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On the Poisson representation of a function harmonic in the upper half-plane

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New conditions of validity of the Poisson representation (in usual and generalized form) for a function harmonic in the upper half-plane are obtained. These conditions differ from known ones by weaker growth restrictions inside the half-plane and stronger restrictions on the behavior in the neighbourhood of the real axis.

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In the theory of entire and subharmonic functions the following Poisson representation of a real-valued harmonic function u in $\mathbf{C}_+ := \{z : \operatorname{Im} z > 0\}$ is very important:

$$u(z) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\nu(t)}{(x-t)^2 + y^2} + cy,\ z = x + iy,\ y > 0, \tag{1}$$

where c is a real constant and ν is a real-valued σ-finite Borel measure on $\mathbf{R}$ such that

$$\int_{-\infty}^{\infty} \frac{d|\nu|(t)}{1+t^2} dt < \infty.$$

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We mention applications to the theory of integral transforms [1, Ch. 4]; the entire functions theory [7, Ch. 5]; [8, Part II]; [5, Ch. 3]; the theory of H_p spaces [6, Ch. 6].

It is well-known (see, e.g., [8, p. 100]; [5, p. 39]; [6, p. 107]) that the necessary and sufficient condition for (1) to be true is representability of u in the form $u = u_1 - u_2$ where u_1 and u_2 are non-negative harmonic functions in $\mathbf{C}_+$. Nevertheless, for several applications (see, e.g., [7, Ch. 5]; [8, Part II]; [5, Ch. 3]), conditions that can be expressed in terms of *growth* of u are more useful. For functions u continuous in the closure $\overline{\mathbf{C}}_+$ of $\mathbf{C}_+$ the strongest version of conditions of this kind is contained in the following result of R. Nevanlinna [9]:

Theorem A. ([9]) *Let u be a real-valued function harmonic in $\mathbf{C}_+$, continuous in $\overline{\mathbf{C}}_+$ and satisfying the conditions:*

(i) there exists a sequence $\{r_k\}$, $r_k \to \infty$, such that

$$\int_0^\pi u^+(re^{i\theta}) \sin\theta d\theta = O(r), \ r = r_k \to \infty; \tag{2}$$

(ii)

$$\int_{-\infty}^{\infty} \frac{u^+(t)}{1+t^2} dt < \infty. \tag{3}$$

Then u admits representation (1) with $d\nu(t) = u(t)dt$.

In [2] and [3], different conditions of validity of representation (1) have been found:

Theorem B. ([2, 3]) *Let u be a real-valued function harmonic in $\mathbf{C}_+$ and satisfy*

(i) there exists a sequence $\{r_k\}$, $r_k \to \infty$, such that

$$\int_0^\pi u^+(re^{i\theta}) \sin\theta d\theta \le \exp(o(r)), \ r = r_k \to \infty; \tag{4}$$

(ii) there exists $H > 0$ such that

$$\sup_{0<s<H} \int_{-\infty}^{\infty} \frac{|u(t+is)|}{1+t^2} dt < \infty. \tag{5}$$

Then (1) holds. If u is continuous in $\overline{\mathbf{C}}_+$ then $d\nu(t) = u(t)dt$.

Comparing conditions of theorems A and B, we see that (2) is more restrictive than (4) whereas (3) is less restrictive than (5). It should also be mentioned that continuity of u in $\overline{\mathbf{C}}_+$ is not assumed in Theorem B. Assumptions (4) and (5) in Theorem B are sharp in the folllowing sense. Example $u(z) = \mathrm{Re}\{\cos z\}$ shows that "o" cannot be replaced by "O" in (4). Moreover, (5) cannot be replaced by

$$\int_{-\infty}^{\infty} \frac{|u(t+iH)|}{1+t^2} dt < \infty,$$

for some $H > 0$, as the example $u(z) = \mathrm{Im}\{(z - iH)^{2n}\},\ n \in \mathbf{N}$, shows. It is also worth to mention that $|u(t+is)|$ cannot be replaced with $u^+(t+is)$ in (5), as the example $u(z) = -\mathrm{Re}\{z^{2n}\},\ n \in \mathbf{N}$, shows.

Present work is devoted to conditions of the validity of more general representations including (1) as a special case. Further we assume that all harmonic functions and Borel measures are real-valued.

This representation has the form

$$u(z) = \int_{-\infty}^{\infty} P_q(z,t)d\nu(t) + \mathrm{Im}P(z), \tag{6}$$

where

$$P_q(z,t) = \mathrm{Im}\left\{\frac{1}{\pi} \cdot \frac{(1+tz)^q}{(t-z)(1+t^2)^q}\right\}, \quad q \in \mathbf{N} \cup \{0\},$$

ν is a σ-finite Borel measure on $\mathbf{R}$ satisfying

$$\int_{-\infty}^{\infty} \frac{d|\nu|(t)}{1+|t|^{q+1}} < \infty,$$

and P is a real polynomial of degree at most q.

For u harmonic in $\mathbf{C}_+$, continuous in $\overline{\mathbf{C}}_+$ and satisfying conditions

$$\int_0^{\pi} u^+(re^{i\theta}) \sin\theta d\theta = O(r^q),\ r \to \infty;$$

$$\int_{-\infty}^{\infty} \frac{u^+(t)}{1+|t|^{q+1}} dt < \infty, \tag{7}$$

representation (6) (with $d\nu(t) = u(t)dt$) belongs to R. Nevanlinna [9]. Without the continuity assumption on u in $\overline{\mathbf{C}}_+$ and under the growth condition

$$\max_{0<\theta<\pi} u^+(re^{i\theta}) = O(r^\alpha), \quad \alpha < q, \tag{8}$$

it belongs to N.V. Govorov [4, p. 25].

Our first result is the following:

Theorem 1. *Let u be a function harmonic in $\mathbf{C}_+$, satisfying condition (4) of Theorem B and the following condition:*
there exists $\alpha > 0$ such that

$$\liminf_{s\to 0^+} \int_{-\infty}^{\infty} \frac{|u(t+is)|}{1+|t|^\alpha} dt < \infty. \tag{9}$$

Then u admits representation (6), where $q = \max\{n \in \mathbf{N} \cup \{0\} : \ n < \alpha\}$, ν is a σ-finite Borel measure on $\mathbf{R}$ satisfying

$$\int_{-\infty}^{\infty} \frac{d|\nu|(t)}{1+|t|^\alpha} < \infty,$$

and P is a real polynomial of degree at most q.

Note that Nevanlinna's [9] and Govorov's [4] results, mentioned above, are not contained in Theorem 1 because condition (9) is more restrictive than both (7) and (8).

It is easy to see that Theorem B is contained in Theorem 1: if $\alpha = 2$ then $q = 1$,

$$P_1(z,t) = \frac{1}{\pi} \frac{y}{(x-t)^2 + y^2}, \quad z = x + iy,$$

condition (9) with $\alpha = 2$ is less restrictive than (5).

Our next result is related to the question: "Can we replace (9) with a condition requiring convergence of the integrals in (9) only over two horizontal lines?" In some sense, the answer is affirmative. To formulate our result we need:

Lemma 1. *Let $u(z)$ be a function harmonic in $\mathbf{C}_+$ and satisfying the condition*

$$\exists H > 0, \ \forall R > 0, \ \sup_{0<y<H} \int_{-R}^{R} |u(x+iy)| dx < \infty. \tag{10}$$

Then there exists a Borel measure ν *on* $\mathbf{R}$ *such that for all* $R > 0$ *it satisfies* $|\nu|([-R, R]) < \infty$, *and the function*

$$u(z) - \int_{-R}^{R} P_q(z, t) d\nu(t), \quad q \in \mathbf{N} \cup \{0\},$$

is harmonic in $\mathbf{C}_+$, *continuous in* $\mathbf{C}_+ \cup (-R, R)$ *and vanishes on* $(-R, R)$.

Our second result is the following:

Theorem 2. *Let* u *be a function harmonic in* $\mathbf{C}_+$ *satisfying condition (10) of Lemma 1 and (4) of Theorem B. Assume additionally that* u *satisfies the following condition:*
there exist $H > 0$ *and* $\alpha > 0$ *such that*

$$\int_{-\infty}^{\infty} \frac{|u(t + iH)|}{1 + |t|^\alpha} dt + \int_{-\infty}^{\infty} \frac{d|\nu|(t)}{1 + |t|^\alpha} < \infty,$$

where ν *is the* σ*-finite Borel measure defined in Lemma 1. Then* u *admits representation (6), where* q, ν *and* P *are as in Theorem 1.*

The following corollary to Theorem 2 is immediate.

Corollary 1. *Let* u *be a function harmonic in* $\mathbf{C}_+$, *continuous in* $\overline{\mathbf{C}}_+$ *and satisfying (4) of Theorem B. Suppose there exist* $H > 0$ *and* $\alpha > 0$ *such that*

$$\int_{-\infty}^{\infty} \frac{|u(t)| + |u(t + iH)|}{1 + |t|^\alpha} dt < \infty.$$

Then u *admits representation (6) with* $d\nu(t) = u(t)dt$.

Remind that, for $\alpha = 2$, representation (6) reduces to (1).

References

[1] *N.I. Akhiezer*, Lectures on integral transformations. KhGU, Kharkov (1984). (Russian)

[2] *S. Gergün, I.V. Ostrovskii, and A. Ulanovskii*, On the Titchmarsh convolution theorem. — *Comptes Rendus Acad. Sci.* Paris, Sér. I (2000), v. 331, p. 41–46.

[3] *S. Gergün, I.V. Ostrovskii, and A. Ulanovskii*, On the Titchmarsh convolution theorem. — *Ark. Mat.* (2002), v. 40, p. 55–71.

[4] *N.V. Govorov*, Riemann boundary problem with infinite index. Nauka, Moscow (1986). (Russian)

[5] *P. Koosis*, The logarithmic integral, I. Cambridge University Press, Cambridge (1998).

[6] *P. Koosis*, Introduction to H_p spaces. Cambridge University Press, Cambridge (1998).

[7] *B.Ya. Levin*, Distribution of zeros of entire functions. GITTL, Moscow (1956). (Russian)

[8] *B.Ya. Levin*, Lectures on entire functions. AMS, Providence, RI (1996).

[9] *R. Nevanlinna*, Über die Eigenschaften meromorpher Funktionen in einem Winkelraum. — *Acta Soc. Sci. Fenn.* (1925), v. 50:12, p. 1–45.