Matematicheskaya fizika, analiz, geometriya
2002, v. 9, No. 2, p. 117–127

The Stokes structure in asymptotic analysis I: Bessel, Weber and hypergeometric functions

V. Gurarii[1], V. Katsnelson[2], V. Matsaev[3], and J. Steiner[4]

[1]*School of Mathematical Sciences, Swinburne University of Technology PO Box 218 Hawthorn VIC 3122 Melbourne, Australia*
E-mail:vgurarii@swin.edu.au

[2]*Department of Mathematics, the Weizmann Institute of Szience Rehovot, 76100, Israel*
E-mail:katze@wisdom.weizmann.ac.il

[3]*School of Mathematical Sciences, Tel Aviv University Ramat Aviv Tel Aviv, 69978, Israel*
E-mail:matsaev@post.tau.ac.il

[4]*Department of Applied Mathematics, JCT-Jerusalem College of Technology 21 Havaad Haleumi St., POB 16031, Jerusalem, 91160, Israel* *
E-mail:steiner@mail.jct.ac.il

Received January 22, 2002
Communicated by I.V. Ostrovskii

This is the first in a series of four papers which are entitled:

I. The Stokes structure in asymptotic analysis I: Bessel, Weber and hypergeometric functions.

II. The Stokes structure in asymptotic analysis II: generalized Fourier (Borel)–Laplace transforms.

III. The Stokes structure in asymptotic analysis III: remainders and principle of functional closure.

IV. The Stokes structure in asymptotic analysis IV: Stokes' phenomenon and connection coefficients.

They introduce a methodology for the asymptotic analysis of differential equations with polynomial coefficients which also provides a further insight into the Stokes' phenomenon. This approach consists of a chain of steps based on the concept of the Stokes structure an algebraic-analytic structure,

Mathematics Subject Classification 2000: 34M25, 34M30, 34M37, 34M40.

*The work was completed when the fourth named author was working at the School of Mathematical Sciences, Swinburne University of Technology, Melbourne, Australia.

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the idea of which emerges naturally from the monodromic properties of the Gauss hypergeometric function, and which can be treated independently of the differential equations, and Fourier-like transforms adjusted to this Stokes structure. Every step of this approach, together with all its exigencies, is illustrated by means of the non-trivial treatment of Bessel's and Weber's differential equations. It will be the aim of our future series of papers to extend this approach to matrix differential equations.

It is our great pleasure to publish this series of papers in our home town and to dedicate it to the memory of our dear teacher, Naum Il'ich Akhiezer, who taught us the basic knowledge of the theory of transcendental functions and inculcated in us the taste and the love for this theory.

In honor of the 100th birthday of Naum Il'ich Akhiezer

1. Introduction

We introduce our principal subject using the three well known function pairs: (i) the incomplete Gamma function $\Gamma(p,z)$ and the constant function $\mathbf{1}$, where

$$\Gamma(p,z) = \int_z^{+\infty} e^{-t} t^{p-1} dt,\ z \neq 0; \tag{1}$$

(ii) the Bessel functions of third kind or the Hankel functions $H_\nu^{(1)}(z), H_\nu^{(2)}(z)$

$$H_\nu^{(j)}(z) = \frac{\Gamma\left(\frac{1}{2}-\nu\right)\left(\frac{z}{2}\right)^\nu}{\pi^{3/2} i} \int_{\gamma_j} e^{izt}\left(t^2-1\right)^{\nu-\frac{1}{2}} dt,\ |\arg z| < \frac{\pi}{2},\ z \neq 0, j = 1,2, \tag{2}$$

with $\gamma_1,\ \gamma_2$ simple loops bypassing $t = \pm 1$ but not enclosing $t = \mp 1$, respectively, and $\nu \neq \frac{1}{2}, \frac{3}{2}, \ldots$;

(iii) the Weber functions $D_{E-\frac{1}{2}}\left(\sqrt{2}z\right), D_{-E-\frac{1}{2}}\left(\sqrt{2}ze^{-\frac{\pi i}{2}}\right)$

$$D_{E-\frac{1}{2}}\left(\sqrt{2}z\right) = -\frac{\Gamma\left(E+\frac{1}{2}\right)}{2\pi i} e^{-\frac{z^2}{2}} 2^{\frac{E}{2}-\frac{1}{4}} \int_\gamma e^{-zt-\frac{t^2}{4}} (-t)^{-E-\frac{1}{2}} dt,\ |\arg z| < \frac{\pi}{2} \tag{3}$$

with γ the simple loop starting at $+\infty$ on the real axis, circling the origin in the counterclockwise direction and returning to the starting point, see, for example, [3, 8].

These function pairs are respectively linearly independent solutions of:

(i) the incomplete Gamma differential equation

$$u''(z) + \left(1 - \frac{p-1}{z}\right) u'(z) = 0; \tag{4}$$

(ii) Bessel's differential equation

$$u''(z)+\frac{1}{z}u'(z)+\left(1-\frac{\nu^2}{z^2}\right)u(z)=0; \tag{5}$$

(iii) Weber's differential equation

$$u''(z)+\left(2E-z^2\right)u(z)=0. \tag{6}$$

Note, that equations (4), (5) have $z=0$ as their only regular singular point. However, all three equations have $z=\infty$ as their only irregular singular point, hence all solutions have exponential behavior at $z=\infty$.

In fact, it follows from (1)–(3) that

$$\text{(i)}\quad \begin{aligned}&\Gamma(p,z)=e^{-z}z^{p-1}(1+o(1)),\ |z|\to\infty,-\tfrac{3\pi}{2}<\arg z<\tfrac{3\pi}{2};\\ &\mathbf{1}=(1+o(1)),\ z\to\infty;\end{aligned} \tag{7}$$

$$\text{(ii)}\quad \begin{aligned}&H_\nu^{(1)}(z)=\left(\tfrac{2}{\pi z}\right)^{1/2}e^{-i(z-\nu\pi/2-\pi/4)}(1+o(1));\\ &\qquad\qquad |z|\to\infty,\ -2\pi<\arg z<\pi,\\ &H_\nu^{(2)}(z)=\left(\tfrac{2}{\pi z}\right)^{1/2}e^{\,i(z-\nu\pi/2-\pi/4)}(1+o(1)),\\ &\qquad\qquad |z|\to\infty,\ -\pi<\arg z<2\pi;\end{aligned} \tag{8}$$

$$\text{(iii)}\quad \begin{aligned}&D_{E-\frac12}\left(\sqrt{2}z\right)=2^{\frac{E}{2}-\frac14}z^{E-\frac12}e^{-\frac{z^2}{2}}(1+o(1)),\\ &\qquad\qquad |z|\to\infty,\ -\tfrac{3\pi}{4}<\arg z<\tfrac{3\pi}{4},\\ &D_{-E-\frac12}\left(\sqrt{2}ze^{-\frac{\pi i}{2}}\right)=2^{-\frac{E}{2}-\frac14}e^{\frac{\pi i}{2}\left(E+\frac12\right)}z^{-E-\frac12}e^{\frac{z^2}{2}}(1+o(1)),\\ &\qquad\qquad |z|\to\infty,\ \ -\tfrac{\pi}{4}<\arg z<\tfrac{5\pi}{4}.\end{aligned} \tag{9}$$

It is worthwhile noting that firstly, the sectors shown in (8), (9) are wider than those in (2), (3) respectively and secondly, the sectors indicated in (7)–(9) are the widest possible sectors in which the above relations are valid.

Two immediate questions arise:

(a) How does one decode properly the symbol $o(1)$ in the above relationships. In fact, the symbols $o(1)$ in each pair of relations (7)–(9) conceals a formal power series in z^{-1} and the logic of our approach is to reveal this formal series to uncover simultaneously their true relation to regular solutions.

(b) What is the behavior of all these functions outside the indicated sectors.

Of course, one could answer these questions in the traditional way using the above integral representations (1)–(3). However, these representations do not provide the means for a deeper unified understanding of the related questions of asymptotic and resurgent analysis for the above differential equations, and moreover such representations are not generally available except for special differential equations. We thus need another point of view.

2. Monodromic relations

For this purpose we have derived other integral representations of these functions solely in terms of Gauss hypergeometric function $F(a, b, c, z)$.

Theorem 1. *The following integral representations are valid in the indicated sectors:*

(i)
$$\Gamma(p, z) = e^{-z} z^p \int\limits_0^{+\infty} e^{-z\xi} F(1-p, 1; 1; -\xi)\, d\xi, \qquad -\tfrac{\pi}{2} < \arg z < \tfrac{\pi}{2}, \quad z \neq 0; \tag{10}$$

(ii)
$$H_\nu^{(1)}(z) = (2z/\pi)^{1/2} e^{-i((z-\nu\pi/2-\pi/4))} \int\limits_0^{+\infty} e^{-z\xi} F\left(\tfrac{1}{2}-\nu, \tfrac{1}{2}+\nu; 1; -\tfrac{\xi}{2i}\right) d\xi, \qquad -\tfrac{\pi}{2} < \arg z < \tfrac{\pi}{2}, \quad z \neq 0, \tag{11}$$

$$H_\nu^{(2)}(z) = (2z/\pi)^{1/2} e^{+i(z-\nu\pi/2-\pi/4)} \int\limits_0^{+\infty} e^{-z\xi} F\left(\tfrac{1}{2}-\nu, \tfrac{1}{2}+\nu; 1; +\tfrac{\xi}{2i}\right) d\xi, \qquad -\tfrac{\pi}{2} < \arg z < \tfrac{\pi}{2}, \quad z \neq 0; \tag{12}$$

(iii)
$$D_{E-\frac{1}{2}}\left(\sqrt{2}z\right) = 2^{\frac{E}{2}+\frac{3}{4}} z^{E+\frac{3}{2}} e^{-\frac{z^2}{2}} \int\limits_0^{+\infty} e^{-(\xi z)^2} F\left(-\tfrac{E}{2}+\tfrac{1}{4}, -\tfrac{E}{2}+\tfrac{3}{4}; 1; -\xi^2\right) \xi d\xi, \qquad -\tfrac{\pi}{4} < \arg z < \tfrac{\pi}{4}, \quad z \neq 0, \tag{13}$$

$$D_{-E-\frac{1}{2}}\left(\sqrt{2}ze^{-\frac{\pi i}{2}}\right) = 2^{-\frac{E}{2}+\frac{3}{4}} e^{\frac{\pi i}{2}\left(E+\frac{1}{2}\right)} z^{-E+\frac{3}{2}} e^{\frac{z^2}{2}} \int\limits_0^{+\infty} e^{-(\xi z)^2} F\left(\tfrac{E}{2}+\tfrac{1}{4}, \tfrac{E}{2}+\tfrac{3}{4}; 1; +\xi^2\right) \xi d\xi, \qquad \tfrac{\pi}{4} < \arg z < \tfrac{3\pi}{4}, \quad z \neq 0. \tag{14}$$

All the above representations contain respectively an exponential factor times the Laplace (or generalized Laplace) transforms of the Gauss hypergeometric functions:

(i) $F(1-p,1;1;-\xi) = (1+\xi)^{p-1}$,
(ii) $F\left(\frac{1}{2}-\nu, \frac{1}{2}+\nu; 1; \mp\frac{\xi}{2i}\right)$,
(iii) $F\left(\mp\frac{E}{2}+\frac{1}{4}, \mp\frac{E}{2}+\frac{3}{4}; 1; \mp\xi^2\right)$.

Interestingly, although these representations are generally unavailable in the literature*, nevertheless for us they provide the primary clue for our investigation to follow.

Applying the well known monodromic property of the Gauss hypergeometric function

$$F\left(a,b;c;1+(\xi-1)e^{\pm 2i\pi}\right) = F(a,b;c;\xi) \mp \frac{2i\pi\Gamma(c)\,e^{\pm i\pi(c-a-b)}}{\Gamma(a)\Gamma(b)\Gamma(1+c-a-b)}(1-z)^{c-a-b}F(c-a,c-b;1+c-a-b;1-\xi) \tag{15}$$

(see, for example, [10], [11]) to (10)–(14) one can obtain the corresponding monodromic-algebraic relations for the above transcendental functions.

Theorem 2. *The incomplete Gamma function* $\Gamma(p,z)$, *Hankel functions* $H_\nu^{(1)}(z), H_\nu^{(2)}(z)$, *and Weber functions* $D_{E-\frac{1}{2}}\left(\sqrt{2}z\right), D_{-E-\frac{1}{2}}\left(\sqrt{2}ze^{-\frac{\pi i}{2}}\right)$ *have the following monodromic properties:*

(i)
$$\Gamma\left(p, ze^{2\pi i}\right) = e^{2\pi i p}\Gamma(p,z) + (-1)^{1-p}\frac{2\pi i}{\Gamma(1-p)}\,; \tag{16}$$

(ii)
$$\begin{aligned} H_\nu^{(1)}\left(ze^{2\pi i}\right) &= -H_\nu^{(1)}(z) - 2e^{+i\nu\pi}\cos(\nu\pi)\,H_\nu^{(2)}\left(ze^{2\pi i}\right)\,, \\ H_\nu^{(2)}\left(ze^{2\pi i}\right) &= -H_\nu^{(2)}(z) - 2e^{-i\nu\pi}\cos(\nu\pi)\,H_\nu^{(1)}(z)\,; \end{aligned} \tag{17}$$

(iii)
$$\begin{aligned} &D_{E-\frac{1}{2}}\left(\sqrt{2}z\right) \\ &= e^{+\pi i\left(E-\frac{1}{2}\right)}D_{E-\frac{1}{2}}\left(\sqrt{2}ze^{-\pi i}\right) + T_1 2^{+E}e^{-\frac{\pi i}{2}\left(E+\frac{1}{2}\right)}D_{-E-\frac{1}{2}}\left(\sqrt{2}ze^{-\frac{\pi i}{2}}\right)\,, \\ &D_{-E-\frac{1}{2}}\left(\sqrt{2}ze^{-\frac{\pi i}{2}}\right) \\ &= e^{-\pi i\left(E+\frac{1}{2}\right)}D_{-E-\frac{1}{2}}\left(\sqrt{2}ze^{-\frac{3\pi i}{2}}\right) + T_2 2^{-E}e^{\frac{3\pi i}{2}E-\frac{\pi i}{4}}D_{E-\frac{1}{2}}\left(\sqrt{2}ze^{-\pi i}\right)\,, \end{aligned} \tag{18}$$

$$T_1 = i\sqrt{\pi}\frac{e^{i\pi E}2^{\frac{1}{2}-E}}{\Gamma\left(\frac{1}{2}-E\right)}\,, \quad T_2 = i\sqrt{\pi}\frac{e^{-i\pi E}2^{\frac{1}{2}+E}}{\Gamma\left(\frac{1}{2}+E\right)}e^{-i\pi E}.$$

* Formulae similar to (11)–(14) were discovered comparatively recently by Marichev [9] (see, also [5]).

Of course, the relations (16)–(18) provide immediately by recurrence the behavior of all these functions as $z \to \infty$ for all z outside the indicated sectors in (10)–(14).

More importantly, we shall regard these algebraic relations as the essence of the *Stokes structure* for the differential equations (4)–(6). We shall consider this structure as an independent object, the primary basis for our investigations.

3. The Stokes structure

We represent all functions pairs (1)–(3) in phase-amplitude form:

(i)
$$\Gamma(p,z) = e^{-z} z^{p-1} P(z) \; ; \tag{19}$$

(ii)
$$\begin{aligned} H_\nu^{(1)}(z) &= \left(\tfrac{2}{\pi z}\right)^{1/2} e^{-i(z-\nu\pi/2-\pi/4)} P_1(z) \; , \\ H_\nu^{(2)}(z) &= \left(\tfrac{2}{\pi z}\right)^{1/2} e^{+i(z-\nu\pi/2-\pi/4)} P_2(z) \; ; \end{aligned} \tag{20}$$

(iii)
$$\begin{aligned} D_{E-\frac{1}{2}}\left(\sqrt{2}z\right) &= 2^{+\frac{E}{2}-\frac{1}{4}} z^{+E-\frac{1}{2}} e^{-\frac{z^2}{2}} Q_1(z) \; , \\ D_{-E-\frac{1}{2}}\left(\sqrt{2}z e^{\frac{\pi i}{2}}\right) &= 2^{-\frac{E}{2}-\frac{1}{4}} e^{\frac{\pi i}{2}\left(E+\frac{1}{2}\right)} z^{-E-\frac{1}{2}} e^{\frac{z^2}{2}} Q_2(z) \; . \end{aligned} \tag{21}$$

Using (19)–(21) we can rewrite the monodromic relations (16)–(18) as

$$\begin{aligned} P\left(z e^{2\pi i}\right) &= P(z) + (-1)^{1-p} \frac{2\pi i}{\Gamma(1-p)} e^{-2\pi i p} e^z z^{1-p} \times 1 \, , \\ 1 &= 1 \qquad + 0 \times P(z) \; , \end{aligned} \tag{22}$$

$$\begin{aligned} P_1(z e^{2\pi i}) &= P_1(z) + 2i\left(\cos\nu\pi\right) e^{+2iz} P_2(z e^{2\pi i}) \, , \\ P_2(z e^{2\pi i}) &= P_2(z) + 2i\left(\cos\nu\pi\right) e^{-2iz} P_1(z) \, , \end{aligned} \tag{23}$$

$$\begin{aligned} Q_1\left(z e^{\pi i}\right) &= Q_1(z) + T_1 e^{-2\pi i E} z^{-2E} e^{+z^2} Q_2\left(z e^{\pi i}\right) \; , \\ Q_2\left(z e^{\pi i}\right) &= Q_2(z) + T_2 e^{+2\pi i E} z^{+2E} e^{-z^2} Q_1(z) \; , \end{aligned} \tag{24}$$

respectively, where T_1 and T_2 are defined in (18).

For the sake of uniformity we perform the change of variable

$$z \to e^{-\frac{\pi i}{4}} \sqrt{2z}$$

to reduce Weber's differential equation (6) to the reduced Weber's equation

$$u''(z) + \frac{1}{2z} u'(z) + \left(-E\frac{i}{2z} + 1\right) u(z) = 0 \tag{25}$$

and rewrite (24) in a form similar to (22), (23)

$$\begin{aligned} \mathcal{P}_1\left(ze^{2\pi i}\right) &= \mathcal{P}_1(z) + T_1(E)\, z^{-E} e^{2iz}\, \mathcal{P}_2\left(ze^{2\pi i}\right) , \\ \mathcal{P}_1\left(ze^{2\pi i}\right) &= \mathcal{P}_1(z) + T_2(E)\, z^{E} e^{-2iz}\, \mathcal{P}_1(z) , \end{aligned} \tag{26}$$

where $\mathcal{P}_1(z)$, $\mathcal{P}_2(z)$ are the phase amplitudes of $D_{E-\frac{1}{2}}\left(2e^{-\frac{\pi i}{4}}\sqrt{z}\right)$, $D_{-E-\frac{1}{2}}\left(2e^{\frac{\pi i}{2}}e^{-\frac{\pi i}{4}}\sqrt{z}\right)$, respectively.

It follows from differential equations (4), (5), (25) that *the phase amplitudes* $P(z)$*;* $P_1(z), P_2(z)$ *;* $\mathcal{P}_1(z), \mathcal{P}_2(z)$ *are analytic on the Riemann surface of* $\log z$

$$\{z : 0 < |z| < \infty,\ -\infty < \arg z < \infty\}$$

all with at most exponential growth at ∞, since equation (4) is a perturbation of $u'' + u' = 0$ with solutions $e^{-z}, 1$ and equations (5), (25) are perturbations of $u'' + u = 0$ with solutions e^{+iz}, e^{-iz}.

Moreover, $P(z)$ is a bounded function in closed subsectors of

$$\{z : -\frac{3\pi}{2} < \arg z < \frac{3\pi}{2},\quad 0 < |z| < \infty\} \tag{27}$$

while $P_1(z), \mathcal{P}_1(z)$ and $P_2(z), \mathcal{P}_2(z)$ are bounded functions in closed subsectors of

$$\begin{aligned} &\{z : -2\pi < \arg z < \pi, \quad 0 < |z| < \infty\}, \\ &\{z : -\pi < \arg z < 2\pi, \quad 0 < |z| < \infty\}, \end{aligned} \tag{28}$$

respectively.

This suggests an algebraic-analytic structure which serves not only the incomplete Gamma, Bessel's and reduced Weber's differential equations but also all their perturbations. For example the differential equation

$$u''(z) + \left(\sum_{m=1} \frac{b_m}{z^m}\right) u'(z) + \left(1 - \frac{p-1}{z} + \sum_{m=2} \frac{c_m}{z^m}\right) u(z) = 0$$

can be considered as a perturbation of (4).

Definition. *A pair of functions* $\{p_1(z), p_2(z)\}$

(i) *analytic on the Riemann surface of* $\log z$ *with at most exponential growth at* ∞ *,*

(ii) *bounded in closed subsectors of*

$$\begin{aligned} S(1) &= \{z : -\frac{3\pi}{2} - \arg\alpha < arg\, z < \frac{3\pi}{2} - \arg\alpha,\ 0 < |z| < \infty\}, \\ S(2) &= \{z : -\frac{\pi}{2} - \arg\alpha < arg\, z < \frac{5\pi}{2} - \arg\alpha,\ 0 < |z| < \infty\}, \end{aligned} \tag{29}$$

(iii) *satisfying the monodromic relations with connection coefficients* T_1, T_2

$$\begin{aligned} p_1(ze^{2\pi i}) &= p_1(z) + T_1 e^{\alpha z} z^{\beta} p_2(ze^{2\pi i}), \\ p_2(ze^{2\pi i}) &= p_2(z) + T_2 e^{-\alpha z} z^{-\beta} p_1(z) \end{aligned} \tag{30}$$

forms the two-element Stokes structure

$$\mathfrak{S}(2) = \mathfrak{S}\{p_1(z), p_2(z)\}$$

generated by $e^{\alpha z} z^{\beta}$ *for given constants* α, β.

Thus, the above phase amplitudes

(i) $$\{P(z), 1\}, \tag{31}$$

(ii) $$\{P_1(z), P_2(z)\}, \tag{32}$$

(iii) $$\{\mathcal{P}_1(z), \mathcal{P}_2(z)\} \tag{33}$$

form the two-element Stokes structures $\mathfrak{S}_{\Gamma}(2)$, $\mathfrak{S}_B(2)$, $\mathfrak{S}_W(2)$ with

(i) $$T_1 = (-1)^{1-p} \frac{2\pi i}{\Gamma(1-p)} e^{-2\pi i p}, \quad T_2 = 0, \quad \alpha = 1, \quad \beta = 1 - p, \tag{34}$$

(ii) $$T_1 = 2i \cos \nu\pi, \quad T_2 = 2i \cos \nu\pi, \quad \alpha = i, \quad \beta = 0, \tag{35}$$

(iii) $$T_1 = i\sqrt{\pi} \frac{e^{i\pi E} 2^{\frac{1}{2} - E}}{\Gamma\left(\frac{1}{2} - E\right)}, \quad T_2 = i\sqrt{\pi} \frac{e^{-i\pi E} 2^{\frac{1}{2} + E}}{\Gamma\left(\frac{1}{2} + E\right)} e^{-i\pi E}, \quad \alpha = i, \quad \beta = -E, \tag{36}$$

respectively.

4. From the Stokes structure to asymptotic expansions

Without any reference to the differential equations, the Stokes structure* contains very important information about the behavior of its elements on the Riemann surface of $\log z$.

For example, we can show that in any closed subsectors of $S(1)$, $S(2)$ respectively $p_1(z)$, $p_2(z)$ tend to finite limits $a_{1,0}$, $a_{2,0}$ as $z \to \infty$. For all the particular cases (31)–(33) considered above clearly $a_{1,0} = a_{2,0} = 1$ as evident from (7), (19);

* The terminology of 'Stokes structure' first appeared in [6]. However this definition appears to be too complicated to use it for practical purposes. Our definition and preliminary results were announced in [1, 2].

(8), (20); (9), (21). However, in general this result is not obvious from the definition the Stokes structure nor do we know how to derive it directly from it in any better way except using our techniques to follow in future papers of this series.

In fact, we will prove that the following theorem is true.

Theorem 3. *Let* $p_1(z), p_2(z)$ *be the elements of the Stokes structure* $\mathfrak{S}_\Gamma(2)$, $\mathfrak{S}_B(2)$. *Then the limits*

$$a_{j,0} = \lim_{z\to\infty,\, z\in S(j)} p_j(z), \quad j = 1, 2, \tag{37}$$

and

$$a_{j,k} = \lim_{z\to\infty,\, z\in S(j)} \left(p_j(z) - \sum_{m=0}^{k-1} \frac{a_{j,m}}{z^m} \right) z^k,\ k = 1, 2, \dots, j = 1, 2, \tag{38}$$

exist, and the formal power series $\hat{p}_1(z), \hat{p}_2(z)$

$$\hat{p}_1(z) = \sum_{m=0}^{\infty} \frac{a_{1,m}}{z^m}, \quad \hat{p}_2(z) = \sum_{m=0}^{\infty} \frac{a_{2,m}}{z^m} \tag{39}$$

are the Poincaré asymptotic expansions of $p_1(z), p_2(z)$ *in sectors* $S(1), S(2)$ *respectively.*

It is interesting to note that traditionally asymptotic expansions arose either from integral representations or from differential equations. For example, Sir Harold Jeffreys writes in his book [7]: "... The function represented by the expansion is always precisely defined at the start, as either a definite integral or a solution of a differential equation with stated terminal conditions...".

This is definitely not the case in our approach, since the functions $p_1(z), p_2(z)$ above are not defined at the start, neither as a definite integral nor as a solution of a differential equation with stated terminal conditions. They are simply elements of the Stokes structure $\mathfrak{S}\{p_1(z), p_2(z)\}$.

It is worth noticing also that the relations (30) can be considered as a jump condition of the form

$$\begin{bmatrix} 1 & -T_1 z^\beta e^{\alpha z} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_1(z) \\ p_2(z) \end{bmatrix}_- = \begin{bmatrix} 1 & 0 \\ T_2 z^{-\beta} e^{-\alpha z} & 1 \end{bmatrix} \begin{bmatrix} p_1(z) \\ p_2(z) \end{bmatrix}_+ \tag{40}$$

on the ray

$$\arg z = -\frac{\pi}{2} - \alpha,\ 0 < |z| < \infty. \tag{41}$$

The jump condition (40) determines a Riemann–Hilbert boundary problem for the pair of functions $\begin{bmatrix} p_1(z) \\ p_2(z) \end{bmatrix}$ holomorphic in the exterior domain $-\frac{\pi}{2} - \alpha < \arg z < \frac{3\pi}{2} - \alpha\,, \ \ 0 < |z| < \infty\,.$

Thus, our approach is to derive the asymptotic expansions (37)–(39) and other properties of the functions p_1 and p_2 directly from this Riemann–Hilbert problem. It should be mentioned that the Riemann–Hilbert problem has already been used in other fields of asymptotic problems([12] and the references there): to obtain asymptotic for orthogonal polynomials.

In the subsequent papers of this series our procedure will be as follows:

(i) derive $\mathfrak{S}$ directly from the differential equation,

(ii) define appropriately generalized Fourier(Borel)-like transforms to the elements of $\mathfrak{S}$ and study their analytic properties,

(iii) extract formal power series associated with the elements of $\mathfrak{S}$ and study their interrelationships,

(iv) obtain asymptotic expansions with precise estimates for the remainders of the elements of $\mathfrak{S}$,

(v) check the validity of the principle of functional closure,

(vi) explain the Stokes phenomena, "asymptotics beyond all orders" and "resurgence" using $\mathfrak{S}$,

(vii) derive from $\mathfrak{S}$ integral representations of transcendental functions,

(viii) evaluate the connection coefficients of $\mathfrak{S}$ using adjusted Fourier(Borel)-like transforms,

(ix) solve spectral and scattering problems using $\mathfrak{S}$.

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