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# Estimates of conformal maps of curvilinear strips

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Distortion estimates for conformal maps of curvilinear strips are obtained. The upper estimate applies to much more general class of the strips than in Warschawski's theorem.

*Dedicated to the 100th anniversary of the birth of Naum Il'ich Akhiezer*

The problem considered here originates from the question of extending the notion of conformality of a map to a boundary point $\zeta$ of its domain. In several papers geometric conditions were found for a neighborhood of $\zeta$ which imply the existence of derivative $f'(\zeta)$ when $z$ tends to $\zeta$ from inside of the domain or from a Stolz angle ("angular derivative"). As most previous authors (see, for example, [8–12], where additional bibliography can be found) we use strip-like canonical regions and place the boundary point $\zeta$ at infinity (see [5, Ch. V, §6]). More general classes of strips were also investigated, for which the angular derivative might not exist but some rougher estimates can be obtained, sometimes sufficient for applications to problems of the theory of entire and meromorphic functions. Thus a proof of the classical Denjoy–Carleman–Ahlfors theorem was obtained for the first time by estimating conformal maps of curvilinear strips in [3] (see also [2]).

In this paper we are not proving conditions of existence of angular derivative but rather obtain rough estimates for a wider class of strips.

Let $z = x + iy$, $w = u + iv$, $\Pi := \{z : \phi_-(x) < y < \phi_+(x),\ x \in \mathbf{R}\}$, where $\phi_-$ and $\phi_+$ are continuously differentiable, $S := \{w : |v| < \pi\}$, and $w = \Phi(z)$ a conformal map of $\Pi$ onto $S$ such that $\Phi(\pm\infty) = \pm\infty$. We introduce the following

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notation:

$$\begin{aligned} \theta(x) &:= (\phi_+(x) - \phi_-(x))/2, \\ \psi(x) &:= (\phi_+(x) + \phi_-(x))/2, \\ a(x) &:= \frac{\pi}{\theta(x)}, \quad \text{and} \quad b(x) := -a(x)\psi(x). \end{aligned}$$

**Theorem 1.** *If $\theta(x), 1/\theta(x), \theta'(x)$ and $\psi'$ are bounded for positive $x$ then there exists a positive constant $C$ such that*

$$C^{-1} < \frac{\Re\Phi(x+iy)}{x} < C, \quad x > 0. \tag{1}$$

P r o o f. Let $\zeta = \xi + i\eta := Q(z)$ be the map from $\Pi$ to $S$ given by the formulas

$$\xi = x, \quad \eta = a(x)y + b(x).$$

We claim that this map is quasiconformal. Its characteristic $p(z)$ (see [8, p. 440, (2.10)] or [1]) can be estimated as

$$p(z) \le K(z) := \frac{E+G}{J},$$

where

$$E := \left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial x}\right)^2 = 1 + (a'y + b')^2,$$

and

$$G = \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 = a^2,$$

and

$$J := \frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x} = a.$$

So we obtain

$$K(z) = a + \frac{1}{a} + \frac{a'^2}{a}(y-\psi)^2 - 2a'\psi'(y-\psi) + a\psi'^2.$$

Using the assumptions of Theorem 1 and $|y - \psi| < \theta$, we obtain that $K(z)$ is bounded for $z \in \Pi$, and thus our map $Q$ is quasiconformal. Let $K = \sup K(z)$.

We put

$$\begin{aligned} \Pi_{x_1x_2} &:= \{z : x_1 < x < x_2,\ \phi_-(x) < y < \phi_+(x)\}, \\ S_{x_1x_2} &:= \{w : x_1 < u < x_2,\ |v| < \pi\}, \\ I_x &:= \{z = x + iy : \phi_-(x) < y < \phi_+(x)\}. \end{aligned}$$

Our function $Q$ maps $\Pi_{x_1x_2}$ onto $S_{x_1x_2}$ for all $x_1$ and $x_2$. Modulus of $S_{x_1x_2}$ equals $x_2 - x_1$ (by definition). Let $\Phi(I_x) = L_x$. Let $S(L_{x_1}, L_{x_2})$ be the quadrilateral cut from $S$ by the sections $L_{x_1}$ and $L_{x_2}$. The function $\Phi \circ Q^{-1}$ maps $S_{x_0x_1}$ onto $S(L_{x_0}, L_{x_1})$. Let $\mu(L_{x_0}, L_{x_1})$ be the modulus of the latter quadrilarteral. Evidently

$$K^{-1}(x_1 - x_0) \le \mu(L_{x_0}, L_{x_1}) \le K(x_1 - x_0) \tag{2}$$

because our quasiconformal map distorts the moduli by a factor at most $K$ (see, for example, [1, p. 25] or [4, p. 11]).

Let

$$u_-(x) := \inf\{u : u + iv \in L_x\} \quad \text{and} \quad u_+(x) := \sup\{u : u + iv \in L_x\}.$$

According to [6] (see also Sections 4–12 and 4–13 of [2] for a similar argument), for $x_1 > x_0 > 0$ we have

$$u_-(x_1) - u_+(x_0) > 2\pi\lambda\left(\frac{1}{2\pi}\mu(L_{x_0}, L_{x_1})\right),$$

where

$$\lambda(t) = \frac{1}{\pi}\log\left(\frac{e^{\pi t}}{16} - \frac{1}{2}\right),$$

while for $e^{\pi t} > 8$ we also have

$$u_+(x_1) - u_-(x_0) < \mu(L_{x_1}, L_{x_2}).$$

Now we fix $x_0$ and let $x = x_1$ tend to infinity. Using (2) we obtain

$$u_+(x) < Kx + O(1), \quad x \to \infty$$

and

$$u_-(x) > x/K + O(1), \quad x \to \infty.$$

This implies (1).

R e m a r k s.

1. The lower extimate in (1) follows from Ahlfors' distortion theorem, see for example [2]. Actually the assumption $\theta(x) \le c$ alone implies that $\Re\Phi(x) \ge (\pi/c)x + O(1)$, $x \to \infty$.

2. If $\psi = \text{const}$, it is enough to assume only that $\theta(x)$ is bounded from below by a positive constant $c > 0$ to obtain the upper estimate in (1). Indeed, if $\theta(x) > c$ then the curvilinear strip $\Pi$ contains the horizontal strip $\{x + iy : |y - \text{const}| < c\}$ and thus the modulus of $\Pi_{x_1x_2}$ is at most $\pi(x_2 - x_1)/c$.

3. If $\phi_+$ and $\phi_-$ are periodic with the same period $T$ then (1) can be improved to $\Re\Phi(z)/\Re z \to \text{const}$ as $z \in \Pi, z \to \infty$. Indeed, from the uniqueness in the Riemann mapping theorem it follows that in this case $\Phi(z + T) = \Phi(z) + \text{const}$.

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## References


[1] *L. Ahlfors*, Lectures on quasiconformal mappings. Van Nostrand, Princeton (1966).

[2] *L. Ahlfors*, Conformal invariants. McGraw–Hill, New York (1973).

[3] *L. Ahlfors*, Untersuchungen zur Theorie der konformen Abbildung und der ganzen Funktionen. — *Acta Soc. Sci. Fenn.* (N. S.) (1930), v. A I, No. 9, p. 1–40.

[4] *P.P. Belinskii*, General properties of quasiconformal mappings. Nauka, Novosibirsk (1974). (Russian)

[5] *M.A. Evgrafov*, Analytic functions. Nauka, Moscow (1991). (Russian)

[6] *F. Huckemann*, An extension of the Ahlfors distortion theorem. — *Proc. Cambridge Phil. Soc.* (1954), v. 50 (2), p. 261–265.

[7] *O.E. Gentosh, A.A. Gol'dberg, and O.N. Moravetskaya*, Geometric criteria of existence of strip and angular derivatives for some classes of conformal mappings. — *Mat. fiz., analiz, geom.* (1994), v. 1, p. 102–115. (Russian)

[8] *A.A. Gol'dberg and I.V. Ostrovskii*, Distribution of values of meromorphic functions. Nauka, Moscow (1970). (Russian)

[9] *B. Rodin and S. Warschawski*, On conformal mapping of $L$-strips. — *J. London Math. Soc.* (1975), v. 14, p. 301–307.

[10] *B. Rodin and S. Warschawski*, Extremal length and univalent functions. — *Math. Z.* (1977), v. 153, p. 1–17.

[11] *B. Rodin and S. Warschawski*, Remarks on the paper of K. Burdzy. — *J. Anal. Math.* (1986), v. 46, p. 251–260.

[12] *S. Warschawski*, Conformal mapping of infinite strips. — *Trans. Am. Math. Soc.* (1942), v. 51, p. 280–335.