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Integrable initial boundary value problems

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The Korteweg–de Vries equation is considered on a half-line with zero boundary conditions at the origin and with arbitrary smooth initial values vanishing rapidly enough. The problem is effectively integrated by means of the inverse scattering method when the associated linear problem has no discrete spectrum. In this case the global solvability theorem is proved.

The paper is devoted to initial boundary value problems consistent with the inverse scattering method. We begin with a simple example (Ablowitz, Segur [1]). The following IBV problem for the nonlinear Schrödinger equation:

$$\begin{aligned} &iu_t = u_{xx} \pm |u|^2 u, \quad x > 0, \\ &u|_{x=0} = 0, \\ &u(0,x) = u_0(x) \end{aligned} \tag{1}$$

is easily reduced to the Cauchy problem on the whole line $-\infty < x < \infty$ by the formula

$$u(x,t) = -u(-x,t).$$

That is why the problem is evidently consistent with the inverse scattering transform method. IBV problems with boundary conditions of more general form are discussed in [2, 3]. The next example is much more complicated (Sklyanin [4, 5])

$$\begin{aligned} &iu_t = u_{xx} \pm |u|^2 u, \quad x > 0, \\ &u_x - cu|_{x=0} = 0, \\ &u(0,x) = u_0(x). \end{aligned} \tag{2}$$

Here the solution is prolonged by use of the Backlund transform $u(x,t) = T(u(-x,t))$.

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But there is a class of integrable IBV problems which are not connected with the continuation procedure. For instance, the following IBV problem for the KdV equation cannot be reduced to the Cauchy problem:

$$u_t = u_{xxx} - 6u_x u, \quad x > 0,\, t > 0, \tag{3}$$

$$u|_{x=0} = a, \quad u_{xx}|_{x=0} = b, \tag{4}$$

$$u|_{t=0} = u_0(x), \quad u_0(x)|_{x\to+\infty} \to 0, \tag{5}$$

where a and b are constants, because the KdV equation does not admit any reflection type symmetry ($x \to -x$, $t \to t$, $u \to h(u)$). It was shown (Adler, Gürel, Gürses, Habibullin [6]) that the problem passes the integrability test based on higher symmetries. Soliton-like solutions of the KdV equation satisfying the boundary conditions (4) are studied in [7]. Below we discuss some analytical properties of this IBV problem in the particular case when the parameters a and b vanish

$$u_t = u_{xxx} - 6u_x u, \quad x > 0,\, t > 0, \tag{6}$$

$$u|_{x=0} = 0, \quad u_{xx}|_{x=0} = 0, \tag{7}$$

$$u|_{t=0} = u_0(x), \quad u_0(x)|_{x\to+\infty} \to 0. \tag{8}$$

The problem admits infinitely many integrals of motion, first three of them are

$$J_1 = \int_0^\infty u\,dx, \quad J_2 = \int_0^\infty (u_x^2 + 2u^3)\,dx,$$

$$J_3 = \int_0^\infty (u_{xx}^2 - 5u^2 u_{xx} + 5u^4)\,dx.$$

Suppose that the initial function $u_0(x)$ decreases rapidly enough as well as its x-derivatives. At the point $x = 0$ the function $u_0(x)$ and derivatives vanish. The following Sturm–Liouville problem $-y'' + u_0 y = \xi^2$, $y'(0) = -i\xi y(0)$, $y(+\infty) = 0$ on the half-line $[0, +\infty)$ is supposed to have no nonzero solutions. Under these assumptions global solvability of the IBV problem (6)–(8) is proved.

The problem (6)–(8) is studied by means of the inverse scattering transform method. Remind its essence. Use the Lax pair of the KdV equation

$$-y_{xx} + uy = \xi^2 y, \tag{9}$$

$$y_t = u_x y - 2(u + 2\xi^2) y_x. \tag{10}$$

Define two eigenfunctions $e^{\pm}(x,\xi)$ of the Sturm–Liouville equation (9) on the half-line $x > 0$ such that

$$e^+(x,\xi) \to e^{i\xi x} \quad \text{for} \quad x \to \infty,$$

$$e^-(x,\xi) \to e^{-i\xi x} \quad \text{for} \quad x \to 0.$$

The scattering matrix $s(\xi) = \begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix}$ is defined in the usual way

$$e^{+}(x,\xi) = a(\xi)e^{-}(x,-\xi) + b(\xi)e^{-}(x,\xi), \tag{11}$$
$$e^{-}(x,\xi) = -b(-\xi)e^{+}(x,\xi) + a(\xi)e^{+}(x,-\xi). \tag{12}$$

The principal obstacle preventing to apply effectively the ISM to IBV problem is connected with the following circumstance. Generally, the scattering matrix $s(\xi,t)$ of the Sturm–Liouville problem on the half line depends on t not explicitly but by means of the following nonlinear equation [8]

$$s_t = 4i\xi^3[s,\sigma_3] + P_1\sigma_1 s + P_2\sigma_2 s + P_3\sigma_3 s, \tag{13}$$

where $P_1 = u_x(0,t)$, $P_2 = \frac{1}{2\xi}(-4u(0,t)\xi^2 + u_{xx}(0,t) - 2u^2(0,t))$, $P_3 = \frac{i}{2\xi}(2u^2(0,t)$ $-u_{xx}(0,t))$. However for boundary conditions of some special kinds this equation may admit discrete transformations which will simplify the problem. For instance, in the case (7) the equation (13) reads as

$$s_t = 4i\xi^3[s,\sigma_3] + u_x(0,t)\sigma_1 s. \tag{14}$$

Actually the problem is nonlinear, because both $s(\xi,t)$ and $u_x(0,t)$ are unknown. The last equation is nothing else but the Zakharov–Shabat system, having the following remarkable property (see [9]): if the solution of the scattering problem is known at a point $t = t_0$ but for all values of the spectral parameter then the scattering solution can be recovered for all t by means of solving a linear problem called Riemann problem of analytic factorization. Generally the columns of the scattering matrix give two vectorial solutions of the system (14) defined on different complex semi-planes. However this is not enough to recover completely the system (14). The discrete symmetries allow one to get new solutions from known ones and in such a way to find complete solution of the scattering problem for $t = 0$.

The columns $s_1(\xi,t)$ and $s_2(\xi,t)$ are analytic vectors in upper and lower half planes respectively. The system (14) is invariant under the change of variables $\xi \to \omega\xi$, where $\omega^3 = 1$. Therefore the vectors $s_1(\omega\xi,t)$ and $s_2(\omega\xi,t)$ are also solutions to this system. One can introduce a new spectral parameter $z = \xi^3$. And then the matrix

$$c_+(z,t) = (s_1(\omega\xi,t), s_2(\xi,t))$$

is a solution of (14) analytic for $\operatorname{Im} z > 0$. Similarly $c_-(z,t) = \sigma_1\bar{c}_+(\bar{z},t)\sigma_1$ is a solution analytic for $\operatorname{Im} z < 0$. These two solutions satisfy the Riemann problem

$$c_+(z,t) = c_-(z,t)p(z,t),$$

where $p(z,t) = e^{-4iz\sigma_3}p(z,0)e^{4iz\sigma_3}$. Note that at the origin $\xi = 0$ the scattering matrix $s(\xi,0)$ has a pole except some degenerate case [10] and by this reason the conjugation matrix of the corresponding Riemann problem has singularities on the conjugation contour and one cannot use the standard Zakharov–Shabat scattering theory.

It is more convenient here to use the equivalent scalar equation. Time evolution of the eigenfunction $e^+(o,\xi,t) = me^{4i\xi^3 t}$ is given again by Sturm–Liouville type equation

$$m_{tt} = (w(t) - 16\xi^6)m,$$

where $w(t) = u_{xt}(0,t) + u_x^2(0,t)$.

An important observation is that the auxiliary potential $w(t)$ is continuous and vanishes. So that the scattering matrix for the last equation is also defined correctly

$$S(z) = \begin{pmatrix} A & \bar{B} \\ B & \bar{A} \end{pmatrix},$$

where $z = \xi^3$. Introduce the scattering data for these two equations as follows

$$r(\xi) = \frac{b(\xi)}{a(\xi)}, \quad R(z) = \frac{B(z)}{A(z)}.$$

Two spectral data are connected with each other by formula

$$r(-\xi) = R(z), \quad \text{for} \quad \frac{-2\pi}{3} < \arg\xi < \frac{-1\pi}{3}.$$

Such kind of connections between t- and x-scattering data are called "global relations" (see [3]).

Lemma. *If the equation $y'' = (u_0(x) - \xi^2)y$ has no discrete eigenvalues then the function $R(z)$ satisfies the conditions:*

R1) $|R(z)|^2 \leq 1 - Cz^2(1+z^2)^{-1}$ and $R(z) = O(z^{-1})$ for $z \to \pm\infty$;

R2) the function

$$k(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iyz}R(z)dz < \infty$$

is absolutely continuous and its derivative $k'(y)$ satisfies inequality

$$\int_{-\infty}^{\infty}(1+|y|)|k'(y)|dy < \infty;$$

R3) the function

$$A(z) = \exp(-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log(1-|R(s)|^2)}{s-z}ds)$$

is continuous in the closed upper half-plane and

$$\lim_{z\to 0} zA(z)(R(z)+1) = 0;$$

R4) $R(z)$ is analytic in $Imz > 0$.

To prove the statements we use the following transformation

$$r(\xi) = \int_0^{\infty} e^{ix\xi}q(x)dx = \int_0^{\infty} e^{-iyz}k(y)dy = R(-z)$$

converting $q(x)$ into $k(y)$. Actually it acts as

$$k(y) = \frac{3}{2\pi}\int_0^{\infty} sy^{-1}q(x)v(s)dx, \quad ||k(y)|| \le c_1||q(x)||\cdot||v(s)||,$$

where $s = xy^{-1/3}$, v is the Airy function, i.e., a solution of the Airy equation $v''(s) = sv(s)$, the norm means L_1-norm. The map is bounded and invertible, the inverse map is also bounded, it is of the form

$$q(x) = \frac{1}{2\pi}\int_0^{\infty} sy^{-1/3}k(y)v(s)dy, \quad ||q(x)|| \le c_2||k(y)||\cdot||v(s)||.$$

Now solving the Gel'fand–Levitan–Marchenko equation [10]

$$m_+(z,t) = m_-(z,t)R(z) + m_+(-z,t)$$

one can recover eigenfunctions $m_\pm$. Then time evolution of the scattering data $r(\xi,t)$ is easily found

$$r(-\xi,t) = e^{2izt}\frac{izm_+(z,t) - m_{+t}(z,t)}{izm_+(z,t) + m_{+t}(z,t)}.$$

The next step is to recover the potential $u(x,t)$ by using known $r(\xi,t)$. This part is done by the standard method. Thus the following fact is established:

Theorem. *Let the initial value satisfy the conditions:*

1) $u(x,0) = u_0(x)$ is smooth and vanishes;

2) the associated Sturm–Liouville operator has no discrete eigenvalues. Then the problem

$$u_t = u_{xxx} - 6u_x u, \quad x > 0,\, t > 0, \tag{15}$$

$$u|_{x=0} = 0, \quad u_{xx}|_{x=0} = 0, \tag{16}$$

$$u|_{t=0} = u_0(x), \quad u_0(x)|_{x\to+\infty} \to 0. \tag{17}$$

is uniquely solvable.

By using the technique above the following asymptotic representation can be obtained

$$u_x(0,t) = \frac{1}{t}(1 + o(1)), \text{ for } t \to \infty$$

in the case of general position when the scattering matrix is unbounded at the point $\xi = 0$. It follows from the formula

$$u_x(0,t) = \frac{C_1\dot{\psi}_1(t) + C_2\dot{\psi}_2(t)}{C_1\psi_1(t) + C_2\psi_2(t)},$$

where ψ_1, ψ_2 are solutions of the equation $\psi'' = w(t)\psi$ having the following behaviour at the infinity

$$\psi_1(t) = 1 + o(1), \quad \psi_2(t) = t + o(1), \quad t \to \infty,$$

The function $u_x(0,t)$ is localized only in the degenerate case $|a(0,0)| < \infty$, and then the condition $u_x(0,t) \in L_1[0,+\infty)$ is valid [11]. The corresponding initial functions form nonempty but rather narrow class.

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