Matematicheskaya fizika, analiz, geometriya
2002, v. 9, No. 2, p. 233–242

The Markov trigonometric moment problem in controllability problems for the wave equation on a half-axis

G.M. Sklyar

Szczecin University, Institute of Mathematics
15 Wielkopolska Str., Szczecin, 70451, Poland
E-mail:sklar@sus.univ.szczecin.pl

Department of Mathematical Analysis, V.N. Karazin Kharkov National University
4 Svobody Sq., Kharkov, 61077, Ukraine
E-mail: sklyar@univer.kharkov.ua

L.V. Fardigola

Department of Mathematical Analysis, V.N. Karazin Kharkov National University
4 Svobody Sq., Kharkov, 61077, Ukraine
E-mail:fardigola@univer.kharkov.ua

Mathematics Division, B. Verkin Institute for Low Temperature Physics and Engineering
National Academy of Sciences of Ukraine
47 Lenin Ave., Kharkov, 61103, Ukraine
E-mail:fardigola@ilt.kharkov.ua

Received November 30, 2001
Communicated by V.I. Korobov

In this paper necessary and sufficient conditions of null-controllability and approximate null-controllability are obtained for the wave equation on a half-axis. Controls solving these problems are found explicitly. Moreover bang-bang controls solving the approximate null-controllability problem are constructed with the aid of solutions of the Markov trigonometric moment problem.

0. Introduction

One of the most general-accepted ways for studying control systems with distributed parameters is to write them in the form

$$\frac{d\mathrm{w}}{dt} = A\mathrm{w} + Bu, \qquad t \in (0, T), \tag{0.1}$$

Mathematics Subject Classification 2000: 93B05, 35L05.

where $T > 0$, w : $(0,T) \longrightarrow \mathcal{H}$ is an unknown function, $u : (0,T) \longrightarrow H$ is a control, $\mathcal{H}$, H are Banach spaces, A is an infinitesimal operator in $\mathcal{H}$, $B : H \longrightarrow \mathcal{H}$ is a linear bounded operator (see, e.g., [1]–[6]). An important advantage of this approach is a possibility to employ ideas and technique of the semigroup operator theory. At the same time it should be noticed that the most substantial and important for applications results on operator semigroups deal with the case when the semigroup generator A has a discrete spectrum or a compact resolvent and therefore the semigroup may be treated by means of eigenelements of A. These assumptions correspond to differential equations in bounded domains only.

In this work we consider the wave equation on a half-axis

$$\frac{\partial^2 w(x,t)}{\partial t^2} - \frac{\partial^2 w(x,t)}{\partial x^2} = 0, \qquad x > 0,\ t \in (0,T), \tag{0.2}$$

controlled by the boundary condition

$$w(0,t) = u(t), \qquad t \in (0,T), \tag{0.3}$$

where $T > 0$. We also assume that the control u satisfies the restriction

$$u \in \mathcal{B}(0,T) = \left\{ v \in L^2(0,T) \mid |v(t)| \le 1 \text{ almost everywhere on } (0,T) \right\}. \tag{0.4}$$

All functions appearing in equation (0.2) are defined for $x \ge 0$. Further, we assume everywhere that a function with domain $\mathcal{D} \subset \mathbb{R}$ are defined for $x \in \mathbb{R}$ and vanish for $x \in \mathbb{R} \backslash \mathcal{D}$.

Let us give definitions of the spaces used in our work. Let $\mathcal{S}$ be the Schwartz space [9]

$$\mathcal{S} = \{\varphi \in C^\infty(\mathbb{R}^n) \mid \forall m \in \mathbb{N}\ \forall l \in \mathbb{N}$$
$$\sup\{|D^\alpha \varphi(x)| \left(1 + |x|^2\right)^l \mid x \in \mathbb{R}^n \wedge |\alpha| \le m\} < +\infty\},$$

and let $\mathcal{S}'$ be the dual space, here $D = (-i\partial/\partial x_1, \dots, -i\partial/\partial x_n)$, $\alpha = (\alpha_1, \dots \alpha_n)$ is a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

A distribution $f \in \mathcal{S}'$ is said to be *odd* if $(f, \varphi(x)) = -(f, \varphi(-x))$, $\varphi \in \mathcal{S}$, and *even* if $(f, \varphi(x)) = (f, \varphi(-x))$, $\varphi \in \mathcal{S}$. Let $\Omega : \mathcal{S}' \longrightarrow \mathcal{S}'$ with $D(\Omega) = \{g \in \mathcal{S}' \mid \operatorname{supp} g \subset [0, +\infty)\}$ be the odd extension operator: $(\Omega g, \varphi(x)) = (g, \varphi(x)) -(g, \varphi(-x))$, $\varphi \in \mathcal{S}$, and $\Theta : \mathcal{S}' \longrightarrow \mathcal{S}'$ with $D(\Theta) = \{g \in \mathcal{S}' \mid \operatorname{supp} g \subset [0, +\infty)\}$ be the even extension operator: $(\Omega g, \varphi(x)) = (g, \varphi(x)) + (g, \varphi(-x))$, $\varphi \in \mathcal{S}$.

It is easy to see that

$$(\Omega f)' = \Theta f', \quad (\Theta f)' = \Omega f', \qquad f \in D(\Omega) = D(\Theta). \tag{0.5}$$

Denote by H_l^s ($l \in \mathbb{R}$, $s \in \mathbb{R}$) the following Sobolev spaces:

$$H_l^s = \left\{ \varphi \in \mathcal{S}' \mid \left(1 + |x|^2\right)^{l/2} \left(1 + |D|^2\right)^{s/2} \varphi \in L^2(\mathbb{R}^n) \right\},$$

$$\|\varphi\|_l^s = \left(\int_{-\infty}^{+\infty} \left| \left(1+|x|^2\right)^{l/2} \left(1+|D|^2\right)^{s/2} \varphi(x) \right|^2 dx \right)^{1/2}.$$

Using [10, Ch. 1] one can see that if $\varphi \in H_l^s$ then $\varphi' \in H_l^{s-1}$ and

$$\left\|\varphi'\right\|_l^{s-1} \leq \|\varphi\|_l^s, \tag{0.6}$$

Further we assume throughout the paper that $s \leq 0$, $l \leq 0$ and use the spaces $\mathcal{H}_l^s(\mu) = \left\{\varphi \in H_l^s \times H_l^{s-1} \mid \operatorname{supp}\varphi \subset [0,\mu]\right\}$, $\mathcal{H}_l^s(+\infty) = \mathcal{H}_l^s$, $\widetilde{H}_l^s = \{\varphi \in H_l^s \times H_l^{s-1} \mid \varphi \text{ is odd}\}$ with the norm $|||\varphi|||_l^s = \left(\left(\|\varphi_0\|_l^s\right)^2 + \left(\|\varphi_1\|_l^{s-1}\right)^2\right)^{1/2}$. Denote by A and B the following operators

$$A = \begin{pmatrix} 0 & 1 \\ d^2/dx^2 & 0 \end{pmatrix}, \qquad A : \widetilde{H}_l^{s-2} \longrightarrow \widetilde{H}_l^{s-2}, \qquad D(A) = \widetilde{H}_l^s, \tag{0.7}$$

$$B = \begin{pmatrix} 0 \\ -2\delta'(x) \end{pmatrix}, \qquad B : \mathbb{R} \longrightarrow \widetilde{H}_l^{s-2}, \qquad D(B) = \mathbb{R}, \tag{0.8}$$

where δ is the Dirac function.

The semigroup generated by A can be explicitly represented by the translation operator and the differentiation operator. In Section 1 it makes possible to obtain necessary and sufficient conditions for null-controllability and approximate null-controllability for system (0.2), (0.3) with restrictions on the control (0.4). Controls solving the problems of null-controllability and approximate null-controllability are found explicitly. But these controls may be of a rather complicated form.

The main goal of Section 2 is to find bang-bang controls solving the approximate null-controllability problem (a control $u \in \mathcal{B}(0,T)$ is called a bang-bang one if it has a finite number of discontinuity points and $|u(t)| = 1$ almost everywhere on $(0,T)$). We show that this problem can be reduced to the Markov trigonometric moment problem. We can construct solutions of the Markov trigonometric moment min-problems for finite sequences by the method given in [7]. These solutions give us bang-bang controls solving the approximate null-controllability problem.

1. Null-controllability problems

Consider control system (0.2), (0.3) with the initial conditions

$$\begin{cases} w(x,0) = w_0^0(x) \\ \partial w(x,0)/\partial t = w_1^0(x) \end{cases}, \qquad x > 0, \tag{1.1}$$

and the steering conditions

$$\left\{\begin{array}{rcl} w(x,T) & = & w_0^T(x) \\ \partial w(x,T)/\partial t & = & w_1^T(x) \end{array}\right., \qquad x>0, \tag{1.2}$$

where $w^0 = \begin{pmatrix} w_0^0 \\ w_1^0 \end{pmatrix} \in \mathcal{H}_l^s$, $w^T = \begin{pmatrix} w_0^T \\ w_1^T \end{pmatrix} \in \mathcal{H}_l^s$. We consider solutions of problem (0.2), (0.3) in the space $\mathcal{H}_l^s$.

Let $T>0$, $w^0 \in \mathcal{H}_l^s$. Denote by $\mathcal{R}_T(w^0)$ the reachability set, i.e. the set of states $w^T \in \mathcal{H}_l^s$ for which there exists a control $u \in \mathcal{B}(0,T)$ such that problem (0.2), (0.3), (1.1), (1.2) has a unique solution. Obviously, if $T' < T$ then $\mathcal{R}_{T'}(w^0) \subset \mathcal{R}_T(w^0)$.

Definition 1.1. *A state $w^0 \in \mathcal{H}_l^s$ is called null-controllable at a given time T if 0 belongs to $\mathcal{R}_T(w^0)$ and approximately null-controllable at this time if 0 belongs to the closure of $\mathcal{R}_T(w^0)$ in $\mathcal{H}_l^s$.*

Problem (0.2), (0.3), (1.1) is equivalent to the Cauchy problem for system (0.1) with the initial condition

$$\mathrm{w}(\cdot,0) = \mathrm{w}^0 \tag{1.3}$$

where $\mathrm{w}^0 \in \widetilde{H}_l^s$, A and B defined by (0.7) and (0.8) respectively.

Let $\mathcal{T}_h$ be the translation operator: $(\mathcal{T}_h f)(x) = f(x+h)$ for a function f with domain $\mathbb{R}$ and $(\mathcal{T}_h f, \varphi) = (f, \mathcal{T}_{-h}\varphi)$, $\varphi \in \mathcal{S}$, for a distribution $f \in \mathcal{S}'$. Then

$$2^{-|l|/2}\left(1+|h|^2\right)^{-|l|/2}\|f\|_l^s \le \|\mathcal{T}_h f\|_l^s \le 2^{|l|/2}\left(1+|h|^2\right)^{|l|/2}\|f\|_l^s, \qquad f \in H_l^s \tag{1.4}$$

Denote

$$S(t) = \frac{1}{2}\begin{pmatrix} \mathcal{T}_t + \mathcal{T}_{-t} & (d/dx)^{-1}\left(\mathcal{T}_t - \mathcal{T}_{-t}\right) \\ (d/dx)\left(\mathcal{T}_t - \mathcal{T}_{-t}\right) & \mathcal{T}_t + \mathcal{T}_{-t} \end{pmatrix} \tag{1.5}$$

where $(d/dx)^{-1}\left(\mathcal{T}_t - \mathcal{T}_{-t}\right) f = \mathcal{F}^{-1}\left[(1/i\sigma)\left(e^{i\sigma t} - e^{-i\sigma t}\right)\mathcal{F}f\right]$, $\mathcal{F} : \mathcal{S}' \longrightarrow \mathcal{S}'$ is the Fourier transform operator. Obviously $S(t)$ is the C_0-semigroup generated by A and

$$\mathrm{w}(\cdot,t) = S(t)\mathrm{w}^0 + \int_0^t S(t-\tau)Bu(\tau)\,d\tau, \qquad t \in [0,T], \tag{1.6}$$

is a unique solution of (0.1), (1.3) in $\widetilde{H}_l^s$.

Taking into account (1.4) it is easy to see that

$$\left\|(1/2)\,(d/dx)^{-1}\,(\mathcal{T}_T-\mathcal{T}_{_T})\,g\right\|_l^{s-1}\le 2^{(2-l)/2}T(1+T^2)^{-l/2}\,\|g\|_l^{s-1}\,,\qquad g\in\widetilde{H}_l^s.$$

Therefore

$$\|S(T)g\|_l^s\le 2^{(2-l)/2}(1+T^2)^{(1-l)/2}\,\|g\|_l^s\,,\qquad g\in\widetilde{H}_l^s\,. \tag{1.7}$$

Denote $R_T\left(\mathrm{w}^0\right)=\left\{S(T)(\mathrm{w}^0+\int_0^T S(-\tau)Bu(\tau)\,d\tau)\mid u\in\mathcal{B}(0,T)\right\}$ for $\mathrm{w}^0\in\widetilde{H}_l^s$. Then from (1.6) we get

Corollary 1.1. *A state $w^0\in\mathcal{H}_l^s$ is null-controllable at a given time $T>0$ iff 0 belongs to $R_T(\mathrm{w}^0)$ and approximately null-controllable at this time iff 0 belongs to the closure of $R_T(\mathrm{w}^0)$ in $\widetilde{H}_l^s$ where $\mathrm{w}^0=\Omega w^0$.*

Theorem 1.1. *A state $w^0\in\mathcal{H}_l^s$ is approximately null-controllable at a given time $T>0$ iff the following three conditions hold:*

$$w_1^0=w_0^{0\prime}\qquad \textit{in } H_l^{s-1}, \tag{1.8}$$

$$w_0^0\in L^2[0,+\infty)\qquad \textit{and}\qquad \left|w_0^0(x)\right|\le 1\quad \textit{almost everywhere on } (0,+\infty). \tag{1.9}$$

$$\operatorname{supp} w_0^0\subset[0,T]. \tag{1.10}$$

Moreover, if conditions (1.8)–(1.10) are valid then the state w^0 is null-controllable at the time T. Under these conditions $T_*=\max\operatorname{supp} w_0^0$ is the optimal time and $u(t)=w_0^0(t)$ $(t\in[0,T_*])$ is the time-optimal control for null-controllability.

P r o o f. *Sufficiency of (1.8)–(1.10).* Let $w^0\in\mathcal{H}_l^s$, $T>0$ and (1.8), (1.9) be satisfied. For all $u\in\mathcal{B}(0,T)$ we have

$$\int_0^T S(-\tau)Bu(\tau)\,d\tau=\mathcal{F}^{-1}\left(\int_0^T\begin{pmatrix}\left(e^{-i\sigma\tau}-e^{i\sigma\tau}\right)\\ \left(e^{-i\sigma\tau}+e^{i\sigma\tau}\right)i\sigma\end{pmatrix}u(\tau)\,d\tau\right)=\begin{pmatrix}\Omega u\\ \Omega u'\end{pmatrix} \tag{1.11}$$

Put $u(t)=w_0^0(t)$ $(t\in[0,T])$ and suppose that $w(\cdot,t)\in H_l^s$ $(t\in[0,T])$ is a solution of problem (0.2), (0.3), (1.1), (1.2). Then $\mathrm{w}(\cdot,t)=\Omega\begin{pmatrix}w(\cdot,t)\\ \partial w(\cdot,t)/\partial t\end{pmatrix}$ $(t\in[0,T])$ is a solution of (0.1), (1.3). It follows from (0.5), (1.6), (1.8), (1.10), (1.11) that

$\mathrm{w}(x,T) = S(T)\left(\begin{array}{c}\left(\Omega(w_0^0-u)\right)(x)\\ \left(\Theta(w_0^0-u)\right)'(x)\end{array}\right) = 0$. By Corollary 1.1 it means that the state w^0 is null-controllable at the time T. Thus sufficiency of (1.8)–(1.10) is shown.

Necessity of (1.8)–(1.10). Let $w^0 \in \mathcal{H}_l^s$ be approximately null-controllable at a time $T > 0$. Then by Corollary 1.1 we have that the origin belongs to the closure of $R(\mathrm{w}^0)$ in $\widetilde{H}_l^s$ where $\mathrm{w}^0 = \Omega w^0$. Hence for each $n \in \mathbb{N}$ there exists a state $\mathrm{w}^n \in R(\mathrm{w}^0)$ such that

$$|||\mathrm{w}^n|||^s < \frac{1}{n}, \tag{1.12}$$

Then with regard to (1.6), (1.11) for some $u_n \in \mathcal{B}(0,T)$ we have

$$\mathrm{w}^n = S(T)\left(\begin{array}{c}\mathrm{w}_0^0 - \Omega u_n\\ \mathrm{w}_1^0 - \Omega u_n'\end{array}\right). \tag{1.13}$$

Taking into account (1.7) and (0.5) we obtain $u_n' \longrightarrow w_1^0$ and $u_n \longrightarrow w_0^0$ as $n \longrightarrow \infty$ in H_0^{s-1}. Therefore (1.8) and (1.10) hold.

Thus u_n weakly converges to w_0^0 as $n \longrightarrow \infty$ in $\mathcal{S}'$ and $\left(L^2(\mathbb{R})\right)'$ (because $\mathcal{S}$ is dense in $L^2(\mathbb{R})$). By the Riesz theorem we conclude that $w_0^0 \in L^2(\mathbb{R})$. Then since $u_n \in \mathcal{B}(0,T)$ we get (1.9). The proof is complete.

Remark 1.1. *Let $w^0 \in \mathcal{H}_l^s$, $\mathrm{w}^0 = \Omega w^0$, $T > 0$, $u \in \mathcal{B}(0,T)$ and let (1.8)–(1.10) be satisfied. Then $\mathrm{w}(\cdot,T) = S(T)\left(\begin{array}{c}\Omega\left(w_0^0-u\right)\\ \left(\Theta\left(w_0^0-u\right)\right)'\end{array}\right)$ in $\widetilde{H}_l^s$ where $\mathrm{w}(\cdot,t) \in \widetilde{H}_l^s$ ($t \in [0,T]$) is the solution of problem (0.1), (1.3).*

2. The Markov trigonometric moment problem in controllability problems

The the approximate null-controllability problem solution found in Section 1 for (0.2), (0.3), (1.1) may be too complicated for practical purposes. In this section we find a bang-bang control solving this problem. To do this we consider the approximate null-controllability problem for the wave equation on $[0,\mu]$ and construct its bang-bang solution by an application of the method for solving the Markov trigonometric moment min-problem [7]. Then we show (Corollary 2.1) that the obtained solution is also a solution of the approximate null-controllability problem for (0.2), (0.3), (1.1).

Consider the following control problem for the wave equation on $[0,\mu]$:

$$\frac{\partial^2 v(x,t)}{\partial t^2} - \frac{\partial^2 v(x,t)}{\partial x^2} = 0, \qquad x \in (0,\mu),\ t \in (0,T), \tag{2.1}$$

$$v(0,t)=u(t),\quad v(\mu,t)=0,\qquad t\in(0,T), \tag{2.2}$$

$$\left\{\begin{array}{rcl} v(x,0) & = & v_0^0(x) \\ \partial v(x,0)/\partial t & = & v_1^0(x) \end{array}\right., \qquad x\in(0,\mu), \tag{2.3}$$

where $T>0$, $u\in\mathcal{B}(0,T)$, $v^0=\begin{pmatrix} v_0^0 \\ v_1^0\end{pmatrix}\in\mathcal{H}_0^s(\mu)$. One can see that this problem can be represented in the form

$$\frac{d\mathrm{v}}{dt}=A\mathrm{v}+\mathcal{B}u,\qquad t\in(0,T), \tag{2.4}$$

$$\mathrm{v}(\cdot,0)=\mathrm{v}^0 \tag{2.5}$$

where A defined by (0.7), $\mathcal{B}=\sum_{k\in\mathbb{Z}}\begin{pmatrix} 0 \\ -2\delta'(x+2k\mu)\end{pmatrix}$, $\mathcal{B}:\mathbb{R}\longrightarrow\widetilde{H}_l^{s-2}$, $D(\mathcal{B})=\mathbb{R}$, $\mathrm{v}=\sum_{k\in\mathbb{Z}}\mathcal{T}_{2k\mu}\Omega\begin{pmatrix} v(\cdot,t) \\ \partial v(\cdot,t)/\partial t\end{pmatrix}\in\widetilde{H}_l^s$, $\mathrm{v}^0=\sum_{k\in\mathbb{Z}}\mathcal{T}_{2k\mu}\Omega v^0\in\widetilde{H}_l^s$, $l<-1/2$ (obviously, $\mathrm{v}(\cdot,t)$, $t\in(0,T)$, and v^0 are odd and periodic). Definitions of the reachability sets $\mathcal{R}^T(v^0)$, $R^T(\mathrm{v}^0)$ null-controllability and approximate null-controllability for problems (2.1)–(2.3) and (2.4), (2.5) are analogous to the definitions of this concepts for problems (0.2), (0.3), (1.1) and (0.1), (1.3) respectively.

Using the same reasonings as in the proof of Theorem 1.1 we obtain

Theorem 2.1. *A state $v^0\in\mathcal{H}_0^s(\mu)$ is approximately null-controllable at a given time $T>0$ iff the following three conditions hold:*

$$v_1^0=v_0^{0\prime}\qquad \textit{in } H_0^{s-1}, \tag{2.6}$$

$$v_0^0\in L^2(0,\mu)\qquad \textit{and}\qquad \left|v_0^0(x)\right|\le 1\quad \textit{almost everywhere on } (0,\mu), \tag{2.7}$$

$$\operatorname{supp} v_0^0\subset[0,T]. \tag{2.8}$$

Moreover, if conditions (2.6)–(2.8) are valid then the state v^0 is null-controllable at the time T. Under these conditions $T_*=\max\operatorname{supp} v_0^0$ is the optimal time and $u(t)=v_0^0(t)$ $(t\in[0,T_*])$ is the time-optimal control for null-controllability.

Remark 2.1. *Let $l<-1/2$, $v^0\in\mathcal{H}_0^s(\mu)$, $T>0$ and let (2.6)–(2.8) be satisfied. Let $\mathrm{v}^0=\sum_{k\in\mathbb{Z}}\mathcal{T}_{2k\mu}\Omega v^0$, $u\in\mathcal{B}(0,T)$, $\operatorname{supp} u\subset[0,\mu]$. Then $\mathrm{v}(\cdot,T)=S(T)\sum_{k\in\mathbb{Z}}\mathcal{T}_{2k\mu}\begin{pmatrix}\Omega\left(v_0^0-u\right) \\ \left(\Theta\left(v_0^0-u\right)\right)'\end{pmatrix}$ in $\widetilde{H}_l^s$ where $\mathrm{v}(\cdot,t)\in\widetilde{H}_l^s$ $(t\in[0,T])$ is the solution of problem (2.4), (2.5).*

If $l < -1/2$, $f \in H_l^s$, $\operatorname{supp} f \subset [0,\mu]$, $k \in \mathbb{Z}$ then

$$\|\mathcal{T}_{2k\mu} f\|_l^s \le \left(1 + (2k\mu)^2\right)^{l/2} \|f\|_0^s . \tag{2.9}$$

Taking into account (1.7), (0.6), (2.9) and Remarks 1.1, 2.1 it is easy to prove

Theorem 2.2. *Assume that $l < -1/2$, a state $w^0 \in \mathcal{H}_l^s$ satisfies conditions (1.8), (1.9),* $\max\operatorname{supp} w_0^0 \le T < \mu$ *and*

$$w_0^0 = v_0^0, \qquad \mathrm{w}^0 = \Omega w^0, \qquad \mathrm{v}^0 = \sum_{k\in\mathbb{Z}} \mathcal{T}_{2k\mu} \Omega v^0, \qquad u \in \mathcal{B}(0,T). \tag{2.10}$$

Then for the solutions w *and* v *of problems (0.1), (1.3) and (2.4), (2.5) we have*

$$\|\mathrm{w}(\cdot,T) - \mathrm{v}(\cdot,T)\|_l^s \le 2^{(6-l)/2}(1+T^2)^{(1-l)/2} \sum_{k\in\mathbb{Z}\setminus\{0\}} \left(1+(2k\mu)^2\right)^{l/2} \left\|v_0^0 - u\right\|_0^s . \tag{2.11}$$

Let $v^0 \in \mathcal{H}_0^s(\mu)$, $u \in \mathcal{B}(0,T)$, $0 < T < \mu$. Denoting

$$\omega_m = \frac{2}{\mu}\int_0^\mu v_0^0(x)\sin\left(\frac{\pi m}{\mu}x\right)\,dx, \quad \nu_m = \frac{2}{T}\int_0^T u(t)\sin\left(\frac{\pi m}{\mu}t\right)\,dt, \quad m = \overline{0,\infty}, \tag{2.12}$$

for $s < -1/2$ we obtain

$$\left(\Omega v_0^0\right)(x) = \sum_{m=0}^\infty \omega_m \sin\left(\frac{\pi m}{\mu}x\right), \quad (\Omega u)(t) = \sum_{m=0}^\infty \nu_m \sin\left(\frac{\pi m}{\mu}t\right) \tag{2.13}$$

and $v_0^0 = u$ in H_0^s iff $\omega_m = \nu_m$, $m = \overline{0,\infty}$.

The problem of determination of a function $u \in \mathcal{B}(0,T)$ satisfying the conditions

$$\int_0^T u(t)\sin\left(\frac{\pi m}{\mu}t\right)\,dt = \frac{2}{T}\omega_m, \qquad m = \overline{0,\infty},$$

for given $\{\omega_m\}_{m=0}^\infty$ and $T > 0$ is called *the Markov trigonometric moment problem on* $(0,T)$ *for the infinite sequence* $\{\omega_m\}_{m=0}^\infty$ [8]. Such a function u is called a solution of this problem.

Thus under the conditions of Theorem 2.1 the null-controllability problem for (2.1)–(2.3) is equivalent to the Markov trigonometric moment problem $(0,T)$ for the infinite sequence $\{\omega_m\}_{m=0}^\infty$ defined by (2.12).

Let $N \in \mathbb{N}$. Consider the problem of determination of a function $u \in \mathcal{B}(0,T)$ satisfying the conditions

$$\int_0^T u(t) \sin\left(\frac{\pi m}{\mu} t\right) dt = \frac{2}{T}\omega_m, \qquad m = \overline{0,N}, \tag{2.14}$$

for given $\{\omega_m\}_{m=0}^N$ and $T > 0$. This problem is called [8] *the Markov trigonometric moment problem on $(0,T)$ for the finite sequence $\{\omega_m\}_{m=0}^N$* and such a function u is called a solution of this problem.

Theorem 2.3. *Let $s < -1/2$. For all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $v^0 \in \mathcal{H}_0^s(\mu)$ satisfies conditions (2.6), (2.7), $\max\operatorname{supp} v_0^0 \le T < \mu$ and the sequence $\{\omega_m\}_{m=0}^N$ defined by (2.12) then for each solution $u \in \mathcal{B}(0,T)$ of moment problem (2.14) the corresponding solution $v(\cdot,t) \in \mathcal{H}_0^s(\mu)$ $(t \in [0,T])$ of problem (2.1)–(2.3) satisfies the condition* $\left\|\left\|\begin{pmatrix} v(\cdot,T) \\ \partial v(\cdot,T)/\partial t \end{pmatrix}\right\|\right\|_0^s \le \sqrt{2}\left\|v_0^0 - u\right\|_0^s < \varepsilon$.

P r o o f. With regard to Remark 2.1 we obtain

$$\|v(\cdot,T)\|_0^s \le \left\|v_0^0 - u\right\|_0^s, \qquad \|\partial v(\cdot,T)/\partial t\|_0^{s-1} \le \left\|v_0^0 - u\right\|_0^s.$$

Let the sequences $\{\omega_m\}_{m=0}^\infty$ and $\{\nu_m\}_{m=0}^\infty$ defined by (2.12). It follows from (2.7) that $|\omega_m| \le 2$, $|\nu_m| \le 2$ $(m = \overline{0,\infty})$. Taking into account (2.13), (2.14) we conclude that

$$\begin{aligned} \left\|\left\|\begin{pmatrix} v(\cdot,T) \\ \partial v(\cdot,T)/\partial t \end{pmatrix}\right\|\right\|_0^s &\le \sqrt{2}\left\|v_0^0 - u\right\|_0^s = \left\|\sum_{m=N+1}^\infty (\omega_m - \nu_m)\sin_m\left(\frac{\pi m}{\mu}x\right)\right\|_0^s \\ &\le 4\sqrt{\mu}\sum_{m=N+1}^\infty \left(1 + \left(\frac{\pi m}{\mu}\right)^2\right)^{s/2} \longrightarrow 0 \text{ as } N \longrightarrow \infty \end{aligned}$$

where $\sin_m(x) = \sin x$ if $x \in [-\pi m, \pi m]$ and $\sin_m(x) = 0$ otherwise. The proof is completed.

From Theorems 2.2, 2.3 we get

Corollary 2.1. *Let $s < -1/2$, $l < -1/2$, $w^0 \in \mathcal{H}_0^s(\mu)$ satisfy conditions (1.8), (1.9), $\max\operatorname{supp} w_0^0 \le T < \mu$. Let also the sequence $\{\omega_m\}_{m=0}^N$ defined by (2.12) with $v_0^0 = w_0^0$. Then for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for each solution $u \in \mathcal{B}(0,T)$ of moment problem (2.14) the corresponding solution $w(\cdot,t) \in \mathcal{H}_l^s$ $(t \in [0,T])$ of problem (0.2), (0.3), (1.1) satisfies the condition* $\left\|\left\|\begin{pmatrix} w(\cdot,T) \\ \partial w(\cdot,T)/\partial t \end{pmatrix}\right\|\right\|_l^s < \varepsilon$.

By the method obtained in [7] under conditions of this corollary for all $N \in \mathbb{N}$ we can find a control $u_N \in \mathcal{B}(0,T)$ satisfying the following three conditions: (i) (2.14) is valid, (ii) $|u_N(t)| = 1$ almost everywhere on $(0,T)$, (iii) u_N has no more than $2N$ points of discontinuity on $(0,T)$. With regard to Corollary 2.1 we conclude that these functions u_N ($N \in \mathbb{N}$) are bang-bang controls solving approximate null-controllability problem for (0.2), (0.3), (1.1).

References

[1] *R.F. Curtain and A.J. Pritchard*, Robust stabilization of infinite-dimensional systems with respect to coprime factor perturbations. In: Differential equations, dynamical systems, and control science. A Festschrift in Honor of Lawrence Markus. New York, NY, Marcel Dekker. — *Lect. Notes Pure Appl. Math.* (1994), v. 152, p. 437–456.

[2] *D.L. Russel and G. Weiss*, A general necessary condition for exact observability. — *SIAM J. Control Optimization* (1994), v. 32, No. 1, p. 1–23.

[3] *R. Triggiani*, Regularity of some structurally damped problems with point control and with boundary control. — *J. Math. Anal. Appl.* (1991), v. 161, No. 2, p. 299–331.

[4] *N. Levan*, The left shift semigroup approach to stability of distributed systems. — *J. Math. Anal. Appl.* (1990), v. 152, No. 2, p. 354–367.

[5] *N.K. Son*, A unified approach to constrained approximate controllability for the heat equations and the retarded equations. — *J. Math. Anal. Appl.* (1990), v. 150, No. 1, p. 1–19.

[6] *Y. You*, Energy decay and exact controllability for the Petrovsky equation in a bounded domain. — *Adv. Appl. Math.* (1990), v. 11, No. 3, p. 372–388.

[7] *V.I. Korobov and G.M. Sklyar*, Time optimality and the trigonometric moment problem. — *Izv. AN SSSR*, Ser. Mat. (1989), v. 53, No. 4, p. 868–885 (Russian).

[8] *M.G. Krein and A.A. Nudel'man*, The Markov moment problem and extremal problems. Nauka, Moscow (1973) (Russian). (Engl. transl.: Amer. Math. Soc. Providence, RI (1977)).

[9] *L. Schwartz*, Théorie des distributions, I, II. Hermann, Paris (1950–1951).

[10] *S.G. Gindikin and L.R. Volevich*, Distributions and convolution equations. Gordon and Breach Sci. Publ., Philadelphia (1992).