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Integrable Markov processes and quantum spin chains

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A set of Markov processes corresponding to systems of hard-core particles interacting along the line are shown to be solvable via a dynamic matrix product ansatz (DMPA). We show that quantum spin Hamiltonians can be treated by the DMPA as well, and demonstrate how the DMPA, originally formulated for systems with open ends, works for periodic systems.

Recent advances in nonequilibrium statistics have allowed for the clarification of many important issues such as exact description of nonequilibrium phase transitions, microscopic structure of shocks, dynamical scaling etc. [1]. Dynamical systems of stochastically hopping hard core particles (th e so-called exclusion processes) have been at the heart of this development, in particular due to remarkable observation that one can find exactly the probabilities of all system configurations in a stationary state. It was subsequently noticed that for the totally symmetric exclusion process one can as well solve the evolution dynamics, namely to find conditional probabilities $P(\nu, t|\nu^0, t^0)$ of the system being at state ν at time t provided that it was in state ν^0 at time t^0 via coordinate Bethe Ansatz [2]. The dynamical matrix product ansatz (DMPA) proposed in [3] has recovered the results of [2] in a more straighforward manner. But for technical reasons to be explained later, application of the DMPA was incomplete in periodicsystems.

In this communication, we show how the DMPA is applied for periodic systems. Alongside, we demonstrate that the DMPA can be used for testing the

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stochastic processes and/or one-dimensional spin chains for integrability. The integrability restrictions come from the consistency conditions on the algebra generated by the DMPA, which has to satisfy the Yang–Baxter equation. The important difference from the conventional approach is that we start directly from the Hamiltonian and do not need a parameter-dependent transfer matrix, commuting with it.

The continuous time dynamics of Markov process is defined via the master equation for the probability $P(\nu,t)$ of the system being at state ν at time t,

$$P(\nu,t)/\partial t = \sum_{\nu'} P(\nu',t) w_{\nu'\to\nu} - P(\nu,t)\sum_{\nu'} w_{\nu\to\nu'}\,, \tag{1}$$

where $w_{\nu\to\nu'}$ is the rate with which the system can change its configuration from ν to ν'.

For definiteness we consider a system of identical particles on a lattice of L sites. Each site k can be either occupied by a particle (local occupation number $n_k = 1$). The set of occupation numbers for all sites constitutes the configuration of the system. The system can change its configuration via hopping of any particle to an empty nearest neighbour site. At any infinitesimally small moment of time dt at most one such process can happen, namely the particle can hop to the empty site to the right/to the left with the rates D_R and D_L respectively. Now, consider a vector space $\mathcal{W} = \prod^L \otimes C^2$. Notice that one can establish one-to-one correspondence between configurational space of our particle system and canonical basis of $\mathcal{W}$; $|\eta\rangle \subset \mathcal{W}$, with the scalar product $\langle\eta'|\eta\rangle = \delta_{\eta\eta'}$. The local particle number operators have form $n_k = I\otimes\ldots I\otimes e_{22}\otimes\ldots I$, and e_{11} respectively for holes. e_{ij} denotes a matrix with $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$, and it stays in the k-th place in the tensor product expression above. I is the unit 2×2 matrix.

With the help of above notations the set of equations (1) can be rewritten as single vector equation (we ask the reader to refer e.g. to [1] for the details of the scheme).

$$\frac{d|P(t)\rangle}{dt} = -H|P(t)\rangle, \tag{2}$$

where $|P(t)\rangle = \sum_{\eta\subset\mathcal{W}} P_\eta(t)\ |\eta\rangle$, $H_{\eta\eta'} = -w_{\eta'\to\eta}$, $\quad H_{\eta\eta} = \sum_{\eta'} w_{\eta\to\eta'}$.

Due to the fact that the hopping is restricted to the nearest-neighbouring sites, the Hamiltonian above is the sum of local two-site Hamiltonians, $H = \sum_{n=1}^{L} h_{n,n+1}$ given by

$$\begin{aligned} 4h_{n,n+1} = {} & D_R\left((1-\sigma_n^z)(1+\sigma_{n+1}^z) - \sigma_n^+\sigma_{n+1}^-\right) \\ & + D_L\left((1+\sigma_n^z)(1-\sigma_{n+1}^z) - \sigma_n^-\sigma_{n+1}^+\right), \end{aligned} \tag{3}$$

where σ^z, $\quad \sigma^\pm = \sigma^x \pm i$.

Note that (2) has a form of a Schrödinger equation in imaginary time, and the Hamiltonian (3) the form of a nonhermitian spin Hamiltonian with nearest neighbour exchange interaction.

The global Hamiltonian H possesses an U(1) symmetry, $[H, \sum_n \sigma_n^z] = 0$. The general hermitian spin Hamiltonian satisfying the latter relation is XXZ Hamiltonian,

$$H = J \sum_n \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta \left(\sigma_n^z \sigma_{n+1}^z - I \right). \tag{4}$$

Thus Schrödinger quantum problem and Markov process for interacting particles are closely connected. In the following we show how to find the spectrum of H and how to single out integrable models.

To demonstrate how the DMPA works, we start from the Hamiltonian of the form

$$H = \sum_{n=1}^{L} h_{n,n+1}; \quad h = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & f & u & 0 \\ 0 & v & g & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5}$$

with arbitrary entries f, g, u, v, which contains both the stochastic (3) and quantum spin system (4). We shall consider a system on a ring with L sites. One proposes to search for time-dependent probability distribution vector $|P(t)\rangle$ in the following form:

$$|P(t)\rangle \sim \mathrm{Tr} \left[\begin{pmatrix} E \\ D \end{pmatrix}^{\otimes L-1} \otimes \begin{pmatrix} EQ \\ DQ \end{pmatrix} \right], \tag{6}$$

where E, D, Q are some time-dependent matrices. The probability of a given configuration ν is given by the scalar product of the above vector with the basis vector $|\nu\rangle$ of the configuration, $P(\nu, t) = \langle \nu | P(t) \rangle$. For example, probability $P_\eta(t)$ of a state $\eta = \{0\ 0\ A\ 0\ A\}$ for a system of 5 sites is sought in the form $P_\eta(\nu, t) \sim \mathrm{Tr}\,(E\ E\ D\ E\ D\ Q)$, up to normalization.

One requires matrices E, D, Q such that (6) satisfies the master equation (2). To simplify matters, first take Q in (6) to be a unit matrix $Q = I$ (this would allow to retrieve the translationally invariant part of the H spectrum as we shall see below). Given that, and that the Hamiltonian in (2) splits into local ones, $H = h_{1,2} + h_{2,3} + \ldots h_{L-1,L} + h_{L,1}$, it is sufficient for the purpose that the matrices E, D satisfy quadratic relation

$$\left(\frac{1}{2} \frac{d}{dt} + \mathbf{h} \right) \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} = \begin{pmatrix} X^0 \\ X \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} - \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} X^0 \\ X \end{pmatrix} \tag{7}$$

conveniently written in short form

$$\left(\frac{1}{2}\frac{d}{dt}+h\right)|\,\mathbf{A}\,\rangle\otimes|\,\mathbf{A}\,\rangle=|\,\mathbf{X}\rangle\otimes|\,\mathbf{A}\rangle-|\,\mathbf{A}\rangle\otimes|\,\mathbf{X},\rangle, \tag{8}$$

where $\langle\mathbf{A}|=(E\ D)$, $\langle\mathbf{X}|=(X^0\ X)$.

For $Q\neq I$ which is generic case one gets two additional relations in addition to (8), connected with the action of the terms $h_{L-1,L}$, $h_{L,1}$ of the Hamiltonian:

$$\left(\frac{1}{2}\frac{d}{dt}+h\right)|\,\mathbf{A}\,\rangle\otimes|\,\mathbf{A}Q\,\rangle = |\,\mathbf{X}\,\rangle\otimes|\,\mathbf{A}Q\,\rangle-|\,\mathbf{A}\,\rangle\otimes|\,\mathbf{X}Q\,\rangle, \tag{9}$$

$$\left(\frac{1}{2}\frac{d}{dt}+h\right)|\,\mathbf{A}Q\,\rangle\otimes|\,\mathbf{A}\,\rangle = |\,\mathbf{X}Q\,\rangle\otimes|\,\mathbf{A}\,\rangle-|\,\mathbf{A}Q\,\rangle\otimes|\,\mathbf{X}\,\rangle. \tag{10}$$

Equation (8) gives the relations for the matrices E, D while Eqs. (9, 10) impose restrictions on Q.

Equation (7) has four components:

$$\frac{1}{2}\left(\dot{E}E+E\dot{E}\right) = X^0E-EX^0, \tag{11}$$

$$\frac{1}{2}\left(\dot{E}D+E\dot{D}\right)+fED+uDE = X^0D-EX, \tag{12}$$

$$\frac{1}{2}\left(\dot{D}E+D\dot{E}\right)+gDE+vED = XE-DX^0, \tag{13}$$

$$\frac{1}{2}\left(\dot{D}D+D\dot{D}\right) = XD-DX. \tag{14}$$

Using the reduction

$$\dot{E}=0;\ X^0=0, \tag{15}$$

Eq. (11) is satisfied trivially. Supposing additionally that E^{-1} exists, we obtain from (12), (13):

$$2X=(g-f)D+vEDE^{-1}-uE^{-1}DE; \tag{16}$$

$$\dot{D}+(g+f)D+vEDE^{-1}+uE^{-1}DE=0. \tag{17}$$

Substituting $X,\dot{D}$ into (14), one gets

$$(g+f)DD+uDE^{-1}DE+vEDE^{-1}D=0. \tag{18}$$

Now, consider the two equations (9, 10) with Q present. First, suppose that Q is time-independent, and Q^{-1} exists. Then, Eq.(9) is satisfied with the choices

(16–18) since multiplication of (9) by Q^{-1} from the right reduces (9) to (8). In order to satisfy Eq.(10) with the choice (15–18),the matrix Q must obey

$$[Q, DE^{-1}] = [Q, E^{-1}D] = 0. \tag{19}$$

Note that in general we assume that Q does not commute neither with D nor with E separately, $[Q, E] \neq 0$; $[Q, D] \neq 0$.

Proceeding along the lines [3], we can get rid of time-dependence of D via a formal Fourier Transform

$$\mathcal{D}_p(t) = \sum_k \mathrm{e}^{ipk} E^{k-1} D(t) E^{-k}. \tag{20}$$

Indeed, it follows from (20) that

$$E^{-1}\mathcal{D}_p(t)E = \mathrm{e}^{ip}\mathcal{D}_p(t). \tag{21}$$

Substituting (20), (21) into (17), one gets

$$\dot{\mathcal{D}}_p + \epsilon_p \mathcal{D}_p = 0;\ \mathcal{D}_p(t) = e^{-\epsilon_p t}\mathcal{D}_p(0)$$

with the "dispersion relation"

$$\epsilon_p = g + f + v\mathrm{e}^{-ip} + u\mathrm{e}^{ip}. \tag{22}$$

The inverse Fourier Transform is

$$D(t) = E\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \mathrm{d}p\mathcal{D}_p(t) = E\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \mathrm{d}p\mathrm{e}^{-\epsilon_p t}\mathcal{D}_p(0). \tag{23}$$

In what follows we s the time argument in initial matrices $\mathcal{D}_p(0)$. Inserting (23) into (14), and using (21), one obtains:

$$\frac{E^2}{2\pi}\int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi} \mathrm{e}^{-(\epsilon_{p_1}+\epsilon_{p_2})t}\mathcal{D}_{p_1}\mathcal{D}_{p_2}a(p_1,p_2)\mathrm{d}p_1\mathrm{d}p_2 = 0, \tag{24}$$

where

$$a(p_1, p_2) = u\mathrm{e}^{ip_1+ip_2} + (f+g)\mathrm{e}^{ip_1} + v. \tag{25}$$

Now, integral in (24) may be split in two parts as

$$\int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi} \ldots \mathrm{d}p_1\mathrm{d}p_2 = \int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{p_1} \ldots + \int\limits_{-\pi}^{\pi}\int\limits_{p_1}^{\pi} \ldots = I_{p2<p1} + I_{p2>p1}\,. \tag{26}$$

By changing the order of integration and interchanging $p_1 \leftrightarrow p_2$ in the last term in (26) we obtain from (24):

$$\int_{-\pi}^{\pi}\int_{-\pi}^{p_1} (a(p_1,p_2)\mathcal{D}_{p_1}\mathcal{D}_{p_2} + a(p_2,p_1)\mathcal{D}_{p_2}\mathcal{D}_{p_1})\, \mathrm{e}^{-(\epsilon_{p_1}+\epsilon_{p_2})t}\mathrm{d}p_1\mathrm{d}p_2 = 0.$$

Since this is satisfied at all times t, we must require the expression inside the integrand to vanish

$$\mathcal{D}_{p_1}\mathcal{D}_{p_2} = -\frac{a(p_2,p_1)}{a(p_1,p_2)}\mathcal{D}_{p_2}\mathcal{D}_{p_1}\,. \tag{27}$$

Finally, we require the matrix Q from (9,10) to satisfy $[Q, E^{k-1}DE^{-k}] = 0$, which contains (19) as a special case and yields

$$[Q, \mathcal{D}_p(t)] = 0. \tag{28}$$

To obtain Bethe Ansatz equations for the spectrum, consider the quantity

$$\mathrm{Tr}\ (\mathcal{D}_{p_1}\mathcal{D}_{p_2}\ldots\mathcal{D}_{p_N}E^L Q), \tag{29}$$

which is the building block of the expression for expectation values. Indeed, for a system with just two particles the expectation value $\langle\, n_x(t)n_y(t)\,\rangle$, according to (6), (21),(23), can be written as

$$\langle\, n_x(t)n_y(t)\,\rangle = \mathrm{Tr}\ (E^{x-1}DE^{y-x-1}DE^{L-y}Q)/Z_L$$
$$= \int \mathrm{d}p_1 \int dp_2 \mathrm{e}^{-(\epsilon_{p_1}+\epsilon_{p_2})t}\mathrm{e}^{-ip_1x-ip_2y}\mathrm{Tr}\ (\mathcal{D}_{p_1}\mathcal{D}_{p_2}E^L Q)/Z_L\,,$$

where Z_L is an appropriate normalization factor, $Z_L = \sum_{x<y=1,2,\ldots L}\langle\, n_x(t)n_y(t)\,\rangle$.

Generally, one is interested in a system with N particles which leads to (29).

Using (27), and commuting $\mathcal{D}_{p_k}$ around the circle, one obtains using (21), (28) and the cyclic invariance property of the trace:

$$\mathrm{Tr}\ (\mathcal{D}_{p_1}\mathcal{D}_{p_2}\ldots\mathcal{D}_{p_N}E^L Q) = (-1)^{N+1}\prod_{n=1}^{N}\frac{a(p_n,p_k)}{a(p_k,p_n)}e^{ip_kL}\mathrm{Tr}\ (\mathcal{D}_{p_1}\mathcal{D}_{p_2}\ldots\mathcal{D}_{p_N}E^L Q)$$

and consequently

$$(-1)^{N+1}\prod_{n=1}^{N}\frac{a(p_n,p_k)}{a(p_k,p_n)}\mathrm{e}^{ip_kL} = 1. \tag{30}$$

One recognizes in (30), (22) the well-known expressions for the spectrum of the XXZ spin model, for $f = g = -2J\Delta$, $u = v = 2J$.

It may seem that the proposed line of argument is not changed if the auxiliary matrix Q that we have introduced is chosen to be a unity matrix $Q = I$. This trivial choice however would bring about an additional relation. Namely let us rewrite (29) as

$$\mathrm{Tr}\ (\mathcal{D}_{p_1}\mathcal{D}_{p_2}\dots\mathcal{D}_{p_N}E^L) = \mathrm{Tr}\ (\mathcal{D}_{p_1}\mathcal{D}_{p_2}\dots\mathcal{D}_{p_N}EE^{L-1}) \tag{31}$$

and then commute single matrix E around the circle. We shall obtain

$$\mathrm{e}^{ip_1+ip_2+\dots ip_N} = 1 \tag{32}$$

then. This means putting $Q = I$ (or more generally allowing for $[Q, E] = 0$) we recover only the part of spectrum with the total quasimomentum equal to zero, $p_1 + p_2 + \dots p_N = 0$.

We have thus shown how from the Dynamis Matrix Product Ansatz one obtains the spectrum for spin 1/2 XXZ chain. Note that until now we have not got any restriction on parameters in (5). This is a consequence of the fact that the Hamiltonian (5) is integrable for any choice of parameters. In the following we show how the the integrability restrictions enter in a more general setting.

Consider nearest neighbour quantum spin 1 Hamiltonians. This case corresponding to three states at a site, e.g. $n_i = 0, 1, 2$ comprises many integrable systems (see e.g. a list given in [4] where however several symmetry restrictions are imposed). With those restrictions being relaxed, there are many more, e.g. [5, 7].

To be specific, we shall consider the hermitian Hamiltonian of the form

$$h = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & g_A & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & g_B & 0 & 0 \\ 0 & g_A & 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c & 0 & g & 0 \\ 0 & 0 & g_B & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e \end{pmatrix} \tag{33}$$

with 8 independent parameters. If $g = g_B = g_A = 1$, $d = e = 0$ and $a = b = c = -1$, it reduces to isotropic Hamiltonian of Sutherland [6] $h = P - I$, where P is permutation operator in $SU(3) \otimes SU(3)$ defined by $P(A \otimes B) = (B \otimes A)P$ for any matrices A, B from $SU(3)$. Again, the complete Hamiltonian has the form $H = \sum_{n=1}^{L} h_{n,n+1}$ and we are interested in a solution of Schrödinger equation $i\frac{d}{dt}|P\rangle = H|P\rangle$.

Analogously to (6) one proposes an Ansatz for $|P(t)\rangle$ as

$$|P(t)\rangle \sim \mathrm{Tr}\left[\begin{pmatrix} E \\ D^A \\ D^B \end{pmatrix}^{\otimes L-1} \otimes \begin{pmatrix} EQ \\ D^AQ \\ D^BQ \end{pmatrix}\right] \tag{34}$$

with matrices D^A, D^B referring to local variable $n_i = 0, 1, 2$ respectively.

Ansatz (34) leads to the equations (8, 9, 10) where now $\langle \mathbf{A}| = (E\ D^A\ D^B)$, $\langle \mathbf{X}| = (X^0\ X^A\ X^B)$. Eq.(8) has 9 components, written analogously to (11–14). The first one is satisfied with the choice (15). Those four that are linear in E, yield analogically to (16, 17):

$$\dot{D}^A + 2aD^A + g_A\left(E^{-1}D^AE + ED^AE^{-1}\right) \tag{35}$$
$$2X^A = g_A\left(ED^AE^{-1} - E^{-1}D^AE\right);$$

and the Eq. for $\dot{D}^B, X^B$ is obtained by exchanging $a \to b$, $A \to B$. The additional Eqs.(9,10) lead to the conditions $[Q, E^{k-1}D^Z(t)E^{-k}] = 0$, $k = 0, \pm 1, \ldots, Z = A, B$ on Q-matrix.

Eliminating time derivatives in the four remaining equations that are quadratic in D^Z we obtain

$$(2a-d)D^AD^A + g_A\left(D^AE^{-1}D^AE + ED^AE^{-1}D^A\right) = 0, \tag{36}$$
$$(a+b-c)D^AD^B - gD^BD^A + g_AED^AE^{-1}D^B + g_BD^AE^{-1}D^BE = 0, \tag{37}$$
$$(a+b-c)D^BD^A - gD^AD^B + g_AED^BE^{-1}D^A + g_AD^BE^{-1}D^AE = 0, \tag{38}$$
$$(2b-e)D^BD^B + g_B\left(D^BE^{-1}D^BE + ED^BE^{-1}D^B\right) = 0. \tag{39}$$

One proceeds analogically then to what we have done in the spin 1/2 case. Introducing generalized Fourier Transform $\mathcal{D}_p^A(t) = \sum_k \alpha^k \mathrm{e}^{ipk} E^{k-1}D^A(t)E^{-k}$, $\mathcal{D}_p^B(t) = \sum_k \beta^k \mathrm{e}^{ipk} E^{k-1}D^B(t)E^{-k}$ leads to eliminating the time-dependence in (35): $\dot{\mathcal{D}}_p^Z + \epsilon_p \mathcal{D}_p^Z = 0$; $\mathcal{D}_p^Z(t) = e^{-\epsilon_p^Z t}\mathcal{D}_p^Z(0)$; $Z = A, B$, where

$$\epsilon_p^A = 2a + g_A(\alpha \mathrm{e}^{ip} + \mathrm{e}^{-ip}/\alpha),\ \epsilon_p^B = 2b + g_B(\beta \mathrm{e}^{ip} + \mathrm{e}^{-ip}/\beta). \tag{40}$$

The inverse Fourier Transform $D^{A,B}(t) = E\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}p \mathcal{D}_p^{A,B}(t)$, being inserted into (36–39), yields

$$0 = \int \mathrm{d}p_1 \int \mathrm{d}p_2 a_{12} \mathcal{D}_{p_1}^A \mathcal{D}_{p_2}^A \mathrm{e}^{-(\epsilon_{p_1}^A + \epsilon_{p_2}^A)t}, \tag{41}$$

$$0 = \int \mathrm{d}p_1 \int \mathrm{d}p_2 [c_{12}\mathcal{D}_{p_1}^A \mathcal{D}_{p_2}^B - g\beta \mathrm{e}^{ip_2} \mathcal{D}_{p_2}^B \mathcal{D}_{p_1}^A] \mathrm{e}^{-(\epsilon_{p_1}^A + \epsilon_{p_2}^B)t}, \tag{42}$$

$$0 = \int \mathrm{d}p_1 \int \mathrm{d}p_2 [d_{12}\mathcal{D}_{p_1}^B \mathcal{D}_{p_2}^A - g\alpha \mathrm{e}^{ip_2} \mathcal{D}_{p_2}^A \mathcal{D}_{p_1}^B] \mathrm{e}^{-(\epsilon_{p_1}^B + \epsilon_{p_2}^A)t}, \tag{43}$$

$$0 = \int \mathrm{d}p_1 \int \mathrm{d}p_2 b_{12} \mathcal{D}_{p_1}^B \mathcal{D}_{p_2}^B \mathrm{e}^{-(\epsilon_{p_1}^B + \epsilon_{p_2}^B)t} \tag{44}$$

with the functions

$$a_{12} \equiv a(p_1,p_2) = \alpha e^{ip_1}(2a-d) + g_A(1+\alpha^2 e^{ip_1+ip_2}), \quad (45)$$

$$b_{12} \equiv b(p_1,p_2) = \beta e^{ip_1}(2b-e) + g_B(1+\beta^2 e^{ip_1+ip_2}), \quad (46)$$

$$c_{12} \equiv c(p_1,p_2) = (a+b-c)\alpha e^{ip_1} + g_B\alpha\beta e^{ip_1+ip_2} + g_A, \quad (47)$$

$$d_{12} \equiv d(p_1,p_2) = (a+b-c)\beta e^{ip_1} + g_A\alpha\beta e^{ip_1+ip_2} + g_B. \quad (48)$$

The trick with the p_1 and p_2 exchange in (41-44) leads consequently (see (27) and the paragraph above it) to

$$\mathcal{D}^A_{p_1}\mathcal{D}^A_{p_2} = -\frac{a_{21}}{a_{12}}\mathcal{D}^A_{p_2}\mathcal{D}^A_{p_1}\,; \qquad \mathcal{D}^B_{p_1}\mathcal{D}^B_{p_2} = -\frac{b_{21}}{b_{12}}\mathcal{D}^B_{p_2}\mathcal{D}^B_{p_1}\,; \quad (49)$$

$$c_{12}\mathcal{D}^A_{p_1}\mathcal{D}^B_{p_2} - g\beta e^{ip_2}\mathcal{D}^B_{p_2}\mathcal{D}^A_{p_1} + c_{21}\mathcal{D}^A_{p_2}\mathcal{D}^B_{p_2} - g\beta e^{ip_1}\mathcal{D}^B_{p_1}\mathcal{D}^A_{p_2} = 0, \quad (50)$$

$$d_{12}\mathcal{D}^B_{p_1}\mathcal{D}^A_{p_2} - g\alpha e^{ip_2}\mathcal{D}^A_{p_2}\mathcal{D}^B_{p_1} + d_{21}\mathcal{D}^B_{p_2}\mathcal{D}^A_{p}\mathcal{D}^A_{p_1}\mathcal{D}^B_{p_2} = 0 \quad (51)$$

and a restriction $g_B = \alpha/\beta g_A$; $\alpha = \pm\beta$. The latter one is a consequence of the constraint $\epsilon^A_{p_1} + \epsilon^B_{p_2} = \epsilon^B_{p_1} + \epsilon^A_{p_2}$, needed to equate the time-dependent factors after $p_1 \leftrightarrow p_2$ exchange in (42, 43).

Eqs. (50, 51) can be written in the form

$$\mathcal{D}^A_{p_1}\mathcal{D}^B_{p_2} = \mathcal{D}^B_{p_2}\mathcal{D}^A_{p_1}\ \frac{d_{12}g\beta e^{ip_2} - d_{21}g\beta e^{ip_1}}{c_{12}d_{12} - g^2\alpha\beta e^{2ip_1}} + \mathcal{D}^A_{p_2}\mathcal{D}^B_{p_1}\ \frac{g^2\alpha\beta e^{ip_1+ip_2} - d_{12}c_{21}}{c_{12}d_{12} - g^2\alpha\beta e^{2ip_1}}\,, \quad (52)$$

$$\mathcal{D}^B_{p_1}\mathcal{D}^A_{p_2} = \mathcal{D}^A_{p_2}\mathcal{D}^B_{p_1}\ \frac{c_{12}g\alpha e^{ip_2} - c_{21}g\alpha e^{ip_1}}{c_{12}d_{12} - g^2\alpha\beta e^{2ip_1}} + \mathcal{D}^B_{p_2}\mathcal{D}^A_{p_1}\ \frac{g^2\alpha\beta e^{ip_1+ip_2} - c_{12}d_{21}}{c_{12}d_{12} - g^2\alpha\beta e^{2ip_1}}\,. \quad (53)$$

The Eqs.(49, 52, 53) between the Fourier components $\mathcal{D}^A_p, \mathcal{D}^B_p$ define an algebra conveniently written via a scattering matrix S as

$$\mathcal{D}^Z_{p_1}\mathcal{D}^{Z'}_{p_2} = S^{ZZ'}_{YY'}(p_1,p_2)\mathcal{D}^{Y'}_{p_2}\mathcal{D}^Y_{p_1}\,. \quad (54)$$

Applying it twice we see that it satisfies the inversion relation (S^T means transposition) $S(p_1,p_2)S^T(p_2,p_1) = 1$.

Now, note that application of (54) exchange the order of indexes p_1 and p_2. For the cubic terms $\mathcal{D}^X_{p_1}\mathcal{D}^Y_{p_2}\mathcal{D}^Z_{p_3}$ the exchange of order of indexes $123 \to 321$ can be done either by subsequent series of $1 \leftrightarrow 2,\ 1 \leftrightarrow 3,\ 2 \leftrightarrow 3$ or $2 \leftrightarrow 3,\ 1 \leftrightarrow 3,\ 1 \leftrightarrow 2$ exchanges. Associativity of the algebra requires the result to be the same in both cases, which amounts to

$$S^{ij}_{i''j''}(p_1,p_2)S^{i''k}_{i'k''}(p_1,p_3)S^{j''k''}_{j'k'}(p_2,p_3) = S^{jk}_{j''k''}(p_2,p_3)S^{ik''}_{i''k'}(p_1,p_3)S^{i''j''}_{i'j'}(p_1,p_2). \quad (55)$$

This is the Yang–Baxter equation (YBE). Summing over repeating indexes is implied. The YBE guarantees the consistency of the algebra on the cubic and further levels.

The YBE impose severe restrictions on the S matrix elements, which depend on the coefficients of the Hamiltonian (33). Nontrivial solutions of the YBE single out integrable models. We have investigated (55) with the help of Mathematica, and found the following solution:

$$\alpha = i_1\beta, \quad g_B = i_1 g_A, \qquad g_A = i_2 g, \quad d = 2a + (i_3 - 1)g,$$
$$e = 2b + (i_4 - 1)g, \qquad c = a + b + i_5 g,$$

where $i_1, i_2, i_3, i_4, i_5 = \pm 1$.

Sutherland's [6] and, more generally, solution # 4 of [4] is contained in the above. To the best of our knowledge, the general manifold describes a new solution to the YBE.

To summarise, the Dynamic Matrix Product State method turns out to be an effective tool to check the integrability of both stochastic processes and quantum Hamiltonians. Applying it saves the conventional tedious step of calculating parameter-dependent transfer matrices satisfyng the the Yang–Baxter equation, commuting with the original Hamiltonian. E.g. for investigating spin 1 chains conventionally, we would need first to solve YBE with in principle 27^2 independent components. In the DMPA approach we needed only the original Hamiltonian to start with, and then additionally to solve at most 8^2 equations (55) for the algebra associativity. (In fact the number of equations is much smaller due to symmetries). At the same time, we believe that for all the integrable models obtained the commuting transfer-matrices do exist, although we do not need them directly. Also, Bethe Ansatz equations for the spectrum can be constructed dir ectly from the algebra, as was demonstrated for XXZ model (see (30)). For the other models the Bethe Ansatz equations are readily constructed [7] by following e.g. the algebraic approach [8].

The DMPA approach does not make reference to the actual representations of the matrices involved. However, it would be interesting to find them. In special cases (namely, when they are not time dependent, which means only stationary (ground) state is investigated) the representations are known [9, 11, 10].

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