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On q-analogues of certain prehomogeneous vector spaces: comparison of several approaches

D. Shklyarov

Mathematics Division, B. Verkin Institute for Low Temperature Physics & Engineering National Academy of Sciences of Ukraine 47 Lenin Ave., Kharkov, 61164, Ukraine

E-mail:vaksman@ilt.kharkov.ua

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There exist several approaches to constructing q-analogues of prehomogeneous vector spaces of commutative parabolic type. In the present paper we compare three approaches developed by H.P. Jakobsen, T. Tanisaki et al., and L. Vaksman et al. Within framework of these three approaches the following problem is solved: a q-analogue of the algebra $\mathbb{C}[V]$ of holomorphic polynomials on an arbitrary irreducible prehomogeneous vector space V (of commutative parabolic type) is constructed, and, moreover, the corresponding (non-commutative) algebra is endowed with a structure of *U-module* algebra with U being certain quantum universal enveloping algebra. We prove that the three q-analogues of $\mathbb{C}[V]$ are isomorphic as U-module algebras.

For the sake of simplicity we consider only the case when V is the space of 2×2 complex matrices. But we present such proof which is transferable to the case of an arbitrary irreducible prehomogeneous vector space of commutative parabolic type.[·]

1. Introduction: prehomogeneous vector spaces of commutative parabolic type

To start with, we remind what a prehomogeneous vector space is. Let G be an algebraic group over $\mathbb{C}$, $G \to GL(V)$ its linear representation. The pair (G, V)

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is called a prehomogeneous vector space if there exists a Zariski open G-orbit in V.

The theory of prehomogeneous vector spaces was initiated by Sato and Shintani [7, 9]. Some important classes of prehomogeneous vector spaces are well studied. For example, Sato and Kimura [8] have classified prehomogeneous vector spaces in the case when G is reductive and the representation $G \to GL(V)$ is irreducible. This class contains a subclass of prehomogeneous vector spaces of *commutative parabolic type* studied by Rubenthaler [6]. They are parameterized by some of Dynkin diagrams with a distinguished vertex. Let us remind a definition of irreducible prehomogeneous vector spaces of commutative parabolic type.

Let $\mathfrak{g}$ be a complex simple Lie algebra, $\mathfrak{h}$ its Cartan subalgebra, $\{\alpha_i\}_{i=\overline{1,l}}$ simple roots with respect to $\mathfrak{h}$. Let us associate to each $\alpha_* \in \{\alpha_i\}_{i=\overline{1,l}}$ a $\mathbb{Z}$-grading in the Lie algebra $\mathfrak{g}$ as follows. Let $H_0 \in \mathfrak{h}$ is given by

$$\alpha_i(H_0) = \begin{cases} 2, & \alpha_i = \alpha_*, \\ 0, & \text{otherwise}. \end{cases}$$

Then the $\mathbb{Z}$-grading in $\mathfrak{g}$ is defined by

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i, \qquad \mathfrak{g}_i \stackrel{\text{def}}{=} \{\xi \in \mathfrak{g}| \quad [H_0, \xi] = 2i\xi\}.$$

If $\mathfrak{g}_i$ is nonzero only for $i \in \{-1, 0, 1\}$ then the subspace $\mathfrak{g}_{-1}$ is said to be a prehomogeneous vector space of commutative parabolic type (see [5]).

R e m a r k 1. Let us explain why $\mathfrak{g}_{-1}$ is 'prehomogeneous'. Let G be the adjoint group for the Lie algebra $\mathfrak{g}$, K the algebraic subgroup corresponding to $\mathfrak{g}_0$. Then K acts in $\mathfrak{g}_{-1}$ and the pair $(K, \mathfrak{g}_{-1})$ is a prehomogeneous vector space.

Since K acts in $\mathfrak{g}_{-1}$ and $\mathfrak{g}_0 = \mathrm{Lie}K$, one may consider the corresponding representation of $U\mathfrak{g}_0$ in the space $\mathbb{C}[\mathfrak{g}_{-1}]$ of polynomials on $\mathfrak{g}_{-1}$.

R e m a r k 2. The Killing form of $\mathfrak{g}$ makes the vector spaces $\mathfrak{g}_{-1}$ and $\mathfrak{g}_{+1}$ dual to each other. This allows one to identify the algebras $\mathbb{C}[\mathfrak{g}_{-1}]$ and $\mathrm{S}(\mathfrak{g}_{+1})$ (the symmetric algebra over $\mathfrak{g}_{+1}$). The latter algebra is isomorphic to $U\mathfrak{g}_{+1}$ for $\mathfrak{g}_{+1}$ is an abelian Lie subalgebra in $\mathfrak{g}$. The action of $U\mathfrak{g}_0$ in $\mathbb{C}[\mathfrak{g}_{-1}]$ we deal with corresponds (under the isomorphism $\mathbb{C}[\mathfrak{g}_{-1}] \simeq U\mathfrak{g}_{+1}$) to the adjoint action of $U\mathfrak{g}_0$ in $U\mathfrak{g}_{+1}$.

There exist several approaches to constructing a q-analogue of the algebra $\mathbb{C}[\mathfrak{g}_{-1}]$. In the present paper we concern with those developed in [3, 4, 10]. Within framework of each approach a (noncommutative) analogue of $\mathbb{C}[\mathfrak{g}_{-1}]$ is endowed with an action of the quantum universal enveloping algebra $U_q\mathfrak{g}_0$. We prove that q-analogue of $\mathbb{C}[\mathfrak{g}_{-1}]$ constructed in [3, 4, 10] are isomorphic as $U_q\mathfrak{g}_0$-module algebras.

We present a proof fit for the case of an arbitrary prehomogeneous vector space of commutative parabolic type. But for the sake of simplicity we suppose that $\mathfrak{g} = \mathfrak{sl}_4$, $\mathfrak{g}_0 = \mathfrak{s}(\mathfrak{gl}_2 \oplus \mathfrak{gl}_2)$ (thus, $\mathfrak{g}_{-1}$ is the space of 2×2 complex matrices).

The paper is organized as follows. In Section 2 we fix a notation and remind some well known facts from the quantum group theory. In Section 3 the three approaches to constructing quantum prehomogeneous vector spaces of commutative parabolic type are described briefly. In Section 4 we formulate and prove our result.

The author thanks L. Vaksman for stating the problem and discussing results.

2. Notation and auxiliary facts

Let $\mathfrak{g} = \mathfrak{sl}_4(\mathbb{C})$, i.e., the Lie algebra of 4×4 complex matrices with zero trace. Let $\mathfrak{h} \subset \mathfrak{g}$ be the Cartan subalgebra of diagonal matrices. Denote by Δ and W the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and the Weyl group of this system, respectively. Let also α_1, α_2, α_3 be the simple roots in Δ given by

$$\alpha_i(H) = a_i - a_{i+1}$$

with $H = \mathrm{diag}(a_1, a_2, a_3, a_4) \in \mathfrak{h}$. There exists an isomorphism of the group W onto the group S_4 such that the simple reflections s_{α_1}, s_{α_2}, s_{α_3} correspond to the transpositions $(1,2)$, $(2,3)$, $(3,4)$. Let $\Delta_+ \subset \Delta$ be the set of positive roots:

$$\Delta_+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}.$$

Denote by $(\cdot|\cdot)$ the W-invariant scalar product in $\mathfrak{h}^*$ such that $(\alpha_i|\alpha_i) = 2$.

The root α_2 plays the role of the 'distinguished' root α_* (see Introduction). The associated element $H_0 \in \mathfrak{h}$ is given by

$$H_0 = H_1 + 2H_2 + H_3 \tag{2.1}$$

with $H_1 = \mathrm{diag}(1, -1, 0, 0)$, $H_2 = \mathrm{diag}(0, 1, -1, 0)$, $H_3 = \mathrm{diag}(0, 0, 1, -1)$.

Let $\Delta_\mathrm{c} \stackrel{\mathrm{def}}{=} \{\alpha_1, \alpha_3, -\alpha_1, -\alpha_3\} \subset \Delta$, $\Delta_\mathrm{n} \stackrel{\mathrm{def}}{=} \Delta \setminus \Delta_\mathrm{c}$. Then

$$\mathfrak{g}_0 = \mathfrak{h} \bigoplus \left(\bigoplus_{\alpha \in \Delta_\mathrm{c}} \mathfrak{g}_\alpha \right),$$

$$\mathfrak{g}_{+1} = \bigoplus_{\alpha \in \Delta_+ \cap \Delta_\mathrm{n}} \mathfrak{g}_\alpha, \qquad \mathfrak{g}_{-1} = \bigoplus_{-\alpha \in \Delta_+ \cap \Delta_\mathrm{n}} \mathfrak{g}_\alpha,$$

with $\mathfrak{g}_\alpha$ being the root subspace in $\mathfrak{g}$ corresponding to $\alpha \in \Delta$.

Let $\mathrm{W_c}$ be the subgroup in W generated by s_{α_1}, s_{α_3}. Thus, $\mathrm{W_c} \simeq \mathrm{S}_2 \times \mathrm{S}_2$ is the Weyl group of the Lie subalgebra $[\mathfrak{g}_0, \mathfrak{g}_0] = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$.

In what follows the ground field is the field of rational functions $\mathbb{C}(q^{1/4})$ unless the contrary is stated explicitly.

Let us recall one some definitions and facts of the quantum group theory (we follow [1]).

The quantum universal enveloping algebra $U_q\mathfrak{g}$ is the algebra with the generators $\{E_i, F_i, K_i^{\pm 1}\}_{i=\overline{1,3}}$ satisfying the following relations:

$$K_iK_j = K_jK_i, \qquad K_iK_i^{-1} = K_i^{-1}K_i = 1,$$

$$K_iE_j = q^{a_{ij}}E_jK_i, \qquad K_iF_j = q^{-a_{ij}}F_jK_i,$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q - q^{-1}},$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}_q E_i^{1-a_{ij}-s}E_jE_i^s = 0, \qquad i \neq j,$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}_q F_i^{1-a_{ij}-s}F_jF_i^s = 0, \qquad i \neq j,$$

where (a_{ij}) is the Cartan matrix for $\mathfrak{g}$:

$$a_{ij} = \begin{cases} 2, & i - j = 0 \\ -1, & |i - j| = 1 \\ 0, & \text{otherwise} \end{cases},$$

$$\begin{bmatrix} n \\ j \end{bmatrix}_q \stackrel{\text{def}}{=} \frac{[n]_q!}{[n-j]_q![j]_q!}, \quad [n]_q! \stackrel{\text{def}}{=} [n]_q \cdot [n-1]_q \cdot \ldots \cdot [1]_q, \quad [n]_q \stackrel{\text{def}}{=} \frac{q^n - q^{-n}}{q - q^{-1}}.$$

The algebra $U_q\mathfrak{g}$ is endowed with Hopf algebra structure as follows

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta(K_i) = K_i \otimes K_i,$$

$$S(E_i) = -K_i^{-1}E_i, \qquad S(F_i) = -F_iK_i, \qquad S(K_i) = K_i^{-1},$$

$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \qquad \varepsilon(K_i) = 1,$$

with Δ, S, ε being the comultiplication, the antipode, and the counit, respectively.

Let us use the short notation $x_{(1)} \otimes x_{(2)}$ for the element $\Delta(x) \in U_q\mathfrak{g} \bigotimes U_q\mathfrak{g}$ ($x \in U_q\mathfrak{g}$). For example, coassociativity of the comultiplication $\Delta : U_q\mathfrak{g} \to U_q\mathfrak{g} \bigotimes U_q\mathfrak{g}$ looks in this notation as follows

$$x_{(1)} \otimes x_{(2)(1)} \otimes x_{(2)(2)} = x_{(1)(1)} \otimes x_{(1)(2)} \otimes x_{(2)}. \tag{2.2}$$

Sometimes we use the notation $x_{(1)} \otimes x_{(2)} \otimes x_{(3)}$ for the right (and left) hand side of (2.2). By analogy $x_{(1)} \otimes x_{(2)} \otimes x_{(3)} \otimes x_{(4)} = x_{(1)} \otimes x_{(2)(1)} \otimes x_{(2)(2)} \otimes x_{(3)}$ etc.

The adjoint representation of the algebra $U_q\mathfrak{g}$ is defined as follows

$$\mathrm{ad}x(y) \stackrel{\mathrm{def}}{=} x_{(1)} \cdot y \cdot S(x_{(2)})$$

with $x, y \in U_q\mathfrak{g}$. This adjoint action makes $U_q\mathfrak{g}$ a $U_q\mathfrak{g}$-module algebra. It means that the product map $U_q\mathfrak{g} \bigotimes U_q\mathfrak{g} \to U_q\mathfrak{g}$ is a morphism of $U_q\mathfrak{g}$-modules and the unit $1 \in U_q\mathfrak{g}$ is $U_q\mathfrak{g}$-invariant.

We fix the following notation for some subalgebras in $U_q\mathfrak{g}$:

$$U_q^{\geq 0} = \langle E_i, K_i^{\pm 1} | \quad i = \overline{1,3} \rangle, \qquad U_q^{\leq 0} = \langle F_i, K_i^{\pm 1} | \quad i = \overline{1,3} \rangle,$$

$$U_q^{+} = \langle E_i | \quad i = \overline{1,3} \rangle, \qquad U_q^{-} = \langle F_i | \quad i = \overline{1,3} \rangle,$$

$$U_q^{0} = \langle K_i^{\pm 1} | \quad i = \overline{1,3} \rangle, \qquad U_q\mathfrak{g}_0 = \langle K_i^{\pm 1}, E_j, F_j | \quad i = \overline{1,3}, \quad j \neq 2 \rangle.$$

Let us recall one a definition of the Lusztig automorphisms T_i, $i = \overline{1,3}$, of the algebra $U_q\mathfrak{g}$. The action of T_i on the subalgebra $U_q^{\geq 0}$ is given by

$$T_i(K_j) = K_j \cdot K_i^{-a_{ij}}, \qquad T_i(E_i) = -F_i K_i,$$

$$T_i(E_j) = (-\mathrm{ad}E_i)^{-a_{ij}}(E_j), \quad i \neq j.$$

To define T_i completely one sets

$$T_i \circ k = k \circ T_i,$$

where k is the conjugate linear antiautomorphism of the $\mathbb{C}(q^{1/4})$-algebra $U_q\mathfrak{g}$ given by

$$k(E_i) = F_i, \quad k(F_i) = E_i, \quad k(K_i) = K_i^{-1}, \quad k(q^{1/4}) = q^{-1/4}.$$

Let $w \in \mathrm{W}$ and $w = s_{i_1} s_{i_2} \ldots s_{i_k}$ be a reduced expression (we write s_i instead of s_{α_i}). It is well known that the automorphism $T_w \stackrel{\mathrm{def}}{=} T_{i_1} T_{i_2} \ldots T_{i_k}$ does not depend on particular choice of a reduced expression of w.

All $U_q\mathfrak{g}$-modules we consider possess the property

$$V = \bigoplus_{\mu \in \mathbb{Z}^3} V_\mu, \quad V_\mu \stackrel{\mathrm{def}}{=} \{v \in V | \quad K_i v = q^{\mu_i} v, i = \overline{1,3}\}$$

with $\mu = (\mu_1, \mu_2, \mu_3)$. This allows one to introduce endomorphisms H_i, $i = \overline{1,3}$, of any $U_q\mathfrak{g}$-module V by $H_i v = \mu_i v$ for $v \in V_\mu$ ($\mu = (\mu_1, \mu_2, \mu_3)$). Formally this can be written as $K_i = q^{H_i}$.

Let $K_0 \stackrel{\mathrm{def}}{=} K_1 \cdot K_2^2 \cdot K_3$ (i.e., $K_0 = q^{H_0}$ with H_0 given by (2.1)). It is an important consequence of definitions that K_0 belongs to the centre of the algebra $U_q\mathfrak{g}_0$:

$$\mathrm{ad}K_0(\xi) = \xi, \qquad \xi \in U_q\mathfrak{g}_0. \tag{2.3}$$

Let us recall one some facts concerning the universal R-matrix for $U_q\mathfrak{g}$. The notion of the universal R-matrix is due to V. Drinfeld. In context of the present paper it have to be understood just as in [1].

The R-matrix satisfies some well known identities. We don't adduce a full list of these identities but recall one those important for us:

$$\mathrm{id}\otimes\Delta^{\mathrm{op}}(R)=R^{12}\cdot R^{13}, \tag{2.4}$$

$$\Delta^{\mathrm{op}}(\eta)\cdot R=R\cdot\Delta(\eta),\quad \eta\in U_q\mathfrak{g}, \tag{2.5}$$

with $\Delta^{\mathrm{op}}(x)\overset{\mathrm{def}}{=}x_{(2)}\otimes x_{(1)}$, $R^{12}\overset{\mathrm{def}}{=}\sum_i a_i\otimes b_i\otimes 1$, $R^{13}\overset{\mathrm{def}}{=}\sum_i a_i\otimes 1\otimes b_i$ provided $R=\sum_i a_i\otimes b_i$.

Remind an explicit formula for the R-matrix (the so-called multiplicative formula). Let $w_0\in \mathrm{W}$ be the maximal length element. Identifying W with S_4 we get

$$w_0=\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}.$$

The length of any reduced expression of w_0 is equal to 6. To a reduced expression $w_0=s_{i_1}s_{i_2}\dots s_{i_6}$ one attaches the following data:

i) the total order on the set Δ_+ of positive roots:

$$\beta_1=\alpha_{i_1},\quad \beta_2=s_{i_1}(\alpha_{i_2}),\quad \dots\quad \beta_6=s_{i_1}s_{i_2}\dots s_{i_5}(\alpha_{i_6});$$

ii) the set of elements $E_{\beta_1},E_{\beta_2},\dots E_{\beta_6}\in U_q^+$, $F_{\beta_1},F_{\beta_2},\dots F_{\beta_6}\in U_q^-$ which are q-analogues of root vectors in $\mathfrak{g}$:

$$E_{\beta_1}=E_{i_1},\quad E_{\beta_2}=T_{i_1}(E_{i_2}),\quad\dots\quad E_{\beta_6}=T_{i_1}T_{i_2}\dots T_{i_5}(E_{i_6}),$$

$$F_{\beta_1}=F_{i_1},\quad F_{\beta_2}=T_{i_1}(F_{i_2}),\quad\dots\quad F_{\beta_6}=T_{i_1}T_{i_2}\dots T_{i_5}(F_{i_6});$$

iii) the multiplicative formula for the R-matrix:

$$\begin{aligned}R=\exp_{q^2}((q^{-1}-q)E_{\beta_6}\otimes F_{\beta_6})\cdot\ldots\cdot\exp_{q^2}((q^{-1}-q)E_{\beta_2}\otimes F_{\beta_2})\\ \cdot\exp_{q^2}((q^{-1}-q)E_{\beta_1}\otimes F_{\beta_1})\cdot q^{\mathrm{t}}\end{aligned} \tag{2.6}$$

with $\mathrm{t}\overset{\mathrm{def}}{=}-\sum_{i,j}c_{ij}H_i\otimes H_j$, the matrix (c_{ij}) being the inverse to the Cartan matrix, and

$$\exp_{q^2}(t)\overset{\mathrm{def}}{=}\sum_{k=0}^{\infty}\frac{t^k}{(k)_{q^2}!},\quad (k)_{q^2}!\overset{\mathrm{def}}{=}\prod_{j=1}^{k}\frac{1-q^{2j}}{1-q^2}.$$

3. Three approaches to quantization of prehomogeneous vector spaces of commutative parabolic type

3.1. First approach

In this subsection we describe an approach to constructing q-analogues of prehomogeneous vector spaces of commutative parabolic type developed by S. Sinel'shchikov and L. Vaksman in [10].

Let us consider the generalized Verma module $V(0)$ over $U_q\mathfrak{g}$ given by its generator $v(0)$ and the relations

$$E_i v(0) = 0, \quad K_i v(0) = v(0), \quad i = \overline{1,3}, \tag{3.1}$$

$$F_i v(0) = 0, \quad i \neq 2. \tag{3.2}$$

$V(0)$ splits into direct sum of its finite dimensional subspaces $V(0)_k$, $-k \in \mathbb{Z}_+$, with

$$V(0)_k \stackrel{\rm def}{=} \{v \in V(0) | \quad K_0 v = q^{2k} v\}.$$

Consider the *graded* dual $U_q\mathfrak{g}$-module:

$$\mathbb{C}[\mathfrak{g}_{-1}]_q \stackrel{\rm def}{=} \bigoplus_{-k\in\mathbb{Z}_+} (V(0)_k)^*.$$

Let us equip the tensor product $V(0) \bigotimes V(0)$ with a $U_q\mathfrak{g}$-module structure via the opposite comultiplication

$$\xi(v_1 \otimes v_2) = \xi_{(2)}(v_1) \otimes \xi_{(1)}(v_2), \quad \xi \in U_q\mathfrak{g}, \quad v_1, v_2 \in V(0). \tag{3.3}$$

Due to (3.1), (3.2) the maps

$$v(0) \mapsto v(0) \otimes v(0), \qquad v(0) \mapsto 1 \tag{3.4}$$

are extendable up to morphisms of $U_q\mathfrak{g}$-modules

$$\Delta_- : V(0) \to V(0) \bigotimes V(0), \qquad \varepsilon_- : V(0) \to \mathbb{C}(q^{1/4}).$$

It can be shown that Δ_- and ε_- make $V(0)$ into a coassociative coalgebra with a counit. Thus, the dual maps

$$m = (\Delta_-)^* : \mathbb{C}[\mathfrak{g}_{-1}]_q \bigotimes \mathbb{C}[\mathfrak{g}_{-1}]_q \to \mathbb{C}[\mathfrak{g}_{-1}]_q, \quad 1 = (\varepsilon_-)^* : \mathbb{C}(q^{1/4}) \to \mathbb{C}[\mathfrak{g}_{-1}]_q \tag{3.5}$$

make $\mathbb{C}[\mathfrak{g}_{-1}]_q$ into an associative unital algebra. Moreover, the product map m is a morphism of $U_q\mathfrak{g}$-modules and the unit 1 is $U_q\mathfrak{g}$-invariant, i.e., $\mathbb{C}[\mathfrak{g}_{-1}]_q$ is a $U_q\mathfrak{g}$-module algebra. In particular, it is a $U_q\mathfrak{g}_0$-module algebra. This $U_q\mathfrak{g}_0$-module structure is just the one we mentioned in the Introduction.

R e m a r k 3. Let us comment this construction. As we have mentioned in Introduction, $\mathfrak{g}_{-1}$ is the space of 2×2 complex matrices. This vector space contains the so called matrix ball

$$\mathbb{U} = \{T \in \mathfrak{g}_{-1} | TT^* < 1\}$$

(with $*$ being the hermitian conjugation and 1 the unit matrix). Evidently, $\mathbb{U}$ is an open subset in $\mathfrak{g}_{-1}$. It is known that the real simple Lie group $SU_{2,2}$ acts in $\mathbb{U}$ via biholomorphic automorphisms. Thus, elements of the universal enveloping algebra $U\mathfrak{su}_{2,2}$ (and hence elements of $U\mathfrak{sl}_4$) act in the space of holomorphic functions in $\mathbb{U}$ via differential operators. These differential operators have a polynomial coefficients and, thus, preserve the subspace of polynomials. Suppose $v(0)$ is the "delta function", i.e., the functional on the space of polynomials which sends a polynomial to its value at the origin of $\mathbb{U}$. It is clear how to apply differential operators to the delta function. Thus, one can consider the $U\mathfrak{sl}_4$-module generated by $v(0)$. It turns out to have a rather simple structure. Specifically, it is a generalized Verma module. This observation suggests the idea (see [10]) to construct quantum prehomogeneous vector spaces of commutative parabolic type and quantum Cartan domains starting from generalized Verma modules over quantum universal enveloping algebra (they have a lot of nice properties) and then passing to dual modules. By the way the above arguments allow one to observe "hidden" $U\mathfrak{sl}_4$-symmetry of the algebra $\mathbb{C}[\mathfrak{g}_{-1}]$.

The next two approaches are related to the construction of Remark 2 (see Introduction).

3.2. Second approach

Now we are going to describe briefly an approach of H.P. Jakobsen [3] to quantization of $\mathbb{C}[\mathfrak{g}_{-1}]$.

It follows from the definition of $U_q\mathfrak{g}$ that

$$\mathrm{ad}K_2(E_2) = q^2 E_2, \qquad \mathrm{ad}K_1(E_2) = \mathrm{ad}K_3(E_2) = q^{-1}E_2,$$
$$\mathrm{ad}F_1(E_2) = \mathrm{ad}F_3(E_2) = 0,$$
$$(\mathrm{ad}E_1)^2(E_2) = (\mathrm{ad}E_3)^2(E_2) = 0.$$

Thus, $\mathrm{ad}U_q\mathfrak{g}_0(E_2)$ is a finite dimensional $U_q\mathfrak{g}_0$-submodule in $U_q\mathfrak{g}$. Let us denote by $\mathbb{C}[\mathfrak{g}_{-1}]_q^{\mathrm{I}}$ the minimal subalgebra in $U_q\mathfrak{g}$ which contains the subspace $\mathrm{ad}U_q\mathfrak{g}_0(E_2)$. Evidently, it is a $U_q\mathfrak{g}_0$-module subalgebra in $U_q\mathfrak{g}$. The algebra $\mathbb{C}[\mathfrak{g}_{-1}]_q^{\mathrm{I}}$ can be treated as a q-analogue of $\mathbb{C}[\mathfrak{g}_{-1}]$ (see Remark 2 above).

3.3. Third approach

Let us turn to description of an approach of A. Kamita, Y. Morita, and T. Tanisaki [4]. Note that notation in [4] differs from ours.

Let $w_0' \in \mathrm{W_c}$ be the maximal length element. Evidently, $w_0' = (1,2)\cdot(3,4)$. Consider the subspace $\mathbb{C}[\mathfrak{g}_{-1}]_q^{\mathrm{II}}$ in $U_q\mathfrak{g}$ defined by

$$\mathbb{C}[\mathfrak{g}_{-1}]_q^{\mathrm{II}} = U_q^+ \cap T_{w_0'}^{-1}\left(U_q^+\right). \tag{3.6}$$

Obviously the subspace $\mathbb{C}[\mathfrak{g}_{-1}]_q^{\mathrm{II}}$ is a subalgebra in $U_q\mathfrak{g}$. It is shown in [4] that $\mathbb{C}[\mathfrak{g}_{-1}]_q^{\mathrm{II}}$ is a $U_q\mathfrak{g}_0$-module subalgebra in $U_q\mathfrak{g}$ with respect to the adjoint action. It is one more q-analog of the algebra $\mathbb{C}[\mathfrak{g}_{-1}]$.

4. Comparison of the approaches

The aim of this paper is to prove

Theorem 4.1. *The algebras $\mathbb{C}[\mathfrak{g}_{-1}]_q$, $\mathbb{C}[\mathfrak{g}_{-1}]_q^{\mathrm{I}}$, and $\mathbb{C}[\mathfrak{g}_{-1}]_q^{\mathrm{II}}$ are isomorphic to each other as $U_q\mathfrak{g}_0$-module algebras.*

Note that the algebras $\mathbb{C}[\mathfrak{g}_{-1}]_q^{\mathrm{I}}$ and $\mathbb{C}[\mathfrak{g}_{-1}]_q^{\mathrm{II}}$ are subalgebras in the quantum universal enveloping algebra $U_q\mathfrak{g}$. Our proof of the above theorem consists of the following steps:

i) we construct an embedding T of the algebra $\mathbb{C}[\mathfrak{g}_{-1}]_q$ into $U_q\mathfrak{g}$ which intertwines the $U_q\mathfrak{g}_0$-action in $\mathbb{C}[\mathfrak{g}_{-1}]_q$ mentioned in subsection 3.1 and the adjoint $U_q\mathfrak{g}_0$-action;

ii) we show that the subalgebras $T(\mathbb{C}[\mathfrak{g}_{-1}]_q)$, $\mathbb{C}[\mathfrak{g}_{-1}]_q^{\mathrm{I}}$, and $\mathbb{C}[\mathfrak{g}_{-1}]_q^{\mathrm{II}}$ in $U_q\mathfrak{g}$ coincide.

4.1. $\mathbb{C}[\mathfrak{g}_{-1}]_q \simeq \mathbb{C}[\mathfrak{g}_{-1}]_q^{\mathrm{I}}$

In this subsection we construct an $U_q\mathfrak{g}_0$-equivariant embedding T of the algebra $\mathbb{C}[\mathfrak{g}_{-1}]_q$ into $U_q\mathfrak{g}$, and then we show that $T(\mathbb{C}[\mathfrak{g}_{-1}]_q) = \mathbb{C}[\mathfrak{g}_{-1}]_q^{\mathrm{I}}$. The embedding T is constructed via a standard technique due to [2].

Let us use in the sequel the notation $\sum_i a_i \otimes b_i$ for the series

$$\sum_{(k_1,\ldots,k_6)\in\mathbb{Z}_+^6} a_{k_1,\ldots,k_6}(E_{\beta_6}^{k_1}\ldots E_{\beta_1}^{k_6} \otimes F_{\beta_6}^{k_1}\ldots F_{\beta_1}^{k_6})\cdot q^{\mathrm{t}}$$

which determines the universal R-matrix for $U_q\mathfrak{g}$. The coefficients $a_{k_1,\ldots,k_6}$ may be calculated via (2.6). Consider the linear map $T : \mathbb{C}[\mathfrak{g}_{-1}]_q \to U_q\mathfrak{g}$ given by

$$T(f) \stackrel{\mathrm{def}}{=} \sum_i a_i \langle b_i v(0), f\rangle \tag{4.1}$$

with $f \in \mathbb{C}[\mathfrak{g}_{-1}]_q$, $v(0)$ being the generator of the generalized Verma module $V(0)$, $\langle \cdot, \cdot \rangle$ being the pairing $V(0) \times \mathbb{C}[\mathfrak{g}_{-1}]_q \to \mathbb{C}(q^{1/4})$ arising from the equality $\mathbb{C}[\mathfrak{g}_{-1}]_q = (V(0))^*$ (see Subsection 3.1).

Proposition 4.2. *The map T is well defined, i.e., the right hand side of (4.1) is a finite sum for any f.*

P r o o f. Denote by $U_q\mathfrak{g}\widehat{\otimes} V(0)$ the vector space $\prod_{j=0}^{\infty} U_q\mathfrak{g} \otimes V(0)_{-j}$. Equip this vector space with a topology by setting the subspaces $\{\prod_{j=N}^{\infty} U_q\mathfrak{g} \otimes V(0)_{-j}\}_{N=0}^{\infty}$ to be a base of neighborhoods of 0.

It follows directly from Lemma 4.5. that $\sum_i a_i \otimes b_i v(0)$ is an element of $U_q\mathfrak{g}\widehat{\otimes} V(0)$. What remains is to remind the definition of $\mathbb{C}[\mathfrak{g}_{-1}]_q$. ■

Proposition 4.3. *T is a homomorphism of algebras.*

P r o o f. Let $f, \varphi \in \mathbb{C}[\mathfrak{g}_{-1}]_q$. Then due to (3.5), (3.4), (3.3)

$$T(f \cdot \varphi) = \sum_i a_i \langle b_i v(0), f \cdot \varphi \rangle = \sum_i a_i \langle b_i v(0), m(f \otimes \varphi) \rangle$$

$$= \sum_i a_i \langle \Delta_-(b_i v(0)), f \otimes \varphi \rangle = \sum_i a_i \langle \Delta^{\mathrm{op}}(b_i)(v(0) \otimes v(0)), f \otimes \varphi \rangle.$$

By (2.4)

$$\sum_i a_i \langle \Delta^{\mathrm{op}}(b_i)(v(0) \otimes v(0)), f \otimes \varphi \rangle = \sum_{i,j} a_i \cdot a_j \langle (b_i \otimes b_j)(v(0) \otimes v(0)), f \otimes \varphi \rangle$$

$$= \left(\sum_i a_i \langle b_i v(0), f \rangle \right) \cdot \left(\sum_j a_j \langle b_j v(0), \varphi \rangle \right) = T(f) \cdot T(\varphi).$$

■

Proposition 4.4. *T is a morphism of $U_q\mathfrak{g}_0$-modules.*

P r o o f. Consider a $U_q\mathfrak{g}$-module structure in the vector space $U_q\mathfrak{g} \otimes V(0)$ given by

$$\xi(\eta \otimes v) \stackrel{\mathrm{def}}{=} \xi_{(3)} \cdot \eta \cdot S^{-1}(\xi_{(1)}) \otimes \xi_{(2)} v, \quad \xi, \eta \in U_q\mathfrak{g}, \quad v \in V(0).$$

Evidently, one can extend this $U_q\mathfrak{g}$-module structure by continuity up to a $U_q\mathfrak{g}$-module structure in $U_q\mathfrak{g}\widehat{\otimes} V(0)$ (see the proof of Proposition 4.2).

Lemma 4.5. *For any $v \in V(0)$ the element $\sum_i a_i \otimes b_i v$ (with $\sum_i a_i \otimes b_i$ being the universal R-matrix) belongs to $U_q\mathfrak{g}\widehat{\otimes} V(0)$.*

P r o o f o f t h e l e m m a. Equip the algebra U_q^- with a $\mathbb{Z}_+$-grading as follows

$$\deg F_i = \begin{cases} 1, & i = 2, \\ 0, & \text{otherwise}. \end{cases} \tag{4.2}$$

This grading can be described in another way:

$$\left(U_q^-\right)_j = \{\xi \in U_q^- | \quad \mathrm{ad}K_0(\xi) = q^{-2j}\xi\}. \tag{4.3}$$

Thus, for $\xi \in U_q^-$

$$\deg(\xi) = 0 \quad \Leftrightarrow \quad \xi \in U_q^- \cap U_q\mathfrak{g}_0. \tag{4.4}$$

Consider the $\mathbb{Z}_+$-grading in the vector space $V(0)$ given by $\deg(v) = k$ for $v \in V(0)_{-k}$. Obviously, $V(0)$ is a graded U_q^--module, i.e.,

$$\deg(\xi v) = \deg(\xi) + \deg(v)$$

for any homogeneous $\xi \in U_q^-$, $v \in V(0)$.

Let $w_0 \in$ W be the maximal length element. We fix a reduced expression for w_0:

$$w_0 = (1,2)(3,4)(2,3)(1,2)(3,4)(2,3) = s_1 s_3 s_2 s_1 s_3 s_2. \tag{4.5}$$

Obviously $w_0 = w_0' s_2 s_1 s_3 s_2$ with w_0' being the maximal length element in $\mathrm{W_c}$. Describe explicitly the order in Δ_+ attached to the expression (4.5) (see Section 2):

$$\beta_1 = \alpha_1, \qquad \beta_2 = \alpha_3, \qquad \beta_3 = \alpha_1 + \alpha_2 + \alpha_3,$$

$$\beta_4 = \alpha_2 + \alpha_3, \qquad \beta_5 = \alpha_1 + \alpha_2, \qquad \beta_6 = \alpha_2.$$

Thus β_3, β_4, β_5, β_6 belong to Δ_n, β_1, β_2 belong to Δ_c. Using (4.3) one shows that $\deg(F_{\beta_1}) = \deg(F_{\beta_2}) = 0$. By (4.4)

$$F_{\beta_1}, F_{\beta_2} \in U_q^- \cap U_q\mathfrak{g}_0. \tag{4.6}$$

Let $v \in V(0)_{-k}$. By (2.6)

$$\sum_i a_i \otimes b_i v =$$

$$\sum_{\mathbf{k}=(k_1,\ldots,k_6)\in\mathbb{Z}_+^6, \mathbf{m}=(m_1,m_2,m_3)\in\mathbb{Z}^3} b_{\mathbf{k},\mathbf{m}} E_{\beta_6}^{k_1} \ldots E_{\beta_2}^{k_5} E_{\beta_1}^{k_6} K_1^{m_1} K_2^{m_2} K_3^{m_3} \otimes F_{\beta_6}^{k_1} \ldots F_{\beta_2}^{k_5} F_{\beta_1}^{k_6} v.$$

Note that $V(0)_{-k}$ is a finite dimensional $U_q\mathfrak{g}_0$-submodule in $V(0)$ for any $k \in \mathbb{Z}_+$. Therefore by (4.6) $F_{\beta_2}^{k_5} F_{\beta_1}^{k_6} v = 0$ for a sufficiently large $k_5 + k_6$. What remains is to use the equalities

$$\deg(F_{\beta_3}) = \ldots = \deg(F_{\beta_6}) = 1.$$

■

Lemma 4.6. *The linear map*

$$V(0) \to U_q\mathfrak{g}\widehat{\otimes} V(0), \qquad v \mapsto \sum_i a_i \otimes b_i v$$

is a morphism of $U_q\mathfrak{g}$-modules.

P r o o f o f t h e l e m m a. Let $\xi \in U_q\mathfrak{g}$, $v \in V(0)$. Then

$$\begin{aligned}
\xi(\sum_i a_i \otimes b_i v)) &= \sum_i \xi_{(3)} a_i S^{-1}(\xi_{(1)}) \otimes \xi_{(2)} b_i v \\
&= \sum_i \varepsilon(\xi_{(3)(2)}) \xi_{(3)(1)} a_i S^{-1}(\xi_{(1)}) \otimes \xi_{(2)} b_i v \\
&= \sum_i \xi_{(3)(1)} a_i S^{-1}(\xi_{(1)}) \otimes \xi_{(2)} b_i \varepsilon(\xi_{(3)(2)}) v \\
&= \sum_i \xi_{(2)(2)} a_i S^{-1}(\xi_{(1)}) \otimes \xi_{(2)(1)} b_i \varepsilon(\xi_{(3)}) v.
\end{aligned}$$

Let us make use of the property (2.5). We get

$$\begin{aligned}
\xi(\sum_i a_i \otimes b_i v) &= \sum_i \xi_{(2)(2)} a_i S^{-1}(\xi_{(1)}) \otimes \xi_{(2)(1)} b_i \varepsilon(\xi_{(3)}) v \\
&= \sum_i a_i \xi_{(2)(1)} S^{-1}(\xi_{(1)}) \otimes b_i \xi_{(2)(2)} \varepsilon(\xi_{(3)}) v \\
&= \sum_i a_i \xi_{(1)(2)} S^{-1}(\xi_{(1)(1)}) \otimes b_i \xi_{(2)(1)} \varepsilon(\xi_{(2)(2)}) v \\
&= \sum_i a_i \varepsilon(\xi_{(1)}) \otimes b_i \xi_{(2)} v = \sum_i a_i \otimes b_i \varepsilon(\xi_{(1)}) \xi_{(2)} v \\
&= \sum_i a_i \otimes b_i \xi v.
\end{aligned}$$

■

Lemma 4.7. *An element $\sum_j \eta_j \otimes v_j \in U_q\mathfrak{g}\widehat{\otimes} V(0)$ is $U_q\mathfrak{g}_0$-invariant iff for any $\xi \in U_q\mathfrak{g}_0$*

$$\sum_j \xi_{(1)} \eta_j S(\xi_{(2)}) \otimes v_j = \sum_j \eta_j \otimes S(\xi) v_j. \tag{4.7}$$

P r o o f o f t h e l e m m a. Let $\sum_j \eta_j \otimes v_j \in U_q\mathfrak{g}\widehat{\otimes} V(0)$ satisfies (4.7) for any $\xi \in U_q\mathfrak{g}_0$. Rewrite (4.7) for $\xi := S^{-1}(\zeta)$:

$$\sum_j S^{-1}(\zeta_{(2)})\eta_j\zeta_{(1)} \otimes v_j = \sum_j \eta_j \otimes \zeta v_j. \tag{4.8}$$

Using (4.8) one gets

$$\begin{aligned}
\xi(\sum_j \eta_j \otimes v_j) &= \sum_j \xi_{(3)}\eta_j S^{-1}(\xi_{(1)}) \otimes \xi_{(2)}v_j \\
&= \sum_j \xi_{(3)} S^{-1}(\xi_{(2)(2)})\eta_j\xi_{(2)(1)} S^{-1}(\xi_{(1)}) \otimes v_j \\
&= \sum_j \xi_{(2)(2)} S^{-1}(\xi_{(2)(1)})\eta_j\xi_{(1)(2)} S^{-1}(\xi_{(1)(1)}) \otimes v_j \\
&= \sum_j \varepsilon(\xi_{(2)})\eta_j\varepsilon(\xi_{(1)}) \otimes v_j = \varepsilon(\xi)\sum_j \eta_j \otimes v_j.
\end{aligned}$$

Thus $\sum_j \eta_j \otimes v_j$ is $U_q\mathfrak{g}_0$-invariant.

Conversely, suppose that $\sum_j \eta_j \otimes v_j$ is $U_q\mathfrak{g}_0$-invariant. Let us prove (4.8) (obviously, it is equivalent to (4.7)).

$$\begin{aligned}
\sum_j S^{-1}(\xi_{(2)})\eta_j\xi_{(1)} \otimes v_j &= \sum_j S^{-1}(\xi_{(2)})\eta_j\varepsilon(\xi_{(1)(2)})\xi_{(1)(1)} \otimes v_j \\
&= \sum_j S^{-1}(\xi_{(3)})\eta_j\varepsilon(\xi_{(2)})\xi_{(1)} \otimes v_j.
\end{aligned}$$

$U_q\mathfrak{g}_0$-invariance of $\sum_j \eta_j \otimes v_j$ implies

$$\sum_j \varepsilon(\xi_{(2)})\eta_j \otimes v_j = \sum_j \xi_{(2)(3)}\eta_j S^{-1}(\xi_{(2)(1)}) \otimes \xi_{(2)(2)}v_j.$$

Thus

$$\begin{aligned}
\sum_j S^{-1}(\xi_{(2)})\eta_j\xi_{(1)} \otimes v_j &= \sum_j S^{-1}(\xi_{(3)})\xi_{(2)(3)}\eta_j S^{-1}(\xi_{(2)(1)})\xi_{(1)} \otimes \xi_{(2)(2)}v_j \\
&= \sum_j S^{-1}(\xi_{(3)(2)})\xi_{(3)(1)}\eta_j S^{-1}(\xi_{(1)(2)})\xi_{(1)(1)} \otimes \xi_{(2)}v_j \\
&= \sum_j \varepsilon(\xi_{(3)})\eta_j\varepsilon(\xi_{(1)}) \otimes \xi_{(2)}v_j = \sum_j \eta_j \otimes \xi v_j.
\end{aligned}$$

■

Let us complete the proof of Proposition 4.4. By (3.1), (3.2) $v(0)$ is $U_q\mathfrak{g}_0$-invariant. Due to Lemma 4.6. the element $\sum_i a_i \otimes b_i v(0) \in U_q\mathfrak{g}\widehat{\otimes}V(0)$ is $U_q\mathfrak{g}_0$-invariant. By (4.7) we have: for $f \in \mathbb{C}[\mathfrak{g}_{-1}]_q$, $\xi \in U_q\mathfrak{g}_0$

$$\begin{aligned} T(\xi f) &= \sum_i a_i \langle b_i v(0), \xi f\rangle = \sum_i a_i \langle S(\xi) b_i v(0), f\rangle \\ &= \sum_i \xi_{(1)} a_i S(\xi_{(2)}) \langle b_i v(0), f\rangle = \mathrm{ad}\xi(T(f)). \end{aligned}$$

■

We have constructed the mapping $T : \mathbb{C}[\mathfrak{g}_{-1}]_q \to U_q\mathfrak{g}$ which is a morphism of $U_q\mathfrak{g}_0$-module algebras. It turns out to be an embedding.

Proposition 4.8. *T is injective.*

P r o o f. Let $w_0 \in \mathrm{W}$ be the maximal length element. Consider the reduced expression (4.5) for w_0.

It follows from (3.1), (3.2), and (4.6) that

$$\sum_i a_i \otimes b_i v(0) = \sum_j A_j \otimes B_j v(0),$$

where $\sum_i a_i \otimes b_i$ is the universal R-matrix,

$$\begin{aligned} \textstyle\sum_j A_j \otimes B_j \stackrel{\mathrm{def}}{=} & \exp_{q^2}((q^{-1}-q)E_{\beta_6} \otimes F_{\beta_6}) \cdot \exp_{q^2}((q^{-1}-q)E_{\beta_5} \otimes F_{\beta_5}) \\ & \times \exp_{q^2}((q^{-1}-q)E_{\beta_4} \otimes F_{\beta_4}) \cdot \exp_{q^2}((q^{-1}-q)E_{\beta_3} \otimes F_{\beta_3}). \end{aligned} \tag{4.9}$$

It is clear that

$$\sum_j A_j \otimes B_j = \sum_{(k_1,\dots,k_4)\in\mathbb{Z}_+^4} a_{k_1,\dots,k_4} E_{\beta_6}^{k_1} \dots E_{\beta_3}^{k_4} \otimes F_{\beta_6}^{k_1} \dots F_{\beta_3}^{k_4}, \tag{4.10}$$

where all $a_{k_1,\dots,k_4}$ are nonzero elements of $\mathbb{C}(q^{1/4})$. Thus we get the formula

$$T(f) = \sum_{(k_1,\dots,k_4)\in\mathbb{Z}_+^4} a_{k_1,\dots,k_4} E_{\beta_6}^{k_1} \dots E_{\beta_3}^{k_4} \langle F_{\beta_6}^{k_1} \dots F_{\beta_3}^{k_4} v(0), f\rangle. \tag{4.11}$$

To complete the proof of Proposition 4.8 it is sufficient to prove that the elements

$$\{E_{\beta_6}^{k_1} \dots E_{\beta_3}^{k_4}\}_{(k_1,\dots,k_4)\in\mathbb{Z}_+^4}$$

are linearly independent, and the elements

$$\{F_{\beta_6}^{k_1} \dots F_{\beta_3}^{k_4} v(0)\}_{(k_1,\dots,k_4)\in\mathbb{Z}_+^4}$$

constitute a basis in $V(0)$. For this purpose we need the following theorem [1, p. 14]:

Theorem 4.9. *i) The monomials* $\{F_{\beta_6}^{k_1}F_{\beta_5}^{k_2}...F_{\beta_1}^{k_6}K_1^{m_1}K_2^{m_2}K_3^{m_3}E_{\beta_1}^{l_1}E_{\beta_2}^{l_2}...E_{\beta_6}^{l_6}\}$ *with* $(k_1,\dots,k_6)\in\mathbb{Z}_+^6,(m_1,m_2,m_3)\in\mathbb{Z}^3,(l_1,\dots,l_6)\in\mathbb{Z}_+^6$ *constitute a basis in* $U_q\mathfrak{g}$*;*

ii) for $i<j$ *one has:*

$$E_{\beta_i}E_{\beta_j}-q^{(\beta_i|\beta_j)}E_{\beta_j}E_{\beta_i}=\sum_{(k_1,\dots,k_6)\in\mathbb{Z}_+^6}a_{k_1,\dots,k_6}E_{\beta_1}^{k_1}E_{\beta_2}^{k_2}\dots E_{\beta_6}^{k_6},$$

where $a_{k_1,\dots,k_6}\neq 0$ *only when* $k_s=0$ *for* $s\le i$ *or* $s\ge j$*.*

It is not hard to prove that this theorem implies linear independence of the vectors $\{E_{\beta_6}^{k_1}\dots E_{\beta_3}^{k_4}\}_{(k_1,\dots,k_4)\in\mathbb{Z}_+^4}$.

Let us prove that the elements $\{F_{\beta_6}^{k_1}\dots F_{\beta_3}^{k_4}v(0)\}_{(k_1,\dots,k_4)\in\mathbb{Z}_+^4}$ constitute a basis in $V(0)$.

Obviously, $V(0)$ is the linear span of $\{F_{\beta_6}^{k_1}\dots F_{\beta_3}^{k_4}v(0)\}_{(k_1,\dots,k_4)\in\mathbb{Z}_+^4}$. Indeed, this follows from statement i) of Theorem 4.9, from the observation that the map $U_q\mathfrak{g}\to V(0)$, $\xi\mapsto\xi v(0)$ is surjective, and from the relations

$$E_{\beta_i}v(0)=0,\quad i=\overline{1,6},$$

$$K_iv(0)=v(0),\quad i=\overline{1,3},$$

$$F_{\beta_1}v(0)=F_{\beta_2}v(0)=0.$$

What remains is to prove that $\{F_{\beta_6}^{k_1}\dots F_{\beta_3}^{k_4}v(0)\}_{(k_1,\dots,k_4)\in\mathbb{Z}_+^4}$ are linearly independent. We prove this statement using its correctness for $q=1$.

Let A be the ring $\mathbb{C}[q^{1/4},q^{-1/4}]$, U_A^- the A-subalgebra in U_q^- generated by $\{F_i\}_{i=\overline{1,3}}$. Evidently, as $\mathbb{C}(q^{1/4})$-algebras

$$U_q^-\simeq\mathbb{C}(q^{1/4})\bigotimes_A U_A^-. \tag{4.12}$$

It is not hard to prove that $\{F_{\beta_i}\}_{i=\overline{1,6}}$ belong to U_A^-. Consider the U_A^--module $V(0)_A\stackrel{\text{def}}{=}U_A^-v(0)\subset V(0)$. Similarly to (4.12)

$$V(0)\simeq\mathbb{C}(q^{1/4})\bigotimes_A V(0)_A \tag{4.13}$$

as $\mathbb{C}(q^{1/4})$-modules.

There is a natural homomorphism of $\mathbb{C}$-algebras

$$J:U_A^-\to U_A^-/(q^{1/4}-1)\cdot U_A^-\simeq U^-,$$

where U^- is the subalgebra in the classical universal enveloping algebra $U\mathfrak{g}$ generated by $\{J(F_i)\}_{i=\overline{1,3}}$. Denote by $\widetilde{V(0)}$ a U^--module given by the generator $\widetilde{v(0)}$ and the relations

$$J(F_i)\widetilde{v(0)} = 0, \qquad i \neq 2.$$

It is clear that the map $v(0) \mapsto \widetilde{v(0)}$ can be extended up to a map

$$J_0 : V(0)_A \to \widetilde{V(0)}$$

such that for any $\xi \in U_A^-$

$$J(\xi)\widetilde{v(0)} = J_0(\xi v(0)), \tag{4.14}$$

(in particular, J_0 is a $\mathbb{C}$-linear map).

It is well known that the vectors $\{J(F_{\beta_6})^{k_1} \dots J(F_{\beta_3})^{k_4}\widetilde{v(0)}\}_{(k_1,\dots,k_4)\in\mathbb{Z}_+^4}$ constitute a basis in the $\mathbb{C}$-module $\widetilde{V(0)}$. Now it is an easy exercise to prove that the vectors $\{F_{\beta_6}^{k_1} \dots F_{\beta_3}^{k_4} v(0)\}_{(k_1,\dots,k_4)\in\mathbb{Z}_+^4}$ are linearly independent over A (and thus over $\mathbb{C}(q^{1/4})$).

We have completed the proof of Proposition 4.8. ■

Corollary 4.10. *Linear span of* $\{E_{\beta_6}^{k_1} \dots E_{\beta_3}^{k_4}\}_{(k_1,\dots,k_4)\in\mathbb{Z}_+^4}$ *is a* $U_q\mathfrak{g}_0$*-module subalgebra in* $U_q\mathfrak{g}$*. It coincides with* $T(\mathbb{C}[\mathfrak{g}_{-1}]_q)$*.*

P r o o f. By formula (4.11) the linear span is the image of T. By Proposition 4.3 the image is a subalgebra in $U_q\mathfrak{g}$. By Proposition 4.4 the image is an $\mathrm{ad}U_q\mathfrak{g}_0$-invariant subspace in $U_q\mathfrak{g}$. It remains to observe that any $\mathrm{ad}U_q\mathfrak{g}_0$-invariant subalgebra in $U_q\mathfrak{g}$ is a $U_q\mathfrak{g}_0$-module subalgebra. ■

R e m a r k 3. The fact that linear span of $\{E_{\beta_6}^{k_1} \dots E_{\beta_3}^{k_4}\}_{(k_1,\dots,k_4)\in\mathbb{Z}_+^4}$ is a subalgebra in U_q^+ easily follows from the statement ii) of the Theorem 4.9.

The main statement of this section is

Proposition 4.11. $T(\mathbb{C}[\mathfrak{g}_{-1}]_q) = \mathbb{C}[\mathfrak{g}_{-1}]_q^{\mathrm{I}}$.

P r o o f. Let us start with

Lemma 4.12. *For some* $c \in \mathbb{C}(q^{1/4})$

$$E_{\beta_6} = c \cdot E_2.$$

P r o o f o f t h e L e m m a. The elements E_{β_6}, E_2 of the U_q^0-module U_q^+ are weight vectors of the weight $\alpha_2 \in \mathfrak{h}^*$. But the subspace in U_q^+ of weight vectors of that weight is 1-dimensional (this follows from linear independence of the weights $\alpha_1, \alpha_2, \alpha_3 \in \mathfrak{h}^*$ of the generators E_1, E_2, E_3). ■

Equip the algebra U_q^+ with a $\mathbb{Z}_+$-grading as follows

$$\deg E_i = \begin{cases} 1, & i = 2, \\ 0, & \text{otherwise}, \end{cases} . \tag{4.15}$$

In other words,

$$\left(U_q^+\right)_j = \{\xi \in U_q^+ | \quad \mathrm{ad} K_0(\xi) = q^{2j}\xi\}. \tag{4.16}$$

Obviously,

$$\deg(E_{\beta_3}) = \deg(E_{\beta_4}) = \deg(E_{\beta_5}) = \deg(E_{\beta_6}) = 1.$$

It follows from (2.3), (4.16) that endomorphisms from $\mathrm{ad} U_q\mathfrak{g}_0$ preserve this grading: for $\xi \in U_q\mathfrak{g}_0$ and $\eta \in U_q^+$

$$\deg(\mathrm{ad}\xi(\eta)) = \deg(\eta)$$

provided $\mathrm{ad}\xi(\eta) \in U_q^+$. Using this observation and Corollary 4.10 we get

$$\mathrm{ad} U_q\mathfrak{g}_0(E_{\beta_6}) \subseteq \text{linear span}\{E_{\beta_3}, E_{\beta_4}, E_{\beta_5}, E_{\beta_6}\}. \tag{4.17}$$

Actually the spaces in the both sides of (4.17) coincide: dimension of $\mathrm{ad} U_q\mathfrak{g}_0(E_{\beta_6})$ is equal to 4 just as in the classical case $q = 1$. Thus, by Lemma 4.12 and by the definition of the algebra $\mathbb{C}[\mathfrak{g}_{-1}]_q^{\mathrm{I}}$

$$\mathbb{C}[\mathfrak{g}_{-1}]_q^{\mathrm{I}} = \langle E_{\beta_3}, E_{\beta_4}, E_{\beta_5}, E_{\beta_6}\rangle. \tag{4.18}$$

What remains is to use Corollary 4.10. ■

4.2. $\mathbb{C}[\mathfrak{g}_{-1}]_q^{\mathrm{I}} = \mathbb{C}[\mathfrak{g}_{-1}]_q^{\mathrm{II}}$

In this subsection we use the notation of the previous one.

Proposition 4.13. $\mathbb{C}[\mathfrak{g}_{-1}]_q^{\mathrm{I}} \subseteq \mathbb{C}[\mathfrak{g}_{-1}]_q^{\mathrm{II}}$.

P r o o f. By Corollary 4.10 and Proposition 4.11 the linear span of $\{E_{\beta_6}^{k_1}...E_{\beta_3}^{k_4}\}$ coincides with $\mathbb{C}[\mathfrak{g}_{-1}]_q^{\mathrm{I}}$. Thus, due to the definition (3.6) of $\mathbb{C}[\mathfrak{g}_{-1}]_q^{\mathrm{II}}$ it is sufficient to prove that

$$T_{w_0'}^{-1}(E_{\beta_k}) \in U_q^+, \qquad k = \overline{3,6}.$$

Let us prove this, for example, for E_{β_6}. By definition

$$E_{\beta_6} = T_1T_3T_2T_1T_3(E_2) = T_{w_0'}T_2T_1T_3(E_2).$$

One gets

$$T_{w_0'}^{-1}(E_{\beta_6}) = T_2T_1T_3(E_2).$$

It remains to make use of the following well known fact.

Lemma 4.14. *If $w \in \mathrm{W}$ and $w(\alpha_i) \in \Delta_+$ for some $i = \overline{1,3}$ then $T_w(E_i) \in U_q^+$.*

■

Now we are ready to prove

Proposition 4.15. $\mathbb{C}[\mathfrak{g}_{-1}]_q^{\mathrm{II}} \subseteq \mathbb{C}[\mathfrak{g}_{-1}]_q^{\mathrm{I}}$.

P r o o f. Let $\sum a_{k_1,\dots,k_6} E_{\beta_6}^{k_1}\dots E_{\beta_1}^{k_6} \in \mathbb{C}[\mathfrak{g}_{-1}]_q^{\mathrm{II}}$, i.e.,

$$\begin{aligned} & T_{w_0'}^{-1}\left(\sum a_{k_1,\dots,k_6} E_{\beta_6}^{k_1}\dots E_{\beta_1}^{k_6}\right) \\ = \quad & \sum a_{k_1,\dots,k_6} T_{w_0'}^{-1}\left(E_{\beta_6}^{k_1}\dots E_{\beta_3}^{k_4}\right) T_{w_0'}^{-1}\left(E_{\beta_2}^{k_5}\cdot E_{\beta_1}^{k_6}\right) \in U_q^+. \end{aligned} \tag{4.19}$$

By Proposition 4.13

$$T_{w_0'}^{-1}\left(E_{\beta_6}^{k_1}\dots E_{\beta_3}^{k_4}\right) \in U_q^+. \tag{4.20}$$

Lemma 4.16. $T_{w_0'}^{-1}(E_{\beta_2}) \in U_q^{\le 0}$, $T_{w_0'}^{-1}(E_{\beta_1}) \in U_q^{\le 0}$.

P r o o f o f t h e L e m m a. Suppose that β_k is a compact root ($\beta_k = \beta_1$ or $\beta_k = \beta_2$). Let $s_{i_1}s_{i_2}\dots s_{i_M}$ be a reduced expression of w_0' (of course, in our case $M = 2$ and there are only two different reduced expression for w_0'). One has

$$\beta_k = s_{i_1}s_{i_2}\dots s_{i_{k-1}}(\alpha_k), \qquad E_{\beta_k} = T_{i_1}T_{i_2}\dots T_{i_{k-1}}(E_k).$$

Since $T_{w_0'} = T_{i_1}T_{i_2}\dots T_{i_M}$ we get

$$T_{w_0'}^{-1}(E_{\beta_k}) = T_{i_M}^{-1}T_{i_{M-1}}^{-1}\dots T_{i_k}^{-1}(E_k).$$

So we have to prove that

$$T_{i_M}^{-1}T_{i_{M-1}}^{-1}\dots T_{i_k}^{-1}(E_k)\in U_q^{\leq 0}. \tag{4.21}$$

Consider the antiautomorphism τ of the algebra $U_q\mathfrak{g}$ given by

$$\tau(K_i)=K_i^{-1},\quad \tau(E_i)=E_i,\quad \tau(F_i)=F_i.$$

It is not hard to prove that

$$\tau\circ T_i=T_i^{-1}\circ\tau.$$

Thus the inclusion

$$T_{i_M}T_{i_{M-1}}\dots T_{i_k}(E_k)\in U_q^{\leq 0} \tag{4.22}$$

is equivalent to (4.21). One has $T_{i_k}(E_k)=-F_kK_k$. Therefore (4.22) is equivalent to

$$T_{i_M}T_{i_{M-1}}\dots T_{i_{k+1}}(F_k)\cdot T_{i_M}T_{i_{M-1}}\dots T_{i_{k+1}}(K_k)\in U_q^{\leq 0}$$

and thus to

$$T_{i_M}T_{i_{M-1}}\dots T_{i_{k+1}}(F_k)\in U_q^{\leq 0}. \tag{4.23}$$

Applying the antiautomorphism k (see Section 2) to (4.23) we get the equivalent inclusion

$$T_{i_M}T_{i_{M-1}}\dots T_{i_{k+1}}(E_k)\in U_q^{\geq 0}. \tag{4.24}$$

But (4.24) is a direct consequence of Lemma 4.14. ∎

The following result is well known.

Lemma 4.17. *The multiplication in $U_q\mathfrak{g}$ induces the isomorphism of vector spaces*

$$U_q^{+}\bigotimes U_q^{\leq 0}\to U_q\mathfrak{g}.$$

It follows from (4.20) and Lemmas 4.16, 4,17 that (4.19) holds iff $a_{k_1,\dots,k_6}=0$ for $k_5\neq 0$ or $k_6\neq 0$. So, by (4.18) $\sum a_{k_1,\dots,k_6}E_{\beta_6}^{k_1}\dots E_{\beta_1}^{k_6}\in\mathbb{C}[\mathfrak{g}_{-1}]_q^{\mathrm{I}}$ provided $\sum a_{k_1,\dots,k_6}E_{\beta_6}^{k_1}\dots E_{\beta_1}^{k_6}\in\mathbb{C}[\mathfrak{g}_{-1}]_q^{\mathrm{II}}$. We have proved Proposition 4.15 and, thus, Theorem 4.1. ∎

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О q-аналогах некоторых предоднородных векторных пространств: сравнение нескольких подходов

Д. Шкляров

Существует несколько подходов к построению q-аналогов предоднородных векторных пространств коммутативного параболического типа. В настоящей работе сравниваются три подхода, развитые Х.П. Джакобсеном, Т. Танисаки и др., Л. Ваксманом и др. В рамках этих трех подходов решена следующая задача: построен q-аналог алгебры $\mathbb{C}[V]$ голоморфных полиномов на произвольном неприводимом предоднородном векторном пространстве V (коммутативного параболического типа), и, более того, соответствующая (некоммутативная) алгебра наделена структурой U-*модульной* алгебры, где U — некоторая квантовая универсальная обертывающая алгебра. Мы доказываем, что три q-аналога алгебры $\mathbb{C}[V]$ изоморфны как U-модульные алгебры.

Для простоты рассматривается только случай, когда V — пространство комплексных матриц второго порядка. Но приводимое доказательство переносится на случай произвольного неприводимого предоднородного векторного пространства коммутативного параболического типа.

Про q-аналоги деяких предоднорідних векторних просторів: порівняння декількох підходів

Д. Шкляров

Існує декілька підходів до побудування q-аналогів предоднорідних векторних просторів комутативного параболічного типу. В цій роботі порівнюються три підходи, які розвинуто Х.П. Джакобсеном, Т. Танісакі та інш., Л. Ваксманом та інш. В рамках цих трьох підходів вирішено наступну задачу: побудовано q-аналог алгебри $\mathbb{C}[V]$ голоморфних поліномів на довільному незвідному предоднорідному векторному просторі V (комутативного параболічного типу), та, більш того, відповідну (некомутативну) алгебру наділено структурою U-*модульної* алгебри, де U — деяка квантова універсальна огортуюча алгебра. Ми доводимо, що три q-аналоги алгебри $\mathbb{C}[V]$ є ізоморфними як U-модульні алгебри.

Для простоти розглянуто тільки випадок, коли V — простір комплексних матриць другого порядку. Але ми наводимо доведення, яке можна перенести на випадок довільного незвідного предоднорідного векторного простору комутативного параболічного типу.