаточные условия представимости функций из класса Смирнова интегралами Шварца и Коши. Эти условия усиливают один результат А. Александрова.

## Питання Олексія Александрова та логарифмічні детермінанти

М. Содін

Побудовано аналітичну функцію $f$ з класу Смірнова в одиничному крузі, таку що $\operatorname{Re} f = 0$ майже всюди на одиничному колі та

$$\liminf_{t \to \infty} t \operatorname{meas}\{\zeta : |\zeta| = 1, |f(\zeta)| \geq t\} = 0\,.$$

Цей приклад дає негативну відповідь на питання О. Александрова. Знайдено також нові достатні умови для зображення функцій класу Смірнова інтегралами Шварца та Коші. Ці умови підсилюють один результат О. Александрова.