Matematicheskaya fizika, analiz, geometriya
2002, v. 9, No. 2, p. 146–167

The representation of a meromorphic function as the quotient of entire functions and Paley problem in $\mathbb{C}^n$: survey of some results

B.N. Khabibullin

Department of Mathematics, Bashkir State University
32 Frunze Str., Ufa, Bashkortostan, 450074, Russia
E-mail:algeom@bsu.bashedu.ru

Received September 6, 2001
Communicated by I.V. Ostrovskii

The classical representation problem for a meromorphic function f in $\mathbb{C}^n$, $n \geq 1$, consists in representing f as the quotient $f = g/h$ of two entire functions g and h, each with logarithm of modulus majorized by a function as close as possible to the Nevanlinna characteristic. Here we introduce generalizations of the Nevanlinna characteristic and give a short survey of classical and recent results on the representation of a meromorphic function in terms such characteristics. When f has a finite lower order, the Paley problem on best possible estimates of the growth of entire functions g and h in the representations $f = g/h$ will be considered. Also we point out to some unsolved problems in this area.

Dedicated to Professor N.I. Akhiezer in the year of his 100th anniversary

Introduction

Let f be a meromorphic function (we write $f \in Mer$) in the complex plane. At the beginning of the last century the definition of the classic Nevanlinna characteristic $T(r; f)$ and the Liouville Theorem on the growth of entire functions were giving an one of first results on the representation of a meromorphic function as the quotient of entire functions.

Mathematics Subject Classification 2000: 32A20; 32A22, 30D30, 30D20, 30D35.
Supported by the Russian Foundation of Basic Research under Grant No. 00-01-00770.

Theorem 0.1 (J. Liouville–R. Nevanlinna). *A meromorphic function f in the complex plane can be represented as the quotient $f = g/h$ of polinomials g and h, i.e.,*

$$\log\big(|g(z)| + |h(z)|\big) \le O(\log r)\,, \quad r \to +\infty\,,$$

iff

$$T(r; f) \le O(\log r)\,, \quad r \to +\infty\,.$$

The classic Nevanlinna theorem on the representation of a meromorphic function f of bounded characteristic $T(r; f) \le const$, $r < 1$, in unit disk $\mathbb{D}$ as the quotient of bounded holomorphic functions in $\mathbb{D}$ can be considered also as one of sources of our survey.

A first essential result for transcendental meromorphic functions follows immediately from the Lindelöf theorem of the beginning of XX century (see, for example, [Ru2], [GO]):

Theorem 0.2 (E. Lindelöf). *A meromorphic function f in the complex plane can be represented as the quotient $f = g/h$ of entire functions g and h of finite type for order ρ, i.e.,*

$$\log\big(|g(z)| + |h(z)|\big) \le O(r^{\rho})\,, \quad r \to +\infty\,,$$

iff

$$T(r; f) \le O(r^{\rho})\,, \quad r \to +\infty\,.$$

Here we consider far-reaching generalizations of these classic results and give a survey of basic and recent results on the *problem of the representation* of a meromorphic function in $\mathbb{C}^n$ as the quotient of entire functions in terms of various characteristics. Also we point out to some unsolved problems in this area.

Let $f \in Mer$ in $\mathbb{C}^n$ and

$$f = \frac{g_f}{h_f} \tag{0.1}$$

be a *canonical representation* of f as the quotient of entire functions g_f and h_f which are locally relatively prime, i.e., at each point $z \in \mathbb{C}^n$, where $g_f(z) = h_f(z) = 0$, the functions g and h are relatively prime in the ring of germs of functions analytic at z. In $\mathbb{C}^n$ such representations always exist [GR], [H1]. Both functions

$$u_f = \max\big\{\log|g_f|, \log|h_f|\big\} \tag{0.2a}$$

or

$$u_f = \log\big(|g_f|^2 + |h_f|^2\big)^{1/2} \tag{0.2b}$$

are plurisubharmonic on $\mathbb{C}^n$. Consideration of these *characteristic functions* u_f is useful and natural, since u_f is used to define the various types of Nevanlinna characteristic in particular in the Shimizu–Ahlfors form etc. [GO], [HK], [Ko], [Ku2], [Kh4], [Sk1]–[Sk2], [St1]–[Ta].

A function g on $\mathbb{C}^n$ is said to be *circular* if $g(e^{i\theta}z) = g(z)$ for all $z \in \mathbb{C}^n$ and $\theta \in \mathbb{R}$.

First we introduce the *circular Nevanlinna characteristic*

$$T_f^c(z) = \frac{1}{2\pi}\int_0^{2\pi} u_f(e^{i\theta}z)\,d\theta\,, \quad z \in \mathbb{C}^n\,, \tag{0.3}$$

which is a circular plurisubharmonic function [Ku2]. For example, if $f(0) = 1$ and $T_f(t;\zeta)$, $t \geq 0$, are the Nevanlinna characteristics (in the Shimizu-Ahlfors form for the case (0.2b)) [GO] of the family of the meromorphic functions $f_\zeta(w) = f(w\zeta)$ of one variable $w \in \mathbb{C}$, where ζ runs through the unit sphere $S^n \subset \mathbb{C}^n$, then $T_f(t;\zeta) \leq T_f^c(z)$, $z = te^{i\theta}\zeta$. Equality holds if $n = 1$ or if g_f and h_f do not have common zeros $(n > 1)$.

Let $\mathbb{L} = (L_1, \dots, L_k)$ be a fixed simply ordered collection of complex vector subspaces in $\mathbb{C}^n$ and $\mathbb{C}^n$ is the direct sum of L_k. As usual, we write

$$\mathbb{C}^n = L_1 \oplus \cdots \oplus L_k\,. \tag{0.4}$$

Denote by $\mathbb{R}_+$ the set of nonnegative numbers.

Let $z \in \mathbb{C}^n$ and

$$z = w_1 + \cdots + w_k, \qquad w_p \in L_p, \quad p = 1, \dots, k, \tag{0.5a}$$

$$r_p = |w_p|, \quad \vec{r} = (r_1, \dots, r_k) \in \mathbb{R}_+^k, \tag{0.5b}$$

where $|w_p|$ is the Euclidean norm w_p in L_p. For every $z \in \mathbb{C}^n$ the representation (0.5a) is unique.

If $u(z)$ is a function on $\mathbb{C}^n$, then we consider the function u also as the function $u(z) = u(w_1, \dots, w_k)$ on the direct sum (0.4) under the notations (0.5). We keep also the same notation u for $u(w_1, \dots, w_k)$.

We introduce *Nevanlinna* $\mathbb{L}$-*characteristic* of $f \in Mer$ in the form

$$T_{\mathbb{L}}(\vec{r}; f) = \int_{S_1} \cdots \int_{S_k} u_f(r_1\zeta_1, \dots, r_k\zeta_k)\,ds^{(1)}(\zeta_1)\cdots ds^{(k)}(\zeta_k), \quad \vec{r} \in \mathbb{R}_+^k\,,$$

where the functions u_f are defined in (0.2), and $ds^{(p)}$ is the area element on the unit sphere S_p in L_p.

The *Nevanlinna* Ł*-characteristic* can be defined also in the form

$$T_{Ł}(\vec{r};f) = \int\limits_{S_1} \cdots \int\limits_{S_k} T_f^c(r_1\zeta_1,\dots,r_k\zeta_k)\, ds^{(1)}(\zeta_1)\cdots ds^{(k)}(\zeta_k). \tag{0.6}$$

It is easy to check that every Nevanlinna Ł-characteristic is defined by (0.1)–(0.2) to within an additive constant.

The Nevanlinna Ł-characteristic gives the various variants of the classical Nevanlinna characteristics of f in $\mathbb{C}^n$. So, if $k=1$ and $L_1 = \mathbb{C}^n$ in (0.4), then we get Nevanlinna characteristic $T(r;f)$, $r \geq 0$, by exhaustion of $\mathbb{C}^n$ by balls $r\mathbb{B}^n$, where $\mathbb{B}^n$ is the unit ball in $\mathbb{C}^n$; if $k=n$ and $L_1 = \cdots = L_n = \mathbb{C}$ in (0.4), then we get Nevanlinna characteristic $T(\vec{r};f)$, $\vec{r} \in \mathbb{R}_+^n$, by exhaustion of $\mathbb{C}^n$ by polydisks $P^n(\vec{r}) = \{z \in \mathbb{C}^n : |w_p| = r_p, p = 1,\dots,n\}$ with the preceding notations (0.5) (see [St1]–[Ta], [Ku2], [Sk1]–[Sk2], [Kh4] for various variants of definitions of such Nevanlinna characteristics).

Below we formulate the well-known and recent main results on the representation of a meromorphic function on $\mathbb{C}^n$, $n \geq 1$.

The author expresses deep gratitude to I.F. Krasichkov–Ternovskiĭ for useful discussion of this review and a correction of the text, and to M.L. Sodin, A.A. Kondratyuk and Ya.V. Vasyl'kiv for a valuable information, and also to the reviewer for helpful remarks.

1. The representation of arbitrary meromorphic functions

A continuous increasing nonnegative function λ on $\mathbb{R}_+$ is called a *growth function* (see [RT], [Ml2] and [Ku2]).

For $n=1$ and for a *meromorphic function f of finite λ-type*, i.e., satisfying the condition

$$T(r;f) \leq const \cdot \lambda(r) + const\,, \quad r > 0\,, \tag{1.1}$$

where λ has either the *slow growth*, i.e., satisfies the condition

$$\lambda(2r) \leq O\big(\lambda(r)\big)\,, \quad r \to +\infty\,, \tag{1.2}$$

or $\log \lambda(\exp t)$ is *convex*, L.A. Rubel and B.A. Taylor [RT, Theorems 5.4, 3.5-6, 3.2] indicated (1968) *a construction of entire functions g and h such that $f = g/h$ and*

$$\log\big(|g(z)| + |h(z)|\big) \leq A\lambda(Br) + C\,, \quad r = |z| > 0\,, \tag{1.3}$$

where A, B and C are constants.

All conditions for λ was removed for $n = 1$ (1970-72) by J.B. Miles in [Ml1], [Ml2] (see also the original proof of J.B. Miles in [Ko, Ch. 4], [Ru2, Ch. 14]):

Theorem 1.1 (L.A. Rubel–B.A. Taylor–J.B. Miles). *Every function $f \in Mer$ in the complex plane $\mathbb{C}$ can be represented as the quotient $f = g/h$ of entire functions g and h such that*

$$\log\bigl(|g(z)| + |h(z)|\bigr) \leq AT(Br; f) + C\ , \quad r = |z| > 0\ , \tag{1.4}$$

where A, B and C are constants* .

Analogous results for functions in the half-plane and in the unit disk can be find in [Mlt] and in [Beck] recp.

The following special case of the representation theorems was established for $n = 1$ in [Gl] (1972):

Theorem 1.2 (A.A. Gol'dberg). *Every meromorphic function f in the complex plane $\mathbb{C}$ can be represented as the quotient $f = g_1/g_2$ of entire functions g_1 and g_2 <u>without common zeros</u> such that*

$$\log T(r; g_k) = o\bigl(T(r; f)\bigr)\ , \quad r \to \infty\ , \quad k = 1, 2\ .$$

This theorem (for functions g_1 and g_2 *without common zeros*) is, in a certain sense, best possible (see also [Gl]). Further results in this direction (<u>without common zeros</u>) can to find in [Sk3], [Be2], [VSh], [VK].

Our last one-dimensional result [Kh13] (2001) is the following

Theorem 1.3 (B.N. Khabibullin). *Let f be meromorphic function in the complex plane. For each nonincreasing convex function $\varepsilon(r) > 0$ in $\mathbb{R}_+$ entire functions g and h in the representation $f = g/h$ can be chosen so that the estimate*

$$\log\bigl(|g(z)| + |h(z)|\bigr) \leq \frac{A_\varepsilon}{\varepsilon(r)} T\bigl((1 + \varepsilon(r)) \cdot r\, ; f\bigr) + B_\varepsilon\ , \quad r = |z| > 0\ , \tag{1.5}$$

holds. Here A_ε and B_ε are constants depending on the function $\varepsilon(r)$ (but independent of r).

In particular for $1 + \varepsilon(r) \equiv B > 1$ in (1.5) the last Theorem 1.3 implies the Rubel–Taylor–Miles theorem 1.1, and for $\varepsilon(r) = \epsilon \,/\, (1 + r)$ in (1.5) with a constant $\epsilon > 0$ we obtain the sharpening of the one-dimensional variant of Skoda theorem 1.6 with $(1+r)$ in place of $(1+r)^3$ in (1.7). Besides the theorem 1.3 can be essentially stronger than the Rubel–Taylor–Miles theorem or the one-dimensional

* The importance of such results for $n > 1$ was marked by W. Stoll in his book [St, p. 85]: *"Unfortunately the theorem of Miles* [Ml2] *has not been proved for several variables".*

Skoda theorem if the Nevanlinna characteristic $T(r;f)$ has rapid growth and the decreasing function $\varepsilon(r)$ is chosen in conformity with the growth of $T(r;f)$.

For $n>1$ the first result on the representation problem for a special growth functions $\lambda(r)=r^\rho$ (that solves the problem of L.A. Rubel [Eh, Problem 8] (1968)) is apparently due to W. Stoll [St1, Proposition 6.1] (1968).

Let $\lambda(\vec{r})=\lambda(r_1,\dots,r_n)$, $\vec{r}\in\mathbb{R}^n_+$, be a positive continuous function which is nondecreasing in each variable, and $T(\vec{r},f)\le\lambda(\vec{r})$, $\vec{r}\in\mathbb{R}^n_+$. B.A. Taylor posed the problem* [Ta, p. 470] (1968):

(TP) *when can a meromorphic function f be represented as the quotient $f=g/h$ of entire functions g and h such that*

$$\log\bigl(|g(z)|+|h(z)|\bigr)\le A\lambda(B\vec{r})+C\,,\ r_j=|z_j|\ge 1\,,\ 1\le j\le n\,, \tag{1.6}$$

where A, B and C are constants ?

He proved the following Theorem [Ta, Theorem] (1968):

Theorem 1.4 (B.A. Taylor). *If $\lambda(\vec{r})$ is slowly increasing in each variable, in the sense that*

$$\lambda(r_1,\dots,2r_j,\dots,r_n)\le A_j\lambda(r_1,\dots,r_n)$$

for some constant $A_j>0$, $1\le j\le n$, and $T(\vec{r};f)\le const\cdot\lambda(\vec{r})$ for all $r_j\ge 1$, $1\le j\le n$, then there are entire functions g and h such that $f=g/h$ and (1.6) *holds for all $r_j\ge 1$.*

Further progress for meromorphic functions f in several variables and for the characteristic $T(r;f)$, i.e. $k=1$, under special conditions on the behavior of λ, was made in [Ku1], [Ku2, Propositions 7.3, 9.10] (1969–71):

Theorem 1.5 (R.O. Kujala). *If there are constants a_0 (resp., a vanishing function $a_0=a_0(r)$, $r\to+\infty$), a, b and R in $\mathbb{R}_+$ and p_0 in $\mathbb{N}$ such that*

$$\int_s^r \lambda(t)t^{-p-1}\,dt\le a_0\lambda(br)r^{-p}+a\lambda(bs)s^{-p}$$

whenever $r\ge s>R$ and $p_0\le p$ in $\mathbb{N}$ (in particular, if λ satisfies the slow growth condition (1.2)*), then under the condition* (1.1) *the meromorphic function f in $\mathbb{C}^n$ can be represented as the quotient $f=g/h$ of entire functions g and h such that* (1.3) *(recp., with a vanishing function $A=A(r)$, $r\to+\infty$) is valid.*

* In the original, "*When is Λ the field of quotients of Λ_E ?*"

Except the Theorems 0.1, 0.2, 1.2, 1.3 and the mentioned result for $\lambda(r) = r^\rho$ from W. Stoll [St1], all these results were obtained by the Fourier series method (see [Ko], [Ku2], [Ml2], [No], [RT]–[Ru2], [Ta]). This method was used first by N.I. Akhiezer in [A] (1927) for the new proof the Lindelöf theorem and later by L.A. Rubel in [Ru1] (1961) (see in this connection also [GO, P. 85–88]).

Using the $\bar{\partial}$-problem method, H. Skoda [Sk1]–[Sk2] (1971–72) obtained the following results (see also [St, Theorem 9.12]):

Theorem 1.6 (H. Skoda). *For every constant $\epsilon > 0$ entire functions g and h in representation $f = g/h \in Mer$ can be chosen to satisfy the estimates*

$$\log\bigl(|g(z)| + |h(z)|\bigr) \leq C(\epsilon, s)(1+r)^{4n-1} T(r+\epsilon; f) \tag{1.7}$$

or

$$\log\bigl(|g(z)| + |h(z)|\bigr) \leq C(\epsilon, s)\bigl(\log(1+r^2)\bigr)^2 T\bigl((1+\epsilon)r; f\bigr) \tag{1.8}$$

for all $r = |z| \geq s > 0$, where $C(\epsilon, s)$ is a constant depending on ϵ and s.

It should be noted [St, p. 295] that the pair (g, h) in the Theorem 1.6 can have a common divisor, can depend on ε but can not depend on s, and can be different in (1.8) from the pair chosen in (1.7). The case (1.7) is good for rapid growth, the case (1.8) is good for slow growth.

An analysis of these results indicates a definite rift between the cases of slow and rapid growth of the majorants λ or of the characteristics $T(\cdot; f)$, both in methods (the Fourier series method in the Theorems 1.1, 1.4–1.5 and the $\bar{\partial}$-problem method in the Theorem 1.6) and the estimates and conditions on λ or $T(\cdot; f)$. Our balayage (or sweeping out) method (1990–93) enables us to get rid of this rift and yields results in a complete form.

To be more specific, first this balayage method was applied for a new short nonconstructive proof of Rubel–Taylor–Miles theorem 1.1 in [Kh1, § 3] (1991). The improvement of this method for $n = 1$ made it possible recently to prove the Theorem 1.3. In our articles [Kh2], [Kh4] we established for functions of several variables the following results that are, in a certain sense, extreme relative to the Nevanlinna characteristics $T(r; f)$, $r \geq 0$, and (0.3).

Theorem 1.7 (B.N. Khabibullin). *Let $f \in Mer$, $n \geq 1$. Then*

(T) [Kh2, Theorem 1.3], [Kh4, Theorem 5] (1992–93).

For each constant $\varepsilon > 0$ the function f can be represented as the quotient

$$f = \frac{g}{h}, \qquad g \text{ and } h \text{ are entire functions}, \tag{1.9}$$

such that

$$\log\bigl(|g(z)| + |h(z)|\bigr) \leq A_\varepsilon T\bigl((1+\varepsilon)r; f\bigr) + C_\varepsilon, \quad r = |z| > 0; \tag{1.10}$$

(Tc) [Kh4, Theorem 4] (1993).

Under the conditions $T_f^c(z) \le \lambda(z)$, $z \in \mathbb{C}^n$, $f(0) = 1$, *where the function* λ *is circular continuous positive increasing on all rays with origin at* 0 *and satisfies the following two Hörmander conditions* [H2]:

(L) $\log\bigl(1+|z|\bigr) \le O\bigl(\lambda(z)\bigr)$ *as* $|z| \to +\infty$, *and*

(H) *for any number* $\epsilon > 0$ *there exist positive constans* c_1, c_2, c_3, c_4 *such that*

$$|z-\zeta| \le \exp\bigl(-c_1\lambda(z) - c_2\bigr) \Longrightarrow \lambda(\zeta) \le c_3\lambda\bigl((1+\epsilon)z\bigr) + c_4,\ z \in \mathbb{C}^n,$$

or

($\hat{\mathbf{H}}$) *for any* $\epsilon > 0$ *there exist positive constants* σ, c_1, *and* c_2 *such that*

$$|z-\zeta| \le \sigma \Longrightarrow \lambda(\zeta) \le c_1\lambda\bigl((1+\epsilon)z\bigr) + c_2, \quad z \in \mathbb{C}^n, \tag{1.11}$$

for any $\varepsilon > 0$ *the function* f *can be represented as a quotient* (1.9) *such that*

$$\log\bigl(|g(z)| + |h(z)|\bigr) \le A_\varepsilon\lambda\bigl((1+\varepsilon)z\bigr) + C_\varepsilon\,, \quad z \in \mathbb{C}^n\,, \tag{1.12}$$

where under the condition (1.11) *functions* g *and* h *can be chosen so that* $g(0) = h(0) = 1$.

Here A_ε *and* C_ε *are constants depending on* ε.

Besides the fact that the extra conditions on the growth function λ was removed in the Theorem 1.7 (Part **(T)**), it also refines the Theorems 1.1, 1.5 and 1.6. For example, the constant B in (1.3) and (1.4) is replaced by a constant $1+\varepsilon > 1$ arbitrary close to 1, and in (1.7) and in (1.8) it is possible to remove the power factor before $T(\cdot; f)$ if in (1.7) $r+\varepsilon$ is replaced by $(1+\varepsilon)r$, where $\varepsilon > 0$. Following Gol'dberg [Gl], we can show that we cannot set $\varepsilon = 0$ in general in (1.10) and in (1.12). An analog of the Theorem 1.7 (Part **(T)**) for δ-subharmonic functions was established by O.V. Veselovskaya [Vs] (1984) by the Fourier series method.

If we have some additional information on the functions g_f and h_f in the initial representation (0.1), then these additional properties can be preserved sometimes in the final representation (1.9) (see [Kh9, Theorems 1–3] (1998)):

Theorem 1.8 (B.N. Khabibullin). *If it is known in addition to conditions of Theorem 1.7 that for* $f \in Mer$ *with* $f(0) = 1$ *there exists the representation* (0.1), *where* g_f *and* h_f *are* <u>*bounded in some open set*</u> $B \subset \mathbb{C}^n$, *then in* **(T)** (*recp.,* *in* **(Tc)**) *functions* g *and* h *can be chosen so that besides* (1.10) (*recp.,* (1.12)) *for any* $\gamma \geq 1$ *the inequlity*

$$\log\bigl(|g(z)| + |h(z)|\bigr) \leq C_\gamma \log\bigl(2 + |z|\bigr)\,, \quad z \in B_\gamma\,,$$

holds, where

$$B_\gamma = \{z \in B:\ dist(z, \partial B) \geq \bigl(1 + |z|\bigr)^{-\gamma}\}\,,$$

$dist(z, \partial B)$ *is the distance from* z *to the boundary* ∂B *of* B*, and* C_γ *is constant.*

If $n = 1$*, then functions* g *and* h *in* (1.9) *can be chosen also so that* g *and* h *are bounded on the set* $\{z \in B : dist(z, \partial B) \geq 1/\gamma\}$.

For the general case of the Nevanlinna $\mathbb{L}$-characteristic we have at present the following result [Kh11] (2000):

Theorem 1.9 (B.N. Khabibullin). *Let* $f \in Mer$ *and* $\mathbb{C}^n$ *is represented as* (0.4)*. Then, with the preceding notations* (0.2)–(0.6)*, for any constant* $\varepsilon > 0$*, there exists a representation* (1.9) *such that*

$$\log\bigl(|g(z)| + |h(z)|\bigr) \leq A_\varepsilon T^+_{\mathbb{L}}\bigl((1+\varepsilon)\vec{r} + B_\varepsilon \cdot \vec{1}\,; f\bigr) + C_\varepsilon \log\bigl(2 + |z|\bigr) \qquad (1.13)$$

for all $z \in \mathbb{C}^n$*, where* $A_\varepsilon, B_\varepsilon, C_\varepsilon$ *are constants and* $\vec{1} = (1, \dots, 1) \in \mathbb{R}^k_+$*,* $T^+ = \max\{T, 0\}$.

It is easy to see, that the Theorem 1.9 implies the Theorem 1.7 (Part **(T)**), because always $\log\bigl(2 + |z|\bigr) \leq O\bigl(T(r, f)\bigr)$, $r = |z|$, $f \not\equiv 0, \infty$. Also, if

$$\log\bigl(2 + |\vec{r}|\bigr) \leq O\bigl(\lambda(\vec{r})\bigr)\,, \quad k = n, \quad |\vec{r}| = \sqrt{r_1^2 + \cdots + r_k^2}\,, \qquad (1.14)$$

then the Theorem 1.9 is the extension of Theorem 1.4. It solves the Taylor problem **(TP)** with minimal estimate (1.14) from below for $\lambda(\vec{r})$.

A general scheme of the solution of the representation problem and some other problems (balayage method) was presented in [Kh7]–[Kh8], [Kh12].

Unsolved problems

Problem 1. Is it true, that *the summand* $C \log\bigl(2+|z|\bigr)$ *in* (1.13) *in the Theorem 1.9 can be removed* ?

In any case it is true for $k = 1$, and in the Theorem 1.4 for $k = n$ if $\lambda(\vec{r})$ is slowly increasing.

Problem 2. *Extend the result of the Theorem 1.3 to* $\mathbb{C}^n$*,* $n > 1$*, for classic Nevanlinna characteristic* $T(r; f)$ *and for Nevanlinna L-characteristic.*

Problem 3. Denote by $\mathcal{PSH}_p$ the cone of all *plurisubharmonic* function on L_p, $p = 1, \dots, k$, where L_p was defined in (0.4). Denote by $\mathcal{M}_p^+$ the cone of all *positive Borel measures* (*Radon measures*) *with compact support* in L_p.

We introduce in $\mathcal{M}_p^+$ the *partial order* $\prec$ by (see [M, Ch. XI])

$$(\,\nu \prec \mu\,) \iff \Big(\int v\, d\nu \leq \int v\, d\mu \ \text{ for all } \ v \in \mathcal{PSH}_p\Big).$$

If $\nu \prec \mu$, then the measure $\mu \in \mathcal{M}_p^+$, is called a *balayage* (also *sweeping out* or *Jensen measure* [Ga]) of the measure ν (with respect to the cone $\mathcal{PSH}_p$).

Further, let $\boldsymbol{\mu} = (\mu_1, \dots, \mu_k)$ be a fixed simply ordered collection of measures, where every measure $\mu_p \in \mathcal{M}_p^+$ is the balayage of the Dirac measure $\delta_p \in \mathcal{M}_p^+$ at the point $0 \in L_p$, i.e., $\int u\, d\delta_p = u(0)$.

Let u_f be a plurisubharmonic function from (0.2). A function

$$T_{Ł,\boldsymbol{\mu}}(\vec{r}; f) = \int\limits_{S_1} \cdots \int\limits_{S_k} u_f(r_1\zeta_1, \dots, r_k\zeta_k)\, d\mu_1(\zeta_1) \cdots d\mu_k(\zeta_k)\,, \quad \vec{r} \in \mathbb{R}_+^k,$$

will be called the *Nevanlinna–Jensen* $(Ł, \boldsymbol{\mu})$*-characteristic* of $f \in Mer$. *How can the Theorem 1.9 be extended for the Nevanlinna–Jensen* $(Ł, \boldsymbol{\mu})$*-characteristic* ?

2. The representation of meromorphic functions with restrictions on the type and on the circled indicator

In this section we consider the problem of the representation of a meromorphic function f in $\mathbb{C}^n$ as a quotient (1.9) with best possible estimates of the circled indicators and of the types of the entire functions g and f.

The representation (or *factorization*) *type* $\sigma_n^*(f, \rho)$ *of a meromorphic function* f *on* $\mathbb{C}^n$ *of order* ρ is defined to be the infimum of the numbers σ for which there are entire functions g and h of order ρ with the type less than σ (see [LG]) such that $f = g/h$.

Let

$$P_1(\rho) = \begin{cases} \dfrac{\pi\rho}{\sin \pi\rho}\,, & \text{if} \quad 0 \leq \rho \leq 1/2\,, \\ \pi\rho\,, & \text{if} \quad \rho > 1/2\,. \end{cases} \tag{2.1}$$

For $n = 1$ the first sharp result in this direction was established apparently in our paper [Kh3, Theorem 2] (1992):

Theorem 2.1 (B.N. Khabibullin). *Let f be a meromorphic function in the complex plane $\mathbb{C}$. If the type of the Nevanlinna characteristic $T(r; f)$ is equal to σ with order ρ, i.e.,*

$$\limsup_{r \to +\infty} r^{-\rho} T(r; f) = \sigma\,, \tag{2.2}$$

then the sharp estimate $\sigma \le \sigma_1^*(f,\rho) \le \sigma P_1(\rho)$ *is valid.*

For $n > 1$ we set

$$P_n(\rho) = P_1(\rho) \prod_{k=1}^{n-1}\Bigl(1 + \frac{\rho}{2k}\Bigr). \tag{2.3}$$

For $n > 1$ we have [Kh5, Theorem 3] (1993):

Theorem 2.2 (B.N. Khabibullin). *If the type of the Nevanlinna characteristic* $T(r;f)$ *is equal to* σ *with order* ρ, *i.e.,* (2.2) *holds, then*

$$\sigma \le \sigma_n^*(f,\rho) \le \sigma P_1(\rho) \prod_{k=1}^{n-1} b\Bigl(\frac{\rho}{2k}\Bigr) \le e^{n-1}P_n(\rho),$$

where

$$b(x) = \begin{cases} ex\,, & \text{if} \quad x \ge 1\,, \\ e^x\,, & \text{if} \quad x < 1\,, \end{cases} \tag{2.4}$$

and for $\rho \le 1$ *the sharp estimate*

$$\sigma \le \sigma_n^*(f,\rho) \le \sigma P_n(\rho) \tag{2.5}$$

is valid.

Let u be a plurisubharmonic function on $\mathbb{C}^n$ of finite type with order ρ. Define the function

$$h_{c\rho}(z,u) = \limsup_{\xi\to\infty} |\xi|^{-\rho} u(\xi z), \quad \xi \in \mathbb{C},\ z \in \mathbb{C}^n\,.$$

Its upper semicontinuous regularization

$$h_{c\rho}^*(z,u) = \limsup_{\zeta\to z} h_{c\rho}(\zeta,u)$$

is called [LG, Definition 1.29] the *circled indicator* of growth of u (with order ρ). If we are dealing with circled indicator of growth $h_{c\rho}^*(\zeta,u)$, then we assume a priori that u is of finite type with order ρ. It is known that the circled indicator is complex-homogeneous of degree ρ, i.e., $h_{c\rho}^*(\xi z,u) = |\xi|^\rho h_{c\rho}^*(z,u)$, $\xi \in \mathbb{C}$, is plurisubharmonic, and $h_{c\rho}^*(z,u) \ge 0$ if $u \not\equiv -\infty$. We remind that the circular Nevanlinna characteristic $T_f^c(z)$ of $f \in Mer$ in (0.3) is plurisubharmonic. Therefore, circled indicator of growth for T_f^c is defined. In terms of the circled indicator of growth $h_{c\rho}^*(\zeta, T_f^c)$ we have the following best possible result [Kh5, Theorem 5.1, Example 5.1] (1993):

Theorem 2.3 (B.N. Khabibullin). *Suppose that* $f \in Mer$, $f(0) = 1$, *and* k *is a continuous circular function on* $S^n \subset \mathbb{C}^n$.

1. *If* $h^*_{c\rho}(\zeta, T^c_f) < k(\zeta)$, $\zeta \in S^n$, *then there exist entire functions* g *and* h *such that* $f = g/h$, $g(0) = h(0) = 1$, *and*

$$\max\{h^*_{c\rho}(\zeta, g), h^*_{c\rho}(\zeta, h)\} < P_1(\rho)k(\zeta), \quad \zeta \in S^n .$$

The constant $P_1(\rho)$ *cannot in general be diminished, not even for any one of the functions* g *and* h.

2. *If* $f = g/h$ *is a representation of* f *as the quotient of entire functions* g *and* h, *then*

$$\max\{h^*_{c\rho}(\zeta, g), h^*_{c\rho}(\zeta, h)\} \geq h^*_{c\rho}(\zeta, T^c_f), \quad \zeta \in S^n ,$$

and this estimate is sharp.

The main difficulty consists in the obtaining the upper estimates in the Theorems 2.1–2.3. The success here was achieved with the use of the balayage method (see the preceding section).

Unsolved problems

Problem 1. *Prove the upper estimate in* (2.5) *for all* ρ. See the commentary to the last Problem in the next section.

Problem 2. *Extend the results of the Theorems 2.2–2.3 to the Nevanlinna* $\mathbb{L}$*-characteristic and to the the Nevanlinna–Jensen* $(\mathbb{L}, \boldsymbol{\mu})$*-characteristic.*

3. Paley problem

The subject of this section is close to the themes of the previous sections in both estimates and methods of proofs.

For a function u in $\mathbb{C}^n$ (recp., in $\mathbb{R}^m$) with the range of values $[-\infty, +\infty]$ or $\mathbb{R} \cup \{\infty\}$, we set

$$M(r; u) = \max\{u(z) : |z| = r\}, \quad u^+(z) = \max\{u(z), 0\} .$$

In 1932 R. Paley supposed that for any entire function f of order $\rho \geq 0$ the inequality

$$\liminf_{r\to+\infty} \frac{M(r; \log|f|)}{T(r; f)} \leq P_1(\rho) \tag{3.1}$$

holds (see the definition of $P_1(\rho)$ in (2.1)).

In (3.1) the equality is achieved, in particular, for the Mittag-Leffler's function (see, for example, [GO, p.111]). Earlier the inequality (3.1) was proved by G. Valiron [V] (1930) and by A. Whalund [Wh] (1929) for $\rho \le 1/2$.

The Paley conjecture was conclusively proved by N.V. Govorov [Gv] only in 1967. In 1968, V.P. Petrenko [Pe] extended this result to meromorphic functions of finite lower order. Different proofs was obtained later by I.V. Ostrovskiĭ [Os] for meromorphic functions and by M.R. Essén [Es] for subharmonic functions in the complex plane.

Theorem 3.1 (N.V. Govorov–V.P. Petrenko). *If f is a meromorphic function in the complex plane $\mathbb{C}$ of finite lower order λ, i.e.,*

$$\lambda = \liminf_{r\to+\infty} \frac{\log T(r;f)}{\log |r|} < +\infty , \tag{3.2}$$

then the inequality

$$\liminf_{r\to+\infty} \frac{M\bigl(r;\log|f|\bigr)}{T(r;f)} \le P_1(\lambda) \tag{3.3}$$

holds.

The relative growth of $T(r;f)$ and $M\bigl(r;\log|f|\bigr)$ for entire and meromorphic functions of infinite order in the complex plane has also been considered by many authors.

For entire functions of infinite order in the complex plane in [Ch](1981), [MSh](1984) and finally in [DDL](1990–1991) was proved the following result.

Theorem 3.2. (C.T. Chuang, I.I. Marchenko–A.I. Shcherba, C.J. Dai–D. Drasin–B.Q. Li) *Let f be an entire function of infinite order in the complex plane. If $\psi(x)$ is increasing and positive for $x \ge x_0 > 0$ and if*

$$\int_{x_0}^{\infty} \frac{dx}{\psi(x)} < \infty , \tag{3.4}$$

then

$$\liminf_{r\to+\infty} \frac{M\bigl(r;\log|f|\bigr)}{T(r;f)\cdot\psi\bigl(\log T(r;f)\bigr)} = 0 \tag{3.5}$$

and even

$$M\bigl(r;\log|f|\bigr) = o\Bigl(T(r;f)\cdot\psi\bigl(\log T(r;f)\bigr)\Bigr)$$

as $r \to +\infty$ through a set of logarithmic density one.

In [MSh] and [DDL] it is also shown that the results are best possible in some sense.

For meromorphic functions in [DDL] (1990–1991) the following was obtained.

Theorem 3.3 (C.J. Dai–D. Drasin–B.Q. Li). *Let f be a meromorphic function of infinite order in the complex plane. If $\psi(x)$ is the same as in the Theorem 3.2 then*

$$M\big(r;\log|f|\big) = o\Big(T(r;f)\cdot\psi\big(\log T(r;f)\big)\cdot\log\psi\big(\log T(r;f)\big)\Big)$$

as $r\to+\infty$ through a set of logarithmic density one.

A different approach has been taken in [Be1], [BB] (1990–1994) where the characteristic $M\big(r;\log|f|\big)$ has been compared with the derivative of $T(r;f)$.

Theorem 3.4 (W. Bergweiler–H. Block). *Let f be a meromorphic function of infinite order in the complex plane. Then*

$$\liminf_{r\to+\infty}\frac{M\big(r;\log|f|\big)}{rT'_-(r;f)}\le\pi\,,$$

where $T'_-(r;\cdot)$ is the left-side derivative of T of r.

Let $\psi(x)$ be positive and continuously diffirentiable for $x\ge x_0>0$ such that $\psi(x)/x$ is non-decreasing, $\psi(x)\le\sqrt{\psi(x)}$, and (3.4) is satisfied. Then (3.5) holds.

The articles of I.I. Marchenko [Ma1]–[Ma2] contains much information in particular on the growth of entire and meromorphic functions of infinite order.

We don't know any results for entire and meromorphic functions of infinite order in $\mathbb{C}^n$, $n>1$, wich are analogs of the Theorems 3.2–3.4.

Let u be a subharmonic function in $\mathbb{R}^m$, $m\ge 2$, (resp., in $\mathbb{C}^n$) and let $|\mathcal{S}^{m-1}|$ be the area of the unit sphere $\mathcal{S}^{m-1}$ in $\mathbb{R}^m$, ds is the area element on the unit sphere $\mathcal{S}^{m-1}$.

$$m_q(r;u) = \left(\frac{1}{|\mathcal{S}^{m-1}|}\int\limits_{\mathcal{S}^{m-1}}\big|u(rx)\big|^q\,ds(x)\right)^{1/q},\quad r>0,\ 1\le q<+\infty.$$

Then Nevanlinna characteristic $T(r;u)$ and $M(r;u)$ are respectively

$$T(r;u)=m_1(r;u^+),\quad M(r;u^+)=m_\infty(r;u^+)=\lim_{q\to+\infty}m_q(r;u^+).$$

In particular if f is a entire function in $\mathbb{C}^n$ then $T(r;f)=T\big(r;\log|f|\big)$, $m=2n$.

An analogue of (3.3) for subharmonic functions of finite lower order λ in $\mathbb{R}^m$, $m \geq 3$, was obtained by B. Dahlberg [D, Theorem 1.2].

To be more specific suppose $\lambda \in (0, +\infty)$ is given. The Gegenbauer functions C_λ^γ are given as the solutions of the differential equation

$$(1-x^2)\,\frac{d^2u}{dx^2} - (2\gamma+1)x\,\frac{du}{dx} + \lambda(\lambda+2\gamma)\,u = 0, \quad -1 < x < 1\,,$$

with the normalization

$$\lim_{x\to 1-0} C_\lambda^\gamma(x) = C_\lambda^\gamma(1) = \frac{\Gamma(\lambda+2\gamma)}{\Gamma(2\gamma)\Gamma(\lambda+1)}.$$

Put $a_\lambda = \sup\bigl\{t : C_\lambda^{\frac{m-2}{2}}(t) = 0\bigr\}$ and define the function u_λ in $\mathbb{R}^m$, $m \geq 3$, by

$$u_\lambda(x) = \begin{cases} 0 & \text{if } x_1 \leq a_\lambda r \\ r^\lambda C_\lambda^{\frac{m-2}{2}}(x_1/r) & \text{if } x_1 > a_\lambda r\,, \end{cases}$$

where $x = (x_1, \dots, x_m)$ and $r = |x|$.

The function u_λ is subharmonic in $\mathbb{R}^m$ and the lower order of u_λ is λ. Set

$$c(\lambda, m) = \liminf_{r\to+\infty} \frac{M(r; u_\lambda)}{T(r; u_\lambda)}\,.$$

Theorem 3.5 (B. Dahlberg). *Let u be a subharmonic function in $\mathbb{R}^m$, $m \geq 3$, of finite lower order $\lambda > 0$. Then we have that*

$$\liminf_{r\to+\infty} \frac{M(r;u)}{T(r;u)} \;\leq\; c(\lambda, m)\,,$$

and this estimate is best possible.

The generalization of the Theorems 3.1, 3.5 is also known.

Theorem 3.6 (M.L. Sodin [So] (1983)**).** *Let u be a subharmonic function of lower order λ in the complex plane. Then*

$$\liminf_{r\to+\infty} \frac{m_q(r;u^+)}{T(r;u)} \leq m_q(S_\lambda), \quad 1 < q \leq +\infty, \tag{3.6}$$

where $m_q(S_\lambda)$ is the Lebesque mean of order q, $1 < q < +\infty$, of the function

$$S_\lambda(\varphi) = \begin{cases} \pi\lambda\cos\lambda\varphi & \text{if} \quad |\varphi| \leq \dfrac{\pi}{2\lambda}, \\ 0 & \text{if} \quad \dfrac{\pi}{2\lambda} < |\varphi| \leq \pi, \end{cases}$$

for $\lambda \geq 1/2$ *and*

$$S_\lambda(\varphi) = \frac{\pi\lambda\cos\lambda\varphi}{\sin\pi\lambda}, \qquad |\varphi| \leq \pi,$$

for $\lambda < 1/2$, $m_\infty(S_\lambda) = \max\{S_\lambda(\varphi) : |\varphi| \leq \pi\}$. *The inequality* (3.6) *is sharp.*

This result was extended to $m \geq 3$ in [KTV, Theorem 1] (1995).
Set

$$Q_\lambda^{\frac{m-2}{2}}(\theta) = \begin{cases} A(\lambda, m)\, C_\lambda^{\frac{m-2}{2}}(\cos\theta) & \text{if} \quad 0 \leq \theta \leq \alpha_\lambda, \\ 0 & \text{if} \quad \alpha_\lambda < \theta \leq \pi, \end{cases}$$

where

$$\alpha_\lambda = \min\{\theta \in (0,\pi) : C_\lambda^{\frac{m-2}{2}}(\cos\theta) = 0\},$$

$$A(\lambda, m) = \left(\frac{|S^{m-2}|}{|S^{m-1}|} \int\limits_0^{\alpha_\lambda} C_\lambda^{\frac{m-2}{2}}(\cos\theta)\sin^{m-2}\theta\, d\theta \right)^{-1}.$$

Theorem 3.7 (A.A. Kondratyuk–S.I. Tarasyuk–Ya.V. Vasyl'kiv). *Let* u *be a subharmonic function of lower order* $\lambda > 0$ *in* $\mathbb{R}^m$, $m \geq 3$. *Then for every* q, $1 < q \leq +\infty$, *the inequality*

$$\liminf_{r\to+\infty} \frac{m_q(r; u^+)}{T(r; u)} \leq M_q(\lambda, m) \tag{3.7}$$

is true, where $M_q(\lambda, m) = m_q\big(Q_\lambda^{\frac{m-2}{2}}\big)$. *There exists a subharmonic function* u *of lower order* $\lambda > 0$ *in* $\mathbb{R}^m$ *for which in* (3.7) *the equality is achieved.*

Besides, from W. Hayman result [HK] it follows that for subharmonic functions of order $\lambda = 0$ the inequlity

$$\liminf_{r\to+\infty} \frac{m_q(r; u^+)}{T(r; u)} \leq 1, \quad 1 < q \leq +\infty,$$

holds.

In [Kh6] (1995) we extended (3.3) to meromorphic functions f in $\mathbb{C}^n$, $n > 1$. Instead of $M(r; \log|f|)$ the corresponding formulation involves in this case the family of characteristics $M(r; \log|f_\zeta|)$, $\zeta \in S^n$, of the slice functions $f_\zeta(w) = f(w\zeta)$, $w \in \mathbb{C}$. This is understanble, for in the case of a meromorphic function the quantity $M(r; \log|f|)$ can be identically equal to ∞ starting from some value of r, whereas the Nevanlinna characteristics of f can have very slow growth (one example is the function $f(z_1, z_2) = (1 + z_1)/(1 + z_2)$ in $\mathbb{C}^2$).

Theorem 3.8 (B.N. Khabibullin). *Let f be a meromorphic function in $\mathbb{C}^n$ of finite lower order λ, i.e.,* (3.2) *holds. Then*

$$\liminf_{r\to+\infty}\frac{M\big(r;\log|f_\zeta|\big)}{T(r;f)}\le P_n(\lambda)\,,\quad \zeta\in S^n, \tag{3.8}$$

where $P_n(\lambda)$ was defined in (2.3), *and this estimate is best possible.*

The estimate (3.8) was new also for entire functions and was not implied in general by the above result of B. Dahlberg for subharmonic functions in $\mathbb{R}^m$, $m\ge 3$.

On the other hand, one can pose the problem of finding a complete and best possible analogue of (3.3) for plurisubharmonic functions and entire functions of several variables. Such result was obtained in our paper [Kh10, Theorem 1] (1999):

Theorem 3.9 (B.N. Khabibullin). *Let u be a plurisubharmonic function in $\mathbb{C}^n$ of finite lower order λ. Then*

$$\liminf_{r\to+\infty}\frac{M(r;u)}{T(r;u)}\le P_n(\lambda) \tag{3.9}$$

for $\lambda\le 1$ and this estimate is best possible. For $\lambda>1$,

$$\liminf_{r\to+\infty}\frac{M(r;u)}{T(r;u)}\le P_1(\rho)\prod_{k=1}^{n-1}b\Big(\frac{\rho}{2k}\Big)\le e^{n-1}P_n(\rho), \tag{3.10}$$

where $b(x)$ was defined in (2.4).

More can be said about the definitive character of (3.9). For each $\rho\ge 0$ there exist in $\mathbb{C}^n$ entire functions of order ρ and normal type such that

$$\lim_{r\to+\infty}\frac{M\big(r;\log|f|\big)}{T\big(r;\log|f|\big)}=P_n(\rho)\,.$$

Comparing with the Dahlberg's theorem 3.5, we can draw the following conclusions:

(a) for $\lambda=0$ or $\lambda=1$ the result of the Theorem 3.9 is a consequence of Dahlberg's theorem;

(b) for $0<\lambda<1$ and $n=2$ Theorem 3.9 does not follow from Dahlberg's theorem;

(c) for $\lambda\ge\pi e-1$ and $n=2$ even the estimate (3.10) in Theorem 3.9 does not follow from Dahlberg's theorem;

(d) as $\lambda\to+\infty$, for $n=2$ the estimate (3.10) is better by order $O(1/\lambda)$ than the one following from Dahlberg's theorem.

Unsolved problems

Problem 1. *Extend the results of the Theorems 3.2–3.4 for entire and meromorphic functions of several variables of infinite order.*

Problem 2. *Extend the results of the Theorems 3.8–3.9 to the Nevanlinna* $\mathbb{L}$*-characteristic and to the the Nevanlinna–Jensen* $(\mathbb{L}, \boldsymbol{\mu})$*-characteristic.*

Problem 3. *Extend the results of the Theorem 3.9 for the* $m_q(r;\cdot)$ *instead of* $M(r;\cdot)$.

Problem 4. *Prove the upper estimate in* (3.9) *for all* λ.

Commentary. In order to prove the upper estimates in (2.5) for all ρ and in (3.9) for all λ, it is sufficient to confirm the following:

Hypothesis. *Let* S *be a nonnegative increasing function on* $\mathbb{R}_+$, $S(0) = 0$, *and the function* $S(t)$ *is convex with respect to* $\log t$, *i.e.,* $S(e^x)$ *is convex on* $[-\infty, +\infty)$. *Further, let* $\lambda \geq 1/2$, $n \in \mathbb{N}$, $n \geq 2$. *If*

$$\int_0^1 S(tx)(1-x^2)^{n-2} x\, dx \leq t^\lambda, \quad 0 \leq t < +\infty, \tag{3.11}$$

then

$$\int_0^{+\infty} S(t) \frac{t^{2\lambda-1}}{(1+t^{2\lambda})^2}\, dt \leq \frac{\pi(n-1)}{2\lambda} \prod_{k=1}^{n-1} \left(1 + \frac{\lambda}{2k}\right). \tag{3.12}$$

This Hypotesis is true if $\lambda \leq 1$. When

$$S(t) = 2(n-1) \prod_{k=1}^{n-1} \left(1 + \frac{\lambda}{2k}\right) t^\lambda, \quad \lambda \geq \frac{1}{2},$$

we have the equalities in (3.11) and in (3.12).

It is not known even, whether the formulated Hypothesis is true for $n = 2$ and $\lambda > 1$?

References

[A] *N.I. Akhiezer*, A new deduction of necessary conditions for an entire function of integer order to belong to a certian class. — *Zap. phys.-mat. otd. AN USSR* (1927), v. 2, No. 3, p. 29–33. (Ukrainian)

[Beck] *W. Beck*, Efficient quotient representations of meromorphic function in the disk (Ph. D., Thesis). Illinois University, Urbana, IL (1970).

[Be1] *W. Bergweiler*, Maximum modulus, characteristic, and area on the sphere. — *Analysis* (1990), v. 10, p. 163–176. (*Erratum: Analysis* (1992), v. 12, p. 67–69.)

[Be2] *W. Bergweiler*, A question of Gol'dberg concerning entire functions with prescribed zeros. — *J. Anal. Math.* (1994), v. 63, p. 121–122.

[BB] *W. Bergweiler and H. Bock*, On the growth of meromorphic functions of infinite order. — *J. Anal. Math.* (1994), v. 64, p. 327–336.

[Ch] *C.T. Chuang*, Sur la croissance des fonctions. — *Kexue Tongbao* (1981), v. 26, p. 677–684.

[D] *B. Dahlberg*, Mean values of subharmonic function. — *Ark. Mat.* (1972), v. 10, p. 293–309.

[DDL] *C.J. Dai, D. Drasin, and B.Q. Li*, On the growth of entire and meromorphic functions of infinite order. — *J. Anal. Math.* (1990), v. 55, p. 217–228. (*Correction: J. Anal. Math.* (1991), v. 57, p. 299.)

[Eh] *L. Ehrenpreis (Ed.)*, Problems, Entire Functions and Related Parts of Analysis. In: Proc. Sympos. Pure Math., v. 11. AMS, Providence, RI (1968).

[Es] *M.R. Essén*, The $\cos \pi\lambda$ theorem. Springer–Verlag, Berlin (1975).

[Ga] *T.W. Gamelin*, Uniform algebras and Jensen measures. Cambridge University Press, Cambridge (1978).

[Gl] *A.A. Gol'dberg*, On the representation of a meromorphic function as a quotient of entire functions. — *Izv. Vyssh. Uchebn. Zaved. Mat.* (1972), v. 10, No. 125, p. 13–17. (Russian)

[GO] *A.A. Gol'dberg and I.V. Ostrovskiĭ*, The distribution of values of meromorphic functions. Nauka, Moscow (1970). (Russian)

[Gv] *N.V. Govorov*, Paley's conjecture. — *Funkt. Anal. i Prilozhen.* (1969), v. 3, No. 2, p. 41–45. (Russian)

[GR] *R.C. Gunning and H. Rossi*, Analytic functions of several complex variables. Prentice–Hall, Englewood Cliffs (1965).

[HK] *W.K. Hayman and P.B. Kennedy*, Subharmonic functions. Acad. Press, London (1976).

[H1] *L. Hörmander*, An introduction to complex analysis in several variables. Van Nostrand, Princeton, NJ (1966).

[H2] *L. Hörmander*, Generators for some rings of analytic function. — *Bull. Soc. Math. France* (1967), v. 73, No. 6, p. 943–949.

[Kh1] *B.N. Khabibullin*, Sets of uniqueness in spaces of entire functions of a single variable. — *Izv. Akad. Nauk SSSR* (1991), v. 55, No. 5, p. 1101–1123. (Engl. transl.: *Math. USSR Izv.* (1992), v. 39, No. 2, p. 1063–1083.)

[Kh2] *B.N. Khabibullin*, The least plurisuperharmonic majorant and multipliers of entire functions. II. The algebras of functions of finite λ-type. — *Sibirsk. Mat. Zh.* (1992), v. 33, No. 3, p. 186–191. (Engl. transl.: *Siberian Math. J.* (1994), v. 33.)

[Kh3] *B.N. Khabibullin*, On the type of entire and meromorphic functions. — *Mat. sb.* (1992), v. 183, No. 11, p. 35–44. (Russian)

[Kh4] *B.N. Khabibullin*, The theorem on the least majorant and its applications. I. Entire and meromorphic functions. — *Izv. Ross. Akad. Nauk* (1993), v. 57, No. 1, p. 129–146. (Engl. transl.: *Russian Acad. Sci. Izv. Math.* (1994), v. 42, No. 1, p. 115–131.)

[Kh5] *B.N. Khabibullin*, The theorem on the least majorant and its applications. II. Entire and meromorphic functions of finite order. — *Izv. Ross. Akad. Nauk* (1993), v. 57, No. 3, p. 70–91. (Engl. transl.: *Russian Acad. Sci. Izv. Math.* (1994), v. 42, No. 3, p. 479–500.)

[Kh6] *B.N. Khabibullin*, The Paley problem for functions that are meromorphic in $\mathbb{C}^n$. — *Izv. Ross. Akad. Nauk* (1995), v. 342, p. 461–463. (Engl. transl.: *Russian Acad. Sci. Izv. Math.* (1995), v. 51.)

[Kh7] *B.N. Khabibullin*, Zero sets for classes of entire function and a representation of meromorphic functions. — *Mat. Zametki* (1996), v. 59, p. 611–617. (Engl. transl.: *Math. Notes* (1996), v. 59, p. 440–444.)

[Kh8] *B.N. Khabibullin*, Dual approach to certain questions for the weighted spaces of holomorphic functions. — In: Israel Math. Conf. Proc. "Entire Functions in Modern Analysis", Tel-Aviv (2001), v. 15, p. 207–219.

[Kh9] *B.N. Khabibullin*, Variant of a problem on the representation of a meromorphic function as a quotient of entire functions. — *Compl. Variables* (1998), v. 37, p. 371–384.

[Kh10] *B.N. Khabibullin*, Paley problem for plurisubharmonic functions of finite lower order. — *Mat. sb.* (1999), v. 190, No. 2, p. 147–157. (Engl. transl.: *Math. Russian Sb.* (1999), v. 190, p. 309–321.)

[Kh11] *B.N. Khabibullin*, Rubel–Taylor problem on the representation of a meromorphic function. — *Funkt. Anal. i Prilozhen.* (2001), v. 35, No. 3, p. 91–94. (Russian)

[Kh12] *B.N. Khabibullin*, Dual representation of superlinear functionals and its application in function theory. Parts I and II. — *Izv. Ross. Akad. Nauk, Ser. Mat.* (2001), v. 65, No. 4, and No. 5, p. 205–224 and p. 167–190. (Russian)

[Kh13] *B.N. Khabibullin*, The growth of entire functions with given zeros and the representation of meromorphic functions. — *Mat. Zametki* (2002). (Russian, to appear)

[Ko] *A.A. Kondratyuk*, Fourier series and meromorphic functions. Vishcha shkola, L'viv (1988). (Russian)

[KTV] *A.A. Kondratyuk, S.I. Tarasyuk, and Ya.V. Vasyl'kiv*, .General Paley problem — *Ukr. Mat. Zh.* (1996), v. 48, No. 1, p. 25–34. (Engl. transl.: *Ukr. Math. J.* (1969), v. 48.)

[Ku1] *R.O. Kujala*, Functions of finite λ-type in several complex variables. — *Bull. Amer. Math. Soc.* (1969), v. 75, p. 104–107.

[Ku2] *R.O. Kujala*, Functions of finite λ-type in several complex variables. — *Trans. Amer. Math. Soc.* (1971), v. 161, p. 327–358.

[LG] *P. Lelong and L. Gruman*, Entire functions of several complex variables. Springer–Verlag, Berlin (1986).

[Mlt] *K.G. Malyutin*, Fourier series and δ-subharmonic functions of finite γ-type in the half-plane. — *Mat. Sb.* (2001), v. 6, No. 6, p. 51–70. (Russian)

[Ma1] *I.I. Marchenko*, On the growth of entire and meromorphic functions. — *Mat. Sb.* (1998), v. 189, No. 6, p. 59–84.

[Ma2] *I.I. Marchenko*, On the asimptotic values of entire functions. — *Izv. Ross. Akad. Nauk, Ser. Mat.* (1999), v. 63, No. 3, p. 133–146.

[MSh] *I.I. Marchenko and A.I. Shcherba*, Growth of entire functions. — *Sibirsk. Mat. Zh.* (1984), v. 25, p. 598–605.

[M] *P.-A. Meyer*, Probability and potentials. Blaisdell, Waltham, MA (1966).

[Ml1] *J.B. Miles*, Representating a meromorphic function as the quotient of two entire functions of small characteristic. — *Bull. Amer. Math. Soc.* (1970), v. 76, p. 1308–1309.

[Ml2] *J.B. Miles*, Quotient representations of meromorphic functions. — *J. Anal. Math.* (1972), v. 25, p. 371–388.

[No] *Ph. Noverraz*, Extension d'une méthode de séries de Fourier aux fonctions sousharmoniques et plurisousharmoniques. Sém. P. Lelong 6e année: *Analyse* (1965/66), No. 3.

[Os] *I.V. Ostrovskiĭ*, On the Paley problem. In: *N.V. Govorov*, "Rieman's boundary value problem with infinite index". Nauka, Moscow (1986), p. 216–232. (Engl. transl.: Birkhäuser, Basel (1994).)

[Pe] *V.P. Petrenko*, Growth of meromorphic functions of finite lower order. — *Izv. Akad. Nauk SSSR, Ser. Mat.* (1969), v. 33, p. 414–454. (Engl. transl.: Math. USSR-Izv. (1969), v. 3.)

[Ru1] *L.A. Rubel*, A Fourier series method for entire functions. — *Duke Math. J.* (1963), v. 30, p. 437–442.

[Ru2] *L.A. Rubel*, Entire and meromorphic functions. Springer–Verlag, New York, Berlin, Heidelberg (1996).

[RT] *L.A. Rubel and B.A. Taylor*, A Fourier series method for meromorphic and entire functions. — *Bull. Soc. Math. France* (1968), v. 96, p. 53–96.

[Sk1] *H. Skoda*, Solution à croissance du second problème de Cousin dans $\mathbb{C}^n$. — *Ann. Inst. Fourier*, Grenoble (1971), v. 21, p. 11–23.

[Sk2] *H. Skoda*, Croissançe des fonctions entiéres s'annulant sur une hypersurface donnee de $\mathbb{C}^n$. Sem. P. Lelong (1970–71). — *Lect. Notes Math.*, Springer–Verlag (1972), v. 275, p. 82–105.

[Sk3] *H. Skoda*, Sous-ensembles analitique ordre fini ou infini dans $\mathbb{C}^n$. — *Bull. Soc. Math. France* (1972), v. 100, No. 4, p. 353–408.

[So] *M.L. Sodin*, Growth in the L^p metric of entire functions of finite lower order. Manuscript No. 420, Uk–D 83 deposited at Ukr. NINTI 02.07.83. (Russian)

[St1] *W. Stoll*, About entire and meromorphic functions of exponential type. Entire functions and related parts of analysis. In: Proc. Symp. Pure Math. AMS, Providence, RI (1968), p. 392–430.

[St] *W. Stoll*, Value distribution theory for meromorphic maps. Friedr. Vieweg & Sohn Verlagsgesellschaft mbH, Braunschweig (1985).

[Ta] *B.A. Taylor*, The fields of quotients of some rings of entire functions. Entire functions and related parts of analysis. In: Proc. Symp. Pure Math. AMS, Providence, RI (1968), p. 468–474.

[V] *G. Valiron*, Sur un théorème de M. Wiman. — *Opuscula Math. A.* (1930), p. 1–12.

[VK] *Ya.V. Vasyl'kiv and A.A. Kondratyuk*, Minimal growth majorants of entire functions and quotient representations of meromorphic functions. — *Dop. NAN Ukr.* (2002). (Ukrainian, to appear)

[VSh] *B.V. Vynnyts'kiĭ and I.B. Sheparovych*, On interpolation sequences of some classes of of entire functions. — *Mat. Stud.* (1999), v. 12, No. 1, p. 76–84. (Ukrainian)

[Vs] *O.V. Veselovskaya*, An analog of Miles theorem for δ-subharmonic functions in $\mathbb{R}^m$. — *Ukr. Mat. Zh.* (1984), v. 36, No. 6, p. 694–698. (Russian)

[Wh] *A. Whalund*, Über einen Zusammenhang zwischen dem Maximalbetrage der ganzen Funktion und seiner unteren Grenze nach dem Jensensche Theoreme. — *Ark. Mat.* (1929), Bd. 21A, No. 23, S. 1–34.