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# Asymptotics of orthogonal polynomials and eigenvalue distribution of random matrices

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We present an informal discussion of recent results on the asymptotic behaviour of orthogonal polynomials, in particular orthogonal polynomials with varying weight, emphasizing links with spectral theory. We apply some of these results to find the leading term of the covariance of linear statistics of certain unitary invariant ensembles of random matrix of large order.

## 1. Introduction

Orthogonal polynomials were always among the subjects of intense interest of N. Akhiezer. He obtained a number of important results in this branch of classical analysis and gave interesting applications. This concerns in particular asymptotics of orthogonal polynomials whose weight is supported on several disjoint intervals of real axis. Respective results by N. Akhiezer and his colleagues being of considerable interests by themselves, motivated also one of his most known and influential result: invention of finite-band potentials, their links with elliptic functions and with Abel inversion problem.

## 2. Asymptotics of orthogonal polynomials

### 2.1. Generalities

We will restrict ourselves to the polynomials orthogonal with respect to an absolutely continuous measure. Let $\sigma$ be a finite union of disjoint intervals of the real

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axis and $w$ be a non-negative measurable function whose support is $\sigma$. Assuming that all the moment of $w$ are finite, i.e., that

$$\int_\sigma |\lambda|^k w(\lambda) d\lambda < \infty, \ k = 0, 1, \dots , \tag{2.1}$$

we can construct the orthogonal polynomials $\{p_l(\lambda)\}_{l=0}^\infty$, uniquely determined by the conditions that $p_l$ is a polynomial of degree $l$ positive at infinity, and that

$$\int p_l(\lambda) p_m(\lambda) w(\lambda) d\lambda = \delta_{lm}. \tag{2.2}$$

Any system of orthogonal polynomials satisfies the three-term recurrent relation

$$\lambda p_l(\lambda) = r_{l-1} p_{l-1}(\lambda) + s_l p_l(\lambda) + r_l p_{l+1}(\lambda), \ \ r_{-1} = 0, \ r_l > 0, \ l \geq 0. \tag{2.3}$$

The relation determines the semi-infinite Jacobi matrix

$$J = \{J_{kl}\}_{k,l=0}^\infty, \ J_{kl} = r_{k-1}\delta_{k-1,l} + s_k \delta_{kl} + r_k \delta_{k+1,l} \tag{2.4}$$

that can be viewed as a self-adjoint operator in $l^2(\mathbb{Z}_+)$ under certain conditions on the coeffcients $r_l$ and $s_l$. In this case the functions

$$\psi_l(\lambda) = (w(\lambda))^{1/2} p_l(\lambda), \ l = 0, 1, \dots \tag{2.5}$$

comprise the system of orthonormalized in $L^2(\sigma)$ generalized eigenfunctions of $J$, i.e. polynomially bounded solutions of the finite-difference equation $(J\psi)_l = \lambda\psi_l, \ l = 0, \dots$. In particular, the resolution of identity $E^J(d\lambda) = \{E^J_{kl}(d\lambda)\}_{k,l=0}^\infty$ of $J$ is

$$E^{(J)}_{kl}(d\lambda) = \psi_k(\lambda)\psi_l(\lambda) d\lambda. \tag{2.6}$$

### 2.2. Asymptotics of "ordinary" orthogonal polynomials

The asymptotic behavior of orthogonal polynomials, defined by (2.2), depends strongly on the number of components of their support. To avoid technical discussions we will assume that the weight is sufficiently regular.

The simplest case corresponds to the support consisting of the interval $[-1, 1]$. In this case we have the classical result by Bernstein–Szegö (see e.g. [25]), according to which if $\lambda \in (-1, 1)$, then

$$\psi_n(\lambda) = \sqrt{\frac{2}{\pi \sin\phi(\lambda)}} \cos\big(n\phi(\lambda) + \gamma(\lambda)\big) + o(1), \ n \to \infty, \tag{2.7}$$

where $\cos\phi(\lambda) = \lambda$, $\phi \in (0,\pi)$, and $\gamma(\lambda)$ is a certain function. This asymptotic formula (and even the simpler polynomial one, corresponding to $\lambda \notin [-1,1]$) implies the asymptotic form of the coefficients:

$$r_n = 1/2 + o(1),\ s_n = o(1),\ n \to \infty. \tag{2.8}$$

Define in $l^2(\mathbb{Z})$ the self-adjoint operator $J_n$ as the Jacobi matrix with coefficients $\{r_{n,k}, s_{n,k}\}_{k\in\mathbb{Z}}$ of the form

$$r_{n,k} = \begin{cases} r_{n+k}, & -n \le k < \infty\,, \\ 0, & k < -n, \end{cases} \qquad s_{n,k} = \begin{cases} s_{n+k}, & -n \le k < \infty\,, \\ 0, & k < -n. \end{cases} \tag{2.9}$$

Then the relation (2.8) implies that the sequence $\{J_n\}$ of selfadjoint operators converges strongly in $l^2(\mathbb{Z})$ to the Jacobi matrix $J_0^{(\lim)}$, defined by the constant coefficients that are equal to the leading terms of asymptotic formulas (2.8):

$$(J_0^{(\lim)})_{kl} = \frac{1}{2}\delta_{k-1,l} + \frac{1}{2}\delta_{k+1,l},\ k,l \in \mathbb{Z}. \tag{2.10}$$

Rewrite the asymptotic formula (2.7) as follows

$$\psi_n(\lambda) = \big(2d_0(\lambda)\big)^{1/2}\cos\big(\pi n\nu_0(\lambda) + \gamma(\lambda)\big) + o(1), \tag{2.11}$$

where

$$\nu_0(\lambda) = \frac{\phi(\lambda)}{\pi} = \int_\lambda^1 d_0(\mu)d\mu, \quad d_0(\lambda) = (\pi(1-\lambda^2))^{-1/2}\chi_{[-1,1]}(\lambda), \tag{2.12}$$

and $\chi_{[-1,1]}$ is the indicator of the interval $[-1,1]$. By using (2.6), we obtain that for any interval $\Delta$ of the spectral axis the spectral measure $E^{J_n}(\Delta)$ of the Jacobi matrix $J_n$ converges strongly to the spectral measure $E^{J_0^{(\lim)}}(\Delta)$ of the matrix $J_0^{(\lim)}$, where

$$E_{kl}^{J_0^{(\lim)}}(d\lambda) = e(\lambda)\cos\big((k-l)\nu_0(\lambda)\big)d\lambda, \quad e(\lambda) = \frac{E_{00}^{J_0^{(\lim)}}(d\lambda)}{d\lambda} = d_0(\lambda)\chi_{[-1,1]}(\lambda),$$

and $\nu_0(\lambda)$, and $d_0(\lambda)$ are defined in (2.12). It is convenient to view $\nu_0(\lambda)$ as defined on the whole axis, continuing $\nu_0$ by zero for $\lambda > 1$, and by 1 for $\lambda < -1$. Then we can say, by using the terminology of spectral theory of operators with ergodic (in particularly constant) coefficients, that $\nu_0$ is the Integrated Density of States of the Jacobi matrix $J_0^{(\lim)}$(see [21] for a definition of this quantity).

In the case, where the support of the weight is a union of $q \geq 2$ finite disjoint intervals

$$\sigma = \bigcup_{l=1}^{q} [a_l, b_l], \tag{2.13}$$

the asymptotic formulas are more complicated. To write these formulas, obtained in papers [4, 23, 28], we recall the notion of the equilibrium (harmonic) measure. Given a compact set $\sigma$ (2.13), consider the functional

$$\mathcal{E}_\sigma(m) = -\int_\sigma \int_\sigma \log|\lambda - \mu| m(d\lambda) m(d\mu) \tag{2.14}$$

defined for any non-negative unit measure $m$ supported on $\sigma$. A unique minimizer $\nu$ of this functional is called the equilibrium (or harmonic) measure of the compact $\sigma$. This classical concept of the theory of logarithmic potential has the following electrostatic interpretation. Suppose that $\sigma$ is a conductor and positive linear charges of total charge 1 are placed on $\sigma$. Then the measure $\nu$ describes the equilibrium distribution of charges on $\sigma$. In particular, if $\sigma = [-1, 1]$, then $\nu$ coincides with the measure $\nu_0$ of (2.12). Denote

$$\nu(\lambda) = \nu([\lambda, \infty)), \ \alpha_l = \nu([a_l, \infty)), \ l = 1, \dots, q-1. \tag{2.15}$$

According to [4, 23, 28], there exist functions $\mathcal{D} : \sigma \times \mathbb{T}^{q-1} \to \mathbb{R}_+$, and $\mathcal{G} : \sigma \times \mathbb{T}^{q-1} \to \mathbb{R}$ such that

$$\psi_n(\lambda) = (2d_n(\lambda))^{1/2} \cos\left(\pi n \nu(\lambda) + \gamma_n(\lambda)\right) + o(1), \ n \to \infty, \tag{2.16}$$

where $\psi_n(\lambda)$ is defined in (2.5), and

$$d_n(\lambda) = \mathcal{D}(\lambda, n\alpha), \ \gamma_n(\lambda) = \mathcal{G}(\lambda, n\alpha), \ n\alpha = (n\alpha_1, \dots n\alpha_{q-1}). \tag{2.17}$$

Besides, there exist functions $\mathcal{R} : \mathbb{T}^{q-1} \to \mathbb{R}_+$, and $\mathcal{S} : \mathbb{T}^{q-1} \to \mathbb{R}$, such that the coefficients of the Jacobi matrix $J$ of (2.4) have the following asymptotic form:

$$r_n = \mathcal{R}(n\alpha) + o(1), \ s_n = \mathcal{S}(n\alpha) + o(1), \ n \to \infty, \tag{2.18}$$

where the functions $\mathcal{R}$ and $\mathcal{S}$ are the "finite-band" coefficients, well known in spectral theory and in integrable systems (see e.g. [26]). Functions $\mathcal{D}, \mathcal{G}, \mathcal{R}$, and $\mathcal{S}$ can be expressed via the Riemann theta-function, associated in the standard way with two-sheeted Riemann surface obtained by gluing together two copies of the complex plane slit along the gaps $(b_1, a_2), \dots, (b_{q-1}, a_q), (b_q, a_1)$ of the support of the measure $\nu$, the last gap goes through the infinity.

The components of the vector $\alpha = \{\alpha_l\}_{l=1}^{q-1}$ are rationally independent generically, thus the functions $\mathcal{D}(\lambda, n\alpha), \mathcal{G}(\lambda, n\alpha), \mathcal{R}(n\alpha)$, and $\mathcal{S}(n\alpha)$ are quasi-periodic in $n$. As an early precursor of this fact we mention a result by N. Akhiezer [1], according to which if $\sigma$ consists of two intervals, then a certain characteristic of respective extremal polynomials of degree $n$ can be expressed via the Jacobi elliptic functions as $n \to \infty$. As a result the characteristic does not converge as $n \to \infty$ but has a set of limit points that fill a specific interval generically in the intervals lengths.

Asymptotic relations (2.18) allow us to define the limiting Jacobi matrix in the multi-interval case. Namely, we again define the matrix $J_n$ by formulas (2.9). Then, by choosing the subsequence $\{n_j\}_{j\geq 0}$ such that

$$\lim_{j\to\infty} \{n_j \alpha_l\} = x_l, \ l = 1, \dots, q-1, \tag{2.19}$$

where $x$ is a point of $\mathbb{T}^{q-1}$, and $\{t\}$ denotes the fractional part of $t \in \mathbb{R}$, we obtain from (2.18) that the sequence $\{J_{n_j}\}$ converges strongly to the Jacobi matrix $J^{(\text{lim})}(x)$, defined in $l^2(\mathbb{Z})$ by the coefficients $\{\mathcal{R}(k\alpha + x), \ \mathcal{S}(k\alpha + x)\}_{k\in\mathbb{Z}}$.

The frequencies $\alpha_1, \dots, \alpha_{q-1}$, defined in (2.15), are rationally independent generically with respect to the weight, hence the Jacobi matrix $J^{(\text{lim})}(x)$ is quasi-periodic generically, and $\alpha_1, \dots, \alpha_{q-1}$ are the basis frequencies. The spectrum of $J^{(\text{lim})}(x)$ is (2.13), and its spectral measure $E^{J^{(\text{lim})}}$ can be obtained from (2.6) and from (2.16) as the strong limit of the spectral measures of matrices $J_n$. In particular, the diagonal entries of $E^{J^{(\text{lim})}}$ are

$$E_{kk}^{J^{(\text{lim})}}(d\lambda) = \mathcal{D}(\lambda, k\alpha + x)d\lambda, \ k \in \mathbb{Z}. \tag{2.20}$$

This gives the spectral meaning of the normalizing factor in asymptotic formula (2.16). The function $\nu(\lambda)$ of (2.15) has also a spectral meaning. Namely, it can be shown that it is the Integrated Density of States of the quasi-periodic (thus ergodic) Jacobi matrix $J^{(\text{lim})}$. In particular, for any interval $\Delta$ of the spectral axis the measure $\nu$ is

$$\nu(\Delta) = \lim_{L\to\infty} \frac{1}{2L} \sum_{l=-L}^{L} E_{ll}(\Delta). \tag{2.21}$$

The density $d$ of this measure, known as the Density of States of the quasi-periodic matrix $J^{(\text{lim})}$ [21], is

$$d(\lambda) = \int_{T^{q-1}} \mathcal{D}(\lambda, x)dx, \tag{2.22}$$

and is a particular case of general formulas of the spectral theory of ergodic operators (see e.g. formula (4.12) of [21]).

### 2.3. Asymptotics of orthogonal polynomials with varying weight

In this subsection we discuss a special class of orthogonal polynomials, known as polynomials with varying weight. They are defined by the weight $w_n$, supported in general on the whole real line and depending on a big parameter $n \in \mathbb{N}$:

$$w_n(\lambda) = e^{-nV(\lambda)}, \tag{2.23}$$

where $V$ is a sufficiently regular and growing on infinity function (the latter condition is necessary to guarantee condition (2.1)). Polynomials $\{p_l^{(n)}(\lambda)\}_{l=0}^{\infty}$, orthogonal with respect to weight (2.23), appear in approximation theory [27], and in studies of eigenvalue distribution of random matrices (see [18, 20] and below). The corresponding Jacobi matrix $J^{(n)}$ has the form

$$J^{(n)} = \{J_{kl}^{(n)}\}_{k,l=0}^{\infty},\ J_{kl}^{(n)} = r_{k-1}^{(n)}\delta_{k-1,l} + \delta_{kl}s_l^{(n)} + r_k^{(n)}\delta_{k+1,l}. \tag{2.24}$$

Typically $V$ is a polynomial of an even degree, positive at infinity. The simplest case of polynomial potentials is

$$V = 2\lambda^2. \tag{2.25}$$

In this case the respective polynomials can be expressed via the Hermite polynomials:

$$p_l^{(n)}(\lambda) = (2n)^{1/4}(2^l l!\sqrt{\pi})^{1/2}H_l(\lambda\sqrt{2n}) := (2n)^{1/4}h_l(\lambda\sqrt{2n}), \tag{2.26}$$

where $H_l$ is the Hermite polynomial of the order $l$, the polynomials $\{h_l\}_{l\geq 0}$ are orthonormal on the whole axis with respect to the weight $e^{-x^2}$, and coefficients of the matrix $J^{(n)}$ are

$$r_l^{(n)} = \sqrt{l/4n},\ s_l^{(n)} = 0,\ l \geq 0. \tag{2.27}$$

Recall the Plancherel–Rotah asymptotic formula [25] for $h_l$, according to which for $x = \sqrt{2l+1}\cos\phi,\ 0 < \phi < \pi$, we have for $l \to \infty$:

$$\psi_l(x) := e^{-x^2/2}p_l(x) = \left(\frac{2}{l}\right)^{1/4}\frac{1}{(\pi\sin\phi)^{1/2}}\Big[\cos\big((l+1/2)A(\phi) - \pi/4\big) + o(1/l)\Big], \tag{2.28}$$

where

$$A(\phi) = \phi - \sin 2\phi/2. \tag{2.29}$$

Outside the interval $|x| \leq \sqrt{2l+1}$ the function $\psi_l$ decays exponentially fast as $l \to \infty$.

This asymptotic formula allows us to find asymptotic formulas for the functions

$$\psi_l^{(n)}(\lambda) = e^{-nV(\lambda)} p_l^{(n)}(\lambda) \tag{2.30}$$

with $V$ given by (2.25). However, since now the orthogonal polynomials and coefficients of the associated Jacobi matrix (2.24) depend on two indices $l$ and $n$, we have to choose a proper asymptotic regime. We will assume that $l = n + k$, where $n \to \infty$ and $k$ is an arbitrary fixed integer (cf (2.9)). In this regime we find for $\lambda = \cos\phi,\ 0 < \phi < \pi$ :

$$\psi_{n+k}^{(n)}(\lambda) = (2d_0(\lambda))^{1/2} \cos\big(nA(\phi) + k\phi + \Gamma_0(\phi)\big) + o(1),\ \Gamma_0(\phi) = \phi/2 - \pi/4, \tag{2.31}$$

where $d_0(\lambda)$ is defined in (2.12). Denoting $N_0(\lambda) = \pi^{-1}A(\phi) = \pi^{-1}(\phi - \sin 2\phi/2)$, we find that (cf (2.12))

$$N_0(\lambda) = \int_\lambda^1 \rho_0(\mu)d\mu,\quad \rho_0(\lambda) = \frac{2}{\pi}\sqrt{1-\lambda^2}\chi_{[-1,1]}(\lambda). \tag{2.32}$$

This allows us to rewrite (2.31) in the form (cf (2.11))

$$\psi_{n+k}^{(n)}(\lambda) = (2d_0(\lambda))^{1/2} \cos\big(\pi n N_0(\lambda) + \pi k \nu_0(\lambda) + \Gamma_0(\lambda)\big) + o(1),\ n \to \infty, \tag{2.33}$$

where $\nu_0(\lambda)$ is defined in (2.12).

A new phenomenon here is that the interval of oscillatory asymptotic behavior is not the whole support of the weight, as it was in the previous subsection, but a part of the support, the interval $(-1, 1)$.

In the regime,

$$l = n + k,\ n \to \infty,\ k \text{ is fixed}, \tag{2.34}$$

that we are considering now, the coefficients $r_l^{(n)}$ of (2.27) are constant:

$$\lim r_l^{(n)} = 1/2. \tag{2.35}$$

Here the symbol $\lim \ldots$ denotes the limit (2.34).

Recalling that in the case of the Hermite polynomials the diagonal coefficients vanish, we see that the coefficients (2.35) define the same limiting matrix $J_0^{(\mathrm{lim})}$ as in (2.10). This fact can be viewed as a justification of the regime (2.34), because the spectrum $[-1,1]$ of $J^{(\lim)}$ coincides with the oscillatory interval of asymptotic formula (2.33), as it was in the Bernstein–Szegö case. Besides, the leading term of this formula is a generalized eigenfunction of the matrix $J_0^{(\lim)}$ (a polynomially bounded solution of respective finite-difference equation $(J_0^{(\lim)}\psi)_k = \lambda\psi_k$, $k \in \mathbb{Z}$), as it was for the Bernstein–Szegö asymptotic formulas (2.7), and (2.11) of the previous subsection.

To describe the asymptotic formula for any polynomial function $V$ in (2.23), we need another variational problem:

$$\mathcal{E}_V[m] = -\int_{\mathbf{R}}\int_{\mathbf{R}} \log|\lambda - \mu| m(d\lambda)m(d\mu) + \int_{\mathbf{R}} V_{ext}(\lambda)m(d\lambda), \tag{2.36}$$

where and $m$ is non-negative unit measure.

The variational problem, defined by (2.36), goes back to Gauss and is called the minimum energy problem in the external field $V_{ext}$. The unit measure $N$ minimizing (2.36) is called the equilibrium measure in the external field $V_{ext}$ because of its evident electrostatic interpretation as the equilibrium distribution of linear charges on the ideal conductor occupying the axis $\mathbf{R}$ and confined by the external electric field of potential $V_{ext}$. We stress that the respective variational procedure determines the both, the (compact) support $\sigma$ of the measure and the form of the measure. This should be compared with the variational problem (2.14) of the theory of logarithmic potential, where the external field is absent but the support $\sigma$ is given. It is easy to see, that in the case (2.25) the measure $N$ is given by (2.32 ). The minimum energy problem in the external field (2.36) arises in various domains of analysis and its applications (see recent book [24] for a rather complete account of results and references concerning the problem).

Assume that $V_{ext}$ is such that the support $\sigma$ of the measure $N$ has the form (2.13) of the union of $q$ disjoint intervals. Introduce the non-increasing function (cf (2.15))

$$N(\lambda) = N([\lambda, \infty)), \tag{2.37}$$

and the $(q-1)$-dimensional vector

$$\beta = \{\beta_l\}_{l=1}^{q-1},\ \beta_l = N(a_{l+1}). \tag{2.38}$$

According to [12], there exist functions $d_{n,\kappa}(\lambda)$, and $\Gamma_{n,\kappa}(\lambda)$, $\kappa = 0,1$, and a number $0 < \tau \le 1$ such that if $\lambda$ belongs to the interior of the support $\sigma$ (2.13), then

$$\psi_{n-\kappa}^{(n)}(\lambda) = (2d_{n,\kappa}(\lambda))^{1/2} \cos\left(\pi n N(\lambda) + \Gamma_{n,\kappa}(\lambda)\right) + O\left(n^{-\tau}\right),\ n \to \infty, \tag{2.39}$$

where $\psi_l^{(n)}$ is defined in (2.30). Moreover, $d_{n,\kappa}(\lambda)$ and $\Gamma_{n,\kappa}(\lambda)$ depend on $n$ via the vector $n\beta$, i.e., there exist continuous functions $\mathcal{D}_\kappa : \sigma \times T^{q-1} \to \mathbb{R}_+$, and $\mathcal{G}_\kappa : \sigma \times T^{q-1} \to \mathbb{R}$ such that (cf (2.17))

$$d_{n,\kappa}(\lambda) = \mathcal{D}_\kappa(\lambda, n\beta),\ \Gamma_{n,\kappa}(\lambda) = \mathcal{G}_\kappa(\lambda, n\beta),\ \kappa = 0, 1. \tag{2.40}$$

If $\lambda$ belongs to the exterior of $\sigma$, then each $\psi_{n-\kappa}^{(n)}$ decays exponentially in $n$ as $n \to \infty$.

Similar asymptotic formulas are valid for coefficients of the Jacobi matrix $J^{(n)}$ of (2.24). For the sake of simplicity we restrict ourselves to the case of an even function $V$ in (3.1), where the coefficients $s_l^{(n)}$ in (2.24) vanish. Then, according to [12], there exist a continuous function $\widetilde{\mathcal{R}} : \mathbb{T}^{q-1} \to \mathbb{R}_+$ such that we have

$$r_{n-1}^{(n)} = \widetilde{\mathcal{R}}(n\beta) + O\left(n^{-\tau}\right),\ n \to \infty. \tag{2.41}$$

The functions $\mathcal{D}_\kappa, \mathcal{G}_\kappa$, and $\widetilde{\mathcal{R}}$ has the same structure as the functions $\mathcal{D}, \mathcal{G}$, and $\mathcal{R}$ of formulas (2.17) and (2.18) of the previous subsection. Hence the main difference in asymptotic formulas of the previous subsection and of this subsection is that in the former the "rotation number" and the frequencies are determined by the measure $\nu$, minimizing the functional (2.14), while in the latter these quantities are determined by the measure $N$, minimizing the functional (2.36).

To describe a connection between these measures, it is useful to consider the following one-parameter family of external fields:

$$V_{ext} = \frac{1}{g} V, \tag{2.42}$$

where $V$ does not depend on $g > 0$. This form of the potential with the explicitly written amplitude is widely used in physical applications of the random matrix theory (see e.g.[14]). In the case of the potential of this form the minimizing measure of (2.36) and related quantities will depend on $g$. In particular, it will be the case for the support $\sigma_g$ of the minimizing measure $N_g$, i.e., we have

$$\sigma_g = \bigcup_{l=1}^{q-1} [a_l(g), b_l(g)]. \tag{2.43}$$

Consider now the variational problem (2.14) on the support $\sigma_g$ , and denote the respective minimizing measure $\nu_g$. Then, according to [11], the measures $N_g$, and $\nu_g$ are related as follows:

$$N_g = g^{-1} \int_0^g \nu_\xi d\xi\,. \tag{2.44}$$

Some particular cases of this formula and its spectral meaning were given in [19, 10].

In addition, according to [16], for every $g > 0$, except at most a countable set of values without a finite accumulating point, the number of intervals of $\sigma_g$ does not depend on $g$, the endpoints of the intervals are analytic in $g$, and, according to [12], the function $\widetilde{\mathcal{R}}$ depends continuously on $g$.

This allows us to give an analogue of the limiting Jacobi matrices, introduced above. Indeed, consider the coefficients $r_l^{(n)}$ of the Jacobi matrix (2.24), associated with polynomials $\{p_l^{(n)}\}_{l\geq 0}$ with varying weight. Introducing explicitly the dependence of the coefficients $r_l^{(n)}$, of the frequencies (2.38), and of the function $\widetilde{\mathcal{R}}$ of (2.41) on $g$, we can write in view of (2.23):

$$r_l^{(n)}(g) = r_l^{(l)}(gl/n). \tag{2.45}$$

Setting here $l = n_j + k$, where

$$\lim_{j\to\infty} \{n_j\beta_l\} = x_l, \; l = 1,..,q-1, \tag{2.46}$$

$x$ is a points of $\mathbb{T}^{q-1}$, and $k$ is an arbitrary fixed integer (cf (2.19)), we obtain in view of (2.41):

$$\begin{aligned}\lim_{j\to\infty} r_{n_j+k-1}^{(n_j)}(g) &= \lim \widetilde{\mathcal{R}}\left(\frac{n_j+k}{n_j}g, (n_j+k)\beta\left(\frac{n_j+k}{n_j}g\right)\right) \\ &= \widetilde{\mathcal{R}}\left(g, (g\beta(g))'k + x\right).\end{aligned}$$

Now, by using formula (2.44), we find the relation

$$(g\beta(g))' = \alpha(g),$$

where $\alpha(g)$ is defined by (2.15) with $\nu_g$ instead of $\nu$. We conclude that the limit in the r.h.s. of the above formula is

$$\lim_{j\to\infty} r_{n_j+k-1}^{(n_j)}(g) = \widetilde{\mathcal{R}}\left(g, k\alpha(g) + x\right), \; k \in \mathbb{Z}.$$

We obtain the quasi-periodic Jacobi matrix $\widetilde{J}^{(\mathrm{lim})}(x)$, defined by the coefficients of the r.h.s. of the last formula and having the same frequencies $(\alpha_1, \dots, \alpha_{q-1})$ as the coefficients of the matrix $J^{(\mathrm{lim})}(x)$ of the previous section (recall, that we are considering the case in which diagonal entries of $J^{(n)}$ vanish).

By applying the same limiting argument to the asymptotic formula (2.39) for $\kappa = 0$, and by using (2.44), we obtain that in the regime (2.34) (cf (2.33))

$$\begin{aligned}\psi_{n+k}^{(n)}(\lambda) &= (2\mathcal{D}_0(\lambda, g, k\alpha + x))^{1/2} \\ &\times \cos\Big(\pi n N(\lambda, g) + \pi k \nu(\lambda, g) + \mathcal{G}_0(\lambda, g, k\alpha + x)\Big) + o(1), \; n \to \infty.\end{aligned} \tag{2.47}$$

Setting here $k = -1$ and comparing with formula (2.39) for $\kappa = 1$, we obtain that the functions $\mathcal{D}_1$, and $\mathcal{G}_1$ of this formula are

$$\mathcal{D}_1(\lambda, g, x) = \mathcal{D}_0(\lambda, g, -\alpha + x),\ \mathcal{G}_1(\lambda, g, x) = \mathcal{G}_0(\lambda, g, -\alpha + x). \tag{2.48}$$

By using these formulas, we can compute the limits of the entries of the spectral measure $E^{J^{(n)}}$of $J^{(n)}$ of (2.24) in the regime (2.46):

$$\lim_{j\to\infty} E^{J^{(n_j)}}_{n_j+k,n_j+k}(d\lambda) = \mathcal{D}_0(\lambda, g, k\alpha + x),\ k \in \mathbb{Z}.$$

The limit here is the weak limit of measures. The support in $\lambda$ of the r.h.s. of this formula is the support $\sigma_g$ of the equilibrium measure $N_g$. This and the strong convergence of $J^{(n)}$ to $\widetilde{J}^{(\lim)}(x)$ in the regime (2.46) imply that the spectrum of the quasi-periodic matrix $\widetilde{J}^{(\lim)}(x)$ is $\sigma_g$. Besides, arguing as in the proof of (2.22), we obtain that the function

$$\widetilde{d}(\lambda) = \int\limits_{\mathbb{T}^{q-1}} \mathcal{D}_0(\lambda, g, x)dx \tag{2.49}$$

is the Density of States of the quasi-periodic matrix $\widetilde{J}^{(\lim)}$.

It is tempting to conjecture that the matrix $J^{(\lim)}(x)$ of the preceding subsection and the matrix $\widetilde{J}^{(\lim)}(x)$ of this subsection are strongly related if not coincide, modulo a certain shift in their arguments.

The measures $\nu_g$, and $N_g$ as well as their supports are not simple to find in general. Here is a class of polynomial functions $V$ of (2.23), for which these measures can be found in elementary functions [10].

Let $v$ be a polynomial of the degree $q$ such that all the zeros of the polynomial $v^2 - 4g$ are real and simple. Assume that there exist a constant $C$ such that the potential $V_{ext}$ can be written in the form

$$V_{ext}(\lambda) = \frac{v^2}{2pg} + C. \tag{2.50}$$

Then the support $\sigma_g$ of measures $\nu_g$ and $N_g$ is

$$\sigma_g = \{\lambda \in \mathbb{R} : v^2(\lambda) - 4g \le 0\},$$

the measures $\nu_g$ and $N_g$ are absolutely continuous and their densities $d(\cdot, g)$, and $\rho(\cdot, g)$ are

$$d(\lambda, g) = \frac{|v'(\lambda)|}{\pi p}|v^2(\lambda) - 4g|^{-1/2}\chi_{\sigma_g}(\lambda), \quad \rho(\lambda, g) = \frac{|v'(\lambda)|}{2\pi pg}|v^2(\lambda) - 4g|^{1/2}\chi_{\sigma_g}(\lambda). \tag{2.51}$$

It can also be shown that respective limiting Jacobi matrices are periodic of period $q$, and that

$$\alpha_l = \beta_l = l/(q-1),\ l = 1, \dots, q-1. \tag{2.52}$$

The fact that the spectrum of a periodic Jacobi matrix is the image of a polynomial map is well known in spectral theory (see e.g. [17]). Here this fact appears in a somewhat different setting.

In conclusion we will give one more application of the differentiation with respect to the inverse amplitude in (2.42) as a tool of obtaining asymptotic formulas for orthogonal polynomials with varying weight.

Consider the case, where the potential is a real analytic and even function, and assume that the support of the respective measure $N$ is an interval $[-a, a]$. In this case we have (see e.g. [19]):

$$\rho(\lambda) = \frac{1}{2\pi} P(\lambda)\sqrt{a^2-\lambda^2}\chi_{[-a,a]}(\lambda),\ P(\lambda) = \frac{1}{\pi}\int\limits_{-a}^{a} \frac{V'(\lambda)-V'(\mu)}{\lambda-\mu}\frac{d\mu}{\sqrt{a^2-\lambda^2}}, \tag{2.53}$$

where $a$ is defined by the equation

$$\int\limits_{-a}^{a} \frac{\lambda V'(\lambda)d\lambda}{\sqrt{a^2-\lambda^2}} = 2\pi g. \tag{2.54}$$

According to [5, 12], in this case we have

$$\lim_{n\to\infty} r_{n-1}^{(n)} = \frac{a}{2}, \tag{2.55}$$

and according to [6]

$$r_{n+k-1}^{(n)} = a/2 + ck/n + o(n^{-1}),\ |k| = O(n^{-2/3}), \tag{2.56}$$

where

$$c^{-1} = aP(a). \tag{2.57}$$

The proof of relations (2.56)–(2.57) requires rather involved arguments. Here is a short heuristic derivation of the formulas. It is widely believed that the remainder term in (2.55) is $O(n^{-2})$ (see e.g. [13], where this fact is proved for $V(\lambda) = \text{const} \cdot |\lambda|^\alpha,\ a \geq 2$). Hence, we can write

$$r_{n-1}^{(n)}(g) = a(g)/2 + O(n^{-2}),\ n \to \infty,$$

and we have for $l = n + k,\ |k| = o(n),\ n \to \infty$:

$$\begin{aligned} r^{(n)}_{l-1}(g) &= r^{(l)}_{l-1}((1 + k/n)g) = a\left((1 + k/n)g\right)/2 + O(n^{-2}) \\ &= \frac{a\,(g)}{2} + \frac{a'\,(g)\,g}{2}\frac{k}{n} + o(n^{-1}). \end{aligned}$$

Comparing this formula with (2.56)–(2.57), we see that the coefficient $c$ should be equal $a'/2$. The derivative $a'$ can be computed from the equation (2.54):

$$\frac{2}{a'} = \frac{1}{\pi}\int_{-a}^{a} \frac{V''(\lambda)d\lambda}{\sqrt{a^2 - \lambda^2}}.$$

Now it can be shown the last expression coincides with (2.57), provided that $a$ satisfies (2.54)–(2.53).

## 3. Eigenvalue distribution of random matrices

In recent years the eigenvalue distribution of various ensembles of random matrices has been extensively studied, being motivated by a number of questions in physics and mathematics (see e.g. review [20] and references therein). In particular, of considerable interest are unitary invariant ensembles of Hermitian matrices, known also as matrix models. Their probability distribution is defined by the density

$$p_n(M) = Z_n^{-1}\exp(-n\mathrm{Tr}V(M)/g) \tag{3.1}$$

with respect to the "uniform" measure

$$dM = \prod_{j=1}^{n} dM_{jj} \prod_{j\le k} d\Re M_{jk} d\Im M_{jk} \tag{3.2}$$

in the space of Hermitian matrices $M = \{M_{jk}\}^n_{j,k},\ M_{j,k} = \overline{M_{k,j}}$. In (3.1), $Z_n^{-1}$ is the normalization constant and $V$ is a real-valued function, bounded below and growing faster than $2\log|\lambda|$ as $|\lambda| \to \infty$. Typically $V$ is a polynomial of degree $2p$ positive at infinity, although much broader classes of potentials are also studied. Among numerous problems of the theory and of its applications we mention three asymptotic problems on the large $n$ behavior the eigenvalue counting measure of random matrices. The measure is defined as follows.

Denote $\lambda^{(n)}_1 \le \ldots \le \lambda^{(n)}_n$ eigenvalues of a random matrix, and define the measure

$$N_n(\Delta) = \sharp\{\lambda^{(n)}_l \in \Delta\}n^{-1}, \tag{3.3}$$

where $\Delta$ is an interval of the real axis. $N_n$ is called the normalized counting measure of eigenvalues. We are interested in the asymptotic behavior of the following three characteristics of this random measure:

$(p_1)$ Expectation

$$\overline{N}_n(\Delta) = \mathbf{E}\{N_n(\Delta)\}, \tag{3.4}$$

where the symbol $\mathbf{E}\{...\}$ denotes the expectation with respect to the distribution (3.1)–(3.2).

$(p_2)$ Covariance

$$\mathbf{Cov}\{N_n(\Delta)_1, N_n(\Delta_2)\} = \mathbf{E}\{N_n(\Delta_1)N_n(\Delta_2)\} - \mathbf{E}\{N_n(\Delta_1)\}\mathbf{E}\{N_n(\Delta_2)\}.$$

$(p_3)$ Probability distribution $\mathbf{P}\{N_n(\Delta) = k/n\}$, in particular the hole probability

$$E_n(\Delta) = \mathbf{P}\{N_n(\Delta) = 0\}.$$

It is well known in random matrix theory (see e.g. [18]) that the above quantities can be expressed via orthogonal polynomials with varying weight $e^{-nV}$. Namely, introduce the reproducing kernel

$$K_n(\lambda, \mu) = \sum_{l=0}^{n-1} \psi_l^{(n)}(\lambda)\psi_l^{(n)}(\mu) \tag{3.5}$$

of the orthonormalized system (2.30). Then we have [18, 22]:

$$\overline{N}_n(\Delta) = \int_\Delta \rho_n(\lambda)d\lambda, \quad \rho_n(\lambda) = n^{-1}K_n(\lambda, \lambda), \tag{3.6}$$

$$\mathbf{Cov}\{N_n(\Delta)_1, N_n(\Delta_2)\} = \frac{1}{2n^2}\int\int [\chi_{\Delta_1}(\lambda_1) - \chi_{\Delta_2}(\lambda_2)]^2 K_n^2(\lambda_1, \lambda_2)d\lambda_1 d\lambda_2, \tag{3.7}$$

and

$$E_n(\Delta) = \det(1 - K_n(\Delta)), \tag{3.8}$$

where $K_n(\Delta)$ is the integral operator defined by

$$(K_n(\Delta)f)(\lambda) = \int_\Delta K_n(\lambda, \mu)f(\mu)d\mu, \ \lambda \in \Delta,$$

and $\det(1 - K_n(\Delta))$ is its Fredholm determinant.

By using the Christoffel–Darboux formula

$$K_n(\lambda, \mu) = r_{n-1}^{(n)} \frac{\psi_n^{(n)}(\lambda)\psi_{n-1}^{(n)}(\mu) - \psi_{n-1}^{(n)}(\lambda)\psi_n^{(n)}(\mu)}{\lambda - \mu}, \tag{3.9}$$

we find that the problems $(p_1)-(p_3)$ can be studied provided that asymptotic formulas for the polynomials $p_{n-1}^{(n)}$ and $p_n^{(n)}$ are known. Since however, the respective asymptotic formulas were not available till the recent paper [12], the progress of the theory was based on other means. In particular, the problem $(p_1)$ was solved in [8, 15] by using the variational method. More precisely, it was proved that for any interval $\Delta$ of the spectral axis the normalized counting measure (3.3) of the ensemble (3.1)–(3.2) converges in probability to the non-random limit, known as the Integrated Density of States (IDS) of the ensemble, and coinciding with the minimizing measure $N_g$ of the functional (2.36)–(2.42). This implies, in particular, that the measure $\overline{N}_n$ of (3.4) converges weakly to the measure $N_g$.

Furthermore, it was found in [22] that under certain conditions on $V$ in (3.1)–(3.2) (valid for all polynomial potentials) the inequality $\rho(\lambda_0, g) > 0$, where $\rho$ is the density of the measure $N$, implies that

$$\lim_{n\to\infty} E_n\left(\left(\lambda_0, \lambda_0 + \frac{s}{n\rho(\lambda_0, g)}\right)\right) = \det(1 - S_s), \tag{3.10}$$

where $S_s$ is the integral operator, defined by

$$(S_s f)(\xi) = \int_0^s \frac{\sin\pi(\xi - \eta)}{\pi(\xi - \eta)} f(\eta) d\eta,\ 0 \le \xi \le s\,. \tag{3.11}$$

We see that the limit (3.10) does not depend on $V$ in (3.1), coinciding, for example, with that for the simplest Gaussian case (2.25 ). This property is called in the random matrix theory the universality of the local eigenvalue statistics [18, 20]. The property was also proved and considerably detailized in paper [12], as an application of asymptotic formulas (2.39) for the polynomials with varying real analytic weight, obtained in the paper.

It is instructive, however, to obtain formulas for the IDS, and for the kernel of the operator $S_s$ of the universality statement (3.10)–( 3.11), by using the above asymptotic formulas for orthogonal polynomials with the varying weight, and the trick of the varying of the amplitude $g$.

We will use formula (3.6) for the density $\rho_n$ of the mean measure $\overline{N}_n$ of (3.4). The formula includes the orthonormalized functions $\psi_l^{(n)}$ of (2.30) for $l = 0, \dots, n-1$. Indicating explicitly the dependence of these functions on $g$ and using the relation (cf (2.45)):

$$\psi_l^{(n)}(\lambda, g) = \psi_l^{(l)}(\lambda, gl/n), \tag{3.12}$$

we obtain from (3.6) and from (2.39) that the leading contribution to $\rho_n$ as $n \to \infty$ is

$$\frac{1}{n}\sum_{l=0}^{n-1} \mathcal{D}_0\left(\lambda, gl/n, l\alpha(gl/n)\right),$$

where the function $\mathcal{D}_0$ is defined in (2.39)–(2.40). Now we will use the fact that the functions $\mathcal{D}_0(\lambda, g, x)$, and $\alpha(g)$ depend continuously $g$. Hence the summand in the last formula is "slow" varying in $l/n$ and is "fast" varying in $l$. This observation results in the limiting formula for the density $\rho(\lambda, g)$ of the Integrated Density of States $N_g$ of the ensemble (3.1)–(3.2) (the minimizing measure of the functional (2.36)–(2.42 )):

$$\rho(\lambda, g) = \int_0^1 dt \int_{\mathbb{T}^{q-1}} \mathcal{D}_0(\lambda, tg, x)dx. \tag{3.13}$$

Comparing this formula with formula (2.49) and with (2.44), we conclude that the Density of States (2.22) of the matrix $J^{(\lim)}$ and the Density of States (2.49) of the matrix $\widetilde{J}^{(\lim)}$ coincide. This fact can be viewed as a support of our conjecture, according to which the quasi-periodic matrices $J^{(\lim)}(x)$ and $\widetilde{J}^{(\lim)}(x)$ coincide up to a shift in their arguments.

Consider now formulas (3.10)–(3.11). It can be shown that the proof of these formulas reduces to the proof of the validity of the limiting relation

$$\lim_{n\to\infty} \left(n\rho_n(\lambda_0)\right)^{-1} K_n(\lambda_0 + \xi_1/n\rho_n(\lambda_0), \lambda_0 + \xi_2/n\rho_n(\lambda_0)) = \frac{\sin\pi(\xi_1 - \xi_2)}{\pi(\xi_1 - \xi_2)} \tag{3.14}$$

uniformly on compacts in $(\xi_1, \xi_2)$, and for any $\lambda_0$ such that $\rho(\lambda_0, g) \neq 0$ (see [22]). According to ( 3.12), (3.5), and (2.39), the expression under the "lim " sign in the last formula is for $\lambda_{1,2} = \lambda_0 + \xi_{1,2}/\rho_n(\lambda_0)$, and for $n \to \infty$:

$$\left(\rho_n(\lambda_0, g)\right)^{-1} \sum_{l=1}^{n-1} \psi_l^{(l)}(\lambda_1, gl/n)\psi_l^{(l)}(\lambda_2, gl/n) = \left(n\rho_n(\lambda_0, g)\right)^{-1}$$

$$\times \frac{1}{n}\sum_{l=1}^{n-1} \mathcal{D}_0(\lambda, gl/n, l\alpha(gl/n)) \cos\left(\pi(\xi_1 - \xi_2)\frac{\rho(\lambda_0, gl/n)}{\rho(\lambda_0, g)} l/n\right) + o(1),$$

where we have used the asymptotic relation

$$\mathcal{D}_0\big(\lambda_1, gl/n, l\alpha(lg)/n\big) - \mathcal{D}_0\big(\lambda_2, gl/n, l\alpha(gl/n)\big) = o(1),$$

the analogous relation for $\mathcal{G}\big(\lambda_{1,2}, gl/n, l\alpha(gl/n)\big)$, and the relation

$$N(\lambda_1, gl/n) - N(\lambda_2, gl/n) = (\xi_1 - \xi_2)\rho(\lambda_0, gl/n)/\rho(\lambda_0, g) + o(1).$$

Taking again into account that the dependence on $l/n$ is "slow", and that the dependence on $l$ is "fast" in the above formulas, we obtain that the leading term of the r.h.s. of the last formula is

$$\rho^{-1}(\lambda_0, g)\int\limits_0^1 \cos\left(\pi(\xi_1 - \xi_2)\frac{t\rho(\lambda_0, gt)}{\rho(\lambda_0, g)}\right) dt \int\limits_{\mathbb{T}^q} \mathcal{D}_0(\lambda, tg, x)dx.$$

This formula and the relations (3.13) and $d(\lambda, tg)dt = d_t\left(t\rho(\lambda, tg)\right)$ imply (3.14).

As for the problem $(p_2)$, it was studied in several physical papers. In particular, it was found in [7, 9] that in the case where the support $\sigma$ of the measure $N$ consists of a single interval $[-a, a]$ and the indicators $\chi_{\Delta_{1,2}}$ in (3.7) are replaced by continuously differentiable functions $\varphi_{1,2}$ of compact support, i.e., $N_n(\Delta_{1,2})$ are replaced by the linear statistics

$$N_n[\varphi_{1,2}] = \frac{1}{n}\sum_{l=1}^{n} \varphi_{1,2}(\lambda_l^{(n)}), \tag{3.15}$$

we have

$$\begin{aligned} &\lim_{n\to\infty} n^{-2}\mathbf{Cov}\{N_n[\varphi_1], N_n[\varphi_2]\} &(3.16)\\ &= \frac{1}{4}\int\limits_\sigma d\lambda \int\limits_\sigma d\mu \frac{(\varphi_1(\lambda) - \varphi_1(\mu))(\varphi_2(\lambda) - \varphi_2(\mu))}{(\lambda - \mu)^2}(a^2 - \lambda\mu)d(\lambda)d(\mu)\,. \end{aligned}$$

The formula was justified in [15]. We see that in the case, where the support of the measure $N$ consists of the single interval, the limiting form of the covariance of linear statistic (3.15) depends on $V$ only via the endpoints of the support, determined by the variational problem for the functional (2.36).

By using formulas (3.9) and (2.39) for a general case, where the support of the Integrated Density of States $N$ consists of $q$ intervals, we obtain that

$$n^{-2}\mathbf{Cov}\{N_n[\varphi_1], N_n[\varphi_2]\} = C_n + o(1),\ n \to \infty, \tag{3.17}$$

where

$$C_n = \widetilde{\mathcal{R}}^2(\lambda,\omega_n)\int_\sigma d\lambda\int_\sigma d\mu\frac{(\varphi_1(\lambda)-\varphi_1(\mu))(\varphi_2(\lambda)-\varphi_2(\mu))}{(\lambda-\mu)^2} \tag{3.18}$$

$$\times \Big(\mathcal{D}_0(\lambda,\omega_n)\mathcal{D}_1(\lambda,\omega_n)) - \mathcal{D}_0^{1/2}(\lambda,\omega_n)\mathcal{D}_1^{1/2}(\lambda,\omega_n)\mathcal{D}_0^{1/2}(\mu,\omega_n)\mathcal{D}_1^{1/2}(\mu,\omega_n)$$

$$\times \cos\big(\mathcal{G}_0(\lambda,\omega_n)-\mathcal{G}_1(\lambda,\omega_n)\big)\cos\big(\mathcal{G}_0(\mu,\omega_n)-\mathcal{G}_1(\mu,\omega_n)\big)\Big),$$

and $\omega_n = n\beta \in \mathbb{T}^{q-1}$.

We mention two differences of this formula from formula (3.16), corresponding to a single interval case (it can be checked that (3.18) reduces to (3.16) if $q=1$).

The first is that the amplitude $C_n$ of the leading term of the covariance depends non-trivially on $n$ if the support of the IDS of respective matrix ensemble consists of $q \geq 2$ intervals. The dependence is quasi-periodic generically, when the numbers $\beta_l,\ l=1,\dots,q-1$ are rationally independent. In the non-generic case of periodic Jacobi matrix $\widetilde{J}^{(\mathrm{lim})}$ the dependence of the amplitude $C_n$ is also periodic. In particular, in the case (2.52), the period is $q-1$ . This does not agree with results of physical papers [2], according to which the amplitude does not depend on $n$.

The second difference is that the amplitude $C_n$ of leading term of the covariance depends on the potential not only via the edges of the support of the IDS of respective random matrix ensemble, but also via the frequencies $\beta_l$, $l=1,\dots,q-1$. The dependence also disappears in the periodic case (2.52), thus in this case the form of $C_n$ is completely determined by the endpoints of the support. The simplest case is where the potential is even, $\sigma=[-b,a]\cup[a,b]$, where $0<a<b<\infty$, and where the matrix $\widetilde{J}^{(\mathrm{lim})}$ is of period 2 (in particular, this is the case for potential (2.50) with $q=2$). In this case it can be shown, by using formula (2.39), that

$$C_n = \frac{1}{2\pi^2}\int_\sigma d\lambda\int_\sigma d\mu\frac{(\varphi_1(\lambda)-\varphi_1(\mu))(\varphi_2(\lambda)-\varphi_2(\mu))}{(\lambda-\mu)^2}$$

$$\times \big((\lambda\mu-a^2)(\lambda\mu-b^2)-(-1)^n ab(\lambda-\mu)^2\big)\Big/X(\lambda)X(\mu),$$

where $X(\lambda)=\sqrt{(b^2-\lambda^2)(\lambda^2-a^2)},\ \lambda\in\sigma$.

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