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On non-quasianalytic representations of Abelian groups

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We study representations T_g of a locally compact Abelian group G with a scattered spectrum satisfying the conditions: there exists $S \subset G$ such that $G = S - S$ and for all $s \in S$

$$||T_{ns}|| = o(n^k),\ k \geq 1,\ \ln ||T_{-ns}|| = o(\sqrt{n}),\ \text{as } n \to +\infty.$$

Let G be a locally compact Abelian group, G^* be its character group, T be a strongly continuous representation of G by bounded linear operators on a Banach space X. We will only consider non-quasianalytic representations, i.e., those for which the function $\omega(g) := \|T(g)\|$ satisfies the condition

$$\sum_{n=-\infty}^{\infty} \frac{\log \omega(ng)}{1+n^2} < \infty \text{ for all } \ g \in G.$$

Following Lyubich [L1], for each representation T one can define the *approximate point spectrum* $\sigma_a(T)$ by $\sigma_a(T) := \{\chi \in G^* :$ there exists a net x_α, such that $\|x_\alpha\| = 1,\ \|T(g)x_\alpha - \chi(g)x_\alpha\| \to 0\}$ for all $g \in G$ and the *spectrum* $\sigma(T)$ by $\sigma(T) := \{\chi \in G^* : \widehat{f}(\chi) = 0$ whenever $\widehat{f}(T) = 0\}$. The spectrum can be identified with the Gelfand space $\Delta_{\mathcal{A}_T}$ of the Banach algebra $\mathcal{A}_T$ generated by $\widehat{f}(T),\ f \in L^1_\omega(G)$ via the natural homeomorphism: $\psi(\widehat{f}(T)) := \widehat{f}(\chi), \psi \in \Delta_{\mathcal{A}_T},\ \chi \in \sigma(T)$.

The general theory of non-quasianalytic representations of locally compact Abelian groups was constructed in [L-M-F] (see also [D]). Below we briefly recall basic facts about this theory that will be used in the sequel.

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First, if the representation T is uniformly continuous, then $\sigma_a(T)$ is a non-empty compact subset of G^*. Moreover, the following *extension theorem* holds: if $\lambda_0 \in \sigma_a(T(g_0))$ for some $g_0 \in G$, where $\sigma_a(T(g_0))$ is the usual approximate point spectrum of the linear operator $T(g_0)$, then there exists $\chi \in \sigma_a(T)$ such that $\chi(g_0) = \lambda_0$. Note that both the above theorems on non-emptyness of spectrum and the extension theorem were proved in [L1] for uniformly continuous representations of arbitrary groups and, in fact, for representations of arbitrary topological semigroups (see [D-L]).

The main result in [L-M-F] is that the spectrum of (strongly continuous) non-quasianalytic representations is *separable* [decomposable]. In particular, for every compact set $Q \subset G^*$ there exists a maximal spectral subspace $L(Q)$ which is invariant for $T(g)$ and such that $T_Q(g) := T(g)|L(Q)$ is a uniformly continuous representation on $L(Q)$ and $\sigma(T_Q) \subset Q$ (thus, T is uniformly continuous if and only if $\sigma(T)$ is compact), and if U_α is an open cover of $\sigma(T)$ such that $\overline{U_\alpha}$ is compact for every α, then the family $L(\overline{Q_\alpha})$ is complete.

It should be noted that $\chi \in \sigma(T)$ if and only if $|\widehat{f}(\chi)| \leq \|\widehat{f}(T)\|$ for all $f \in L^1_\omega(G)$ (this is immediate from the spectral mapping theorem in [L-M-F]). Thus, $\sigma(T)$ coincides with the spectrum of a semigroup representation as defined in [B-V]. Therefore, the spectrum $\sigma(T)$ coincides with $\sigma_a(T)$ for non-quasianalytic representations, as is shown in [B-V, Proposition 2.5] (the result in [B-V] was stated for Shilov's boundary Γ_T only, but since $\mathcal{A}_T$ is regular, it is well known that $\Gamma_T = \sigma(T)$, so that it remains true for arbitrary non-quasianalytic representations of G). Therefore, in the above theory of decomposable spectrum of non-quasianalytic representations, it doesn't matter whether we consider the spectrum $\sigma(T)$ or $\sigma_a(T)$ (note that this answers affirmatively a question in [L2], Ch. V, Sect. 2). In particular, the extension theorem holds for $\sigma(T)$ as well.

Atzmon proved in [A] the following theorem.

Theorem (Atzmon [A]). *If T is an invertible operator on a Banach space X such that $\sigma(T) = \{1\}$, $\|T^n\| = o(n^k)$, $k \geq 1$, and $\ln \|T^{-n}\| = o(\sqrt{n})$ as $n \to +\infty$, then $(T - I)^k = 0$.*

Using this result and the extension theorem for uniformly continuous representations we prove

Proposition 1. *If T is a strongly continuous representation of G such that $\sigma(T) = \{\chi_0\}$, and if $\omega(g) := \|T(g)\|$ satisfies the following condition: there exists a semigroup $S \subset G$ such that $G = S - S$ and for all $s \in S$*

$$\|\omega(ns)\| = o(n^k), \ k \geq 1, \ \ln \|\omega(-ns)\| = o(\sqrt{n}), \ \text{as } n \to +\infty, \qquad (1)$$

then $(T(g) - \chi_0(g)I)^k = 0$ for all $g \in G$.

R e m a r k 1. Atzmon proved in [A] also the following theorem. Let T be an invertible operator on a Banach space X such that $n||(T-I)^n||^{\frac{1}{n}} \to 0$ as $n \to +\infty$ and $||T^n|| = o(n^k)$ as $n \to +\infty$ then $(T-I)^k = 0$. Using the same reasoning as in the proof of Proposition 1 one can prove the following group generalisation of Atzmon's theorem.

If T is a strongly continuous representation of G such that $\sigma(T) = \{\chi_0\}$, and if $\omega(g) := \|T(g)\|$ satisfies the following condition: there exists a semigroup $S \in G$ such that $G = S - S$ and for all $s \in S$

$$n||(T(s) - \chi_0(s)I)^n||^{\frac{1}{n}} \to 0 \quad \text{and} \quad \|\omega(ns)\| = o(n^k),\ k \geq 1, \quad \text{as } n \to +\infty,$$

then $(T(g) - \chi_0(g)I)^k = 0$ for all $g \in G$.

We now use the above result in order to obtain a new fact for functions on G whose Beurling spectrum is a single point.

Let $a = a(g) \in L^\infty_\omega(G)$, denote by $\sigma(a)$ its Beurling spectrum and consider the subspace $L_a \subset L^\infty_\omega(G)$ which is spanned by translations of a. Consider the representation V of G on L_a given by $(V(g)b)(h) = b(g+h),\ b = b(g) \in L_a$. We assume that $\sigma(a)$ is compact, so that V is a strongly continuous (in fact, uniformly continuous) representation. Using the definitions of the Beurling spectrum of the function and the spectrum of representation it is easy to see that $\sigma(V) = \sigma(a)$.

Recall that a function $b = b(g)$ is a polynomial of order $< k$ on G if

$$\sum_{j=0}^{k} (-1)^j C_k^j b(h + jg) = 0$$

for all $h,\ g \in G$.

Proposition 2. *Let $a = a(g) \in L^\infty_\omega(G)$, where the weight $\omega(g)$ satisfies the conditions* (1). *Then $\sigma(a) = \{\chi_0\}$ if and only if $\chi_0(-g)a(g)$ is a polynomial of degree $< k$.*

R e m a r k 2. Let us take note of the fact that some researches were published lately where both the representations with $\omega(g) := \|T(g)\|$ satisfying (1), and the space $L^\infty_\omega(G)$ itself were studied (see [H], [H-N-R]). We refer the reader to [Z] for an exhaustive review of the results devoted to the operators T with the only point of spectrum $\sigma(T) = \{1\}$.

Let $T : G \to \mathrm{Aut}(X)$ be a strongly continuous representation such that the weight $\omega(g) := \|T(g)\|$ satisfies the conditions: there exists a subset $S \in G$ such that $G = S - S$ and for all $s \in S$

$$\omega(ns) = o(n), \quad \ln \omega(-ns) = o(\sqrt{n}), \text{ as } \ n \to +\infty. \tag{2}$$

Denote by $\mathcal{A}_T$ the Banach algebra generated by $\widehat{f}(T),\ f \in L^1_\omega(G)$.

Theorem 1. *Let $T : G \to \mathrm{Aut}(X)$ be a strongly continuous representation satisfying conditions* (2) *and assume that $\sigma(T)$ is a scattered set. Then the algebra $\mathcal{A}_T$ generated by $\widehat{f}(T)$ is semisimple.*

To prove Theorem 1 we need some preliminary lemmas. For each element $a \in \mathcal{A}_T$, let

$$I_a := \{b \in \mathcal{A}_T : ba = 0\}.$$

Then I_a is a closed ideal of $\mathcal{A}_T$, and we define the spectrum of a, $Sp(a)$, by

$$Sp(a) = h(I_a),$$

where $h(J)$ is the hull of the ideal J.

Lemma 1. *(i) $Sp(a) = \emptyset$ if and only if $a = 0$.*
(ii) $Sp(ab) \subset Sp(a) \cap Sp(b)$ for each $a, b \in \mathcal{A}_T$.

We may assume, without loss of generality, that $\overline{span}\{\widehat{f}(T)x : f \in L^1_\omega(G), x \in X\}$ is dense in X. This implies that the following holds.

Lemma 2. *If $f \in L^1_\omega(G)$ and $\widehat{f} = 1$ on a neighborhood of $\sigma(T)$, then $\widehat{f}(T) = I$.*

For a subset $U \subset \sigma(T)$, let

$$J(U) := \{a \in \mathcal{A}_T : Sp(a) \subset U\}$$

Lemma 3. *If U_1, U_2 is an open cover of $\sigma(T)$, then $\mathcal{A}_T = J(U_1) + J(U_2)$.*

For every $\gamma \in G^*$ we define

$$X\{\gamma\} := \{x \in X : T(g)x = \gamma(g)x \text{ for all } \ g \in G\}.$$

Lemma 4 (compare with [M-V]). *Assume $a \in \mathcal{A}_T$ and $Sp(a) = \{\gamma_0\}$. Then $X\{\gamma_0\} \neq \{0\}, \overline{aX} \subset X\{\gamma_0\}$, and there is $\lambda_0 \neq 0$ such that $ax = \lambda_0 x$ for all $x \in X\{\gamma_0\}$.*

We add some remarks about a new proof of the above mentioned Atzmon theorem. This proof is based on Faber's theorem. It seems to us that the application of Faber's theorem can be useful in the studying the operators in Banach space. This approach allow us from one hand to use well-known results from function theory and from the other hand to avoid estimating the resolvent of the operator at a neighborhood of the spectrum.

Theorem (Faber, [Bi]). *The function*

$$f(z)=\sum_{n=0}^{\infty} a_n z^n$$

can be extended to the complex plane as an entire function of the order ρ *of* $\frac{1}{1-z}$, *i.e.* $f(z)=G\left(\frac{1}{1-z}\right)$, *where* $G(\lambda)$ *is an entire function if and only if there exists an entire function* $A(\lambda)$ *of the order* σ, $0\le\sigma<1$ *such that*

$$a_n=A(n), \quad n=0,\ 1,\ 2\dots .$$

The orders ρ *and* σ *are connected by the equality*

$$\rho=\frac{\sigma}{1-\sigma}.$$

Moreover $G(\lambda)$ *has the minimal, normal or maximal type if and only if the same type has* $A(z)$.

The point is that the resolvent $R_z(T^{-1})=(zI-T^{-1})^{-1}$ of the operator T^{-1} is an entire function of $\frac{1}{1-z}$. This implies that for arbitrary $x\in X,\ x^*\in X^*$ the function $f(z)=x^*(R_z(T^{-1})x)$ is an entire function of $\frac{1}{1-z}$ as well. Using the conditions of Atzmon's theorem one can get the estimate of the growth of $f(z)$, to apply Faber's theorem and to receive the wanted conclusion.

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