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On growth and domains of holomorphy of functions that are analytic in a disk and generate Pólya frequency sequences

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There are related the author's published and new results concerning the growth and the distribution of singularities of functions, that are analytic in the unit disk and generate Pólya frequency sequences.

The Pólya frequency sequences were first introduced by Fekete [5] in 1912 in connection with Laguerre's problem of precise determination of the number of positive roots of a real polynomial with the help of Descartes' rule. These sequences represent a particular case of totally positive kernels which have played a very important role in mathematics, statistics and mechanics. The sequences themselves have found many applications in Analysis. An extensive monograph [6] about many of these applications and generalizations of the concept of multiple positivity has been written by S. Karlin.

Definition 1. *A sequence $\{a_k\}_{k=0}^{\infty}$ is called a Pólya frequency sequence of order r, $r \in \mathbb{N} \cup \{\infty\}$, or a multiply positive sequence, if all minors of order $\le r$ (all minors if $r = \infty$) of the infinite matrix*

$$\begin{Vmatrix} a_0 & a_1 & a_2 & a_3 & \dots \\ 0 & a_0 & a_1 & a_2 & \dots \\ 0 & 0 & a_0 & a_1 & \dots \\ 0 & 0 & 0 & a_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{Vmatrix}$$

are non-negative.

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This class of sequences is denoted by PF_r. The class of corresponding generating functions

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

is also denoted by PF_r. The radius of convergence of a PF_r generating function (PF_r g.f.) is positive provided $r \geq 2$ ([6], p. 394). It is supposed, without loss of generality, that $a_0 = 1$.

The class PF_∞ has been exhaustively described (see [1] or [6], p. 412):

Theorem 2 (Aissen, Edrei, Schoenberg, Whitney; 1953). *The class PF_∞ consists of the functions*

$$e^{\gamma z} \prod_{k=1}^{\infty} (1 + \alpha_k z)/(1 - \beta_k z),$$

where $\gamma \geq 0, \alpha_k \geq 0, \beta_k \geq 0$ and $\sum(\alpha_k + \beta_k) < \infty$.

The problem of the full description of PF_r, $r \in \mathbb{N}$, has not been solved yet. The study of possible zero-sets and growth of polynomials and entire functions belonging to PF_r, $r \in \mathbb{N}$, has been carried out by I.J. Schoenberg, O.M. Katkova and I.V. Ostrovskii (see [9, 8] and [7]).

The aim of this contribution is to relate what is known about the singularities of non-entire functions belonging to PF_r, $r \in \mathbb{N}$. It is clear from Theorem 2 that a PF_∞ g.f. can have only poles located on the positive ray that cannot be "too close" to each other. Therefore, near its singularities this function is of order of growth equal to 0 and its domain of holomorphy extends to the whole complex plane with the exception of a countable set of points on the positive ray. The situation for PF_r g.f.'s with $r \in \mathbb{N}$ is quite different (see [2–4]).

The following result shows that the growth of a PF_r g.f., $r \in \mathbb{N}$, near its singularities can be rather arbitrary:

Theorem 3. *Suppose that an integer $r \geq 2$ and the numbers ρ, $0 < \rho \leq \infty$, and σ, $0 \leq \sigma \leq \infty$, are given. There exists a function $f(z) \in PF_r$ analytic in $\overline{\mathbb{C}} \backslash \{1\}$ and possessing an essential singularity at $z = 1$ such that*

$$\limsup_{y \to 1^-} \frac{\log \log M(y, f)}{\log(1/(1-y))} = \rho$$

and, for $\rho < \infty$, such that

$$\limsup_{y \to 1^-} \frac{\log M(y, f)}{((1/(1-y)))^\rho} = \sigma.$$

The case of non-entire PF_r g.f. of order 0 has been treated separately due to the fact that some restrictions on the growth arise:

Theorem 4. *Suppose that an integer $r \geq 2$ and the numbers $\rho_{\log}$, $1 \leq \rho_{\log} < \infty$, and $\sigma_{\log}$, $0 \leq \sigma_{\log} \leq \infty$, are given. There exists a function $F(z) \in PF_r$ analytic in $\overline{\mathbb{C}}\backslash\{1\}$ and possessing an essential singularity at $z = 1$ such that*

$$\limsup_{y \to 1^-} \frac{\log\log M(y, F)}{\log\log(1/(1-y))} = \rho_{\log}$$

and, for $\rho_{\log} > 1$, such that

$$\limsup_{y \to 1^-} \frac{\log M(y, F)}{(\log(1/(1-y)))^{\rho_{\log}}} = \sigma_{\log}.$$

There are no non-constant PF_r generating functions smaller than of logarithmic order 1 and normal logarithmic type, so Theorem 4 cannot be extended to the case $\rho_{\log} < 1$. We show this with the help of the following result of [4].

Theorem 5. *Let $f(z)$ be a PF_r g.f., $r \geq 2$ with radius of convergence of its power series equal to 1. Then $(1-z)f(z)$ is a PF_{r-1} g.f.*

It immediately follows that $(1-z)f(z)$ belongs to PF_1 and, therefore, $\lim_{y \to 1^-}(1-y)M(y, f) > 0$ and $\log M(y, f) > \log[1/(1-y)] + O(1)$, $y \to 1^-$.

The set of singularities of a PF_r g.f., $r \geq 2$, can also be rather arbitrary. The only necessary restrictions on singularity sets of PF_r g.f.'s are due to the fact that the Taylor coefficients of these functions are non-negative: if G is the domain of holomorphy of the PF_r g.f. $f(z) = \sum_{n=0}^{\infty} a_k z^k$ then the following conditions must be satisfied:

(A) G contains the point $z = 0$ (this condition is assured by $f(z) \in PF_r \subset PF_2$);

(B) G is symmetric with respect to the real axis (since $a_k \in \mathbb{R}, k = 0, 1, 2, \dots$);

(C) $T = dist(0, \partial G) \in \partial G$ (by the well-known Pringsheim's theorem on singularities of power series with non-negative coefficients).

The following full description was proved in [4]:

Theorem 6. *A domain G is the domain of holomorphy of a PF_r g.f., $r \geq 2$, if and only if G satisfies the conditions (A), (B) and (C).*

The problem of the full description of the class PF_r, $r \in \mathbb{N}$, remains open.

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