Matematicheskaya fizika, analiz, geometriya
2002, v. 9, No. 3, p. 446–454

Stabilization of solutions of nonlinear parabolic equations on thin two-layer domains

A.M. Rekalo

Department of Mechanics and Mathematics, V.N. Karazin National University
4 Svobody Sq., Kharkov, 61077, Ukraine
E-mail:rekalo@univer.kharkov.ua

Received December 27, 2001
Communicated by I.D. Chueshov

The paper is concerned with the large time behavior of solutions of the heat equation on a thin two-layer domain. Such systems may arise from modeling thermal emission (as a result of chemical reaction) and heat transfer between two thin films. It is shown that every solution converges as $t \to +\infty$ to a single equilibrium point.

Hale and Raugel [1] proved that every bounded solution of the problem

$$\partial_t u = \triangle u + f(u), \quad t > 0,\ (x, y) \in \Omega,$$

$$\frac{\partial u}{\partial n} = 0, \quad t > 0,\ (x, y) \in \partial\Omega,$$

converges to a single equilibrium point, provided that the bounded domain $\Omega \subset \mathrm{R}^2$ is close (in some sense) to a line segment.

In this paper we consider a gradient-like parabolic equation on *two-layer* thin domains. Since this equation admits a Lyapunov functional then the ω-limit set of any bounded solution belongs to the set of equilibria. Our main result shows that every solution actually converges to *a single* equilibrium point.

Note that Poláčik and Rybakowski [2] have shown that there are scalar parabolic equations on a disk in $\mathbf{R}^2$ having bounded solutions that approach, as $t \to +\infty$, a subset of equilibria homeomorphic to the circle. This implies that we have to impose additional conditions of some kind (for instance, thinness of the domain) in order to guarantee the convergence of solutions.

Our definition of thin domains is somewhat different from that given in [1] and allows us to avoid some tedious technicalities.

Mathematics Subject Classification 2000: 35B40, 35K57.
Supported in part by INTAS grant No. 2000-899.

Let $l,\ a_1,\ a_2,\ \varepsilon_0$ be positive numbers and $g_1,\ g_2 : [0,l] \times [0,\varepsilon_0] \to \mathbf{R}$ be functions of class C^2 such that

$$g_i(X,\varepsilon) > 0, \tag{1}$$

$$g_i(X,0) = a_i, \quad \partial_x g_i(X,0) = 0 \tag{2}$$

for $i = 1,\ 2,\ X \in [0,l],\ \varepsilon \in [0,\varepsilon_0]$. We set $\Omega_\varepsilon = \Omega_{1,\varepsilon} \cup \Omega_{2,\varepsilon}$, where

$$\Omega_{1,\varepsilon} = \{(X,Y) \in \mathbf{R}^2 : 0 < Y < g_1(X,\varepsilon), \quad 0 < X < l\},$$

$$\Omega_{2,\varepsilon} = \{(X,Y) \in \mathbf{R}^2 : -g_2(X,\varepsilon) < Y < 0, \quad 0 < X < l\},$$

and consider the following equations on Ω_ε

$$\begin{cases} \partial_t u_i = \triangle u_i - \alpha u_i + f(u_i), \quad t > 0, \\ u_i|_{t=0} = u_{i0}(X,Y), \quad (X,Y) \in \Omega_{i,\varepsilon},\ i = 1,2, \end{cases} \tag{3}$$

with the boundary conditions

$$\left(\frac{\partial u_i}{\partial n} - (-1)^i k(X)(u_1 - u_2) \right)\bigg|_\Gamma = 0, \quad i = 1,\ 2, \tag{4}$$

$$\frac{\partial u}{\partial n}\bigg|_{\partial\Omega_\varepsilon \setminus \Gamma} = 0. \tag{5}$$

(We set $\Gamma = \partial\Omega_{1,\varepsilon} \cap \partial\Omega_{2,\varepsilon}$, $u(X,Y) \equiv u_i(X,Y)$ for $(X,Y) \in \Omega_{i,\varepsilon}$ and denote by n the unit outward normal to $\partial\Omega_{i,\varepsilon}$.) We assume that $\alpha > 0$, $f \in C^1(\mathbf{R})$ and that there are $\beta,\ \gamma,\ C > 0$ such that

$$|f'(u) - f'(v)| \le C(1 + |u|^\gamma + |v|^\gamma)|u - v|^\beta, \qquad u, v \in \mathbf{R}. \tag{6}$$

The function $f(u)$ is also assumed to satisfy the dissipation condition

$$\limsup_{|u|\to\infty} \frac{f(u)}{u} \le 0.$$

Finally, let us suppose that $k(X) \in C^1[0,l]$ and

$$\kappa = \inf_{x\in[0,l]} k(x) > 0. \tag{7}$$

Under these assumptions, we can apply to (3)–(5) a standard technique of abstract parabolic equations theory [3] in order to obtain the global strong solution $u(t) = u(t,u_0)$ for

$$u_0 \in H^1(\Omega_\varepsilon) = H^1(\Omega_{1,\varepsilon}) \oplus H^1(\Omega_{2,\varepsilon}).$$

This solution is unique so the map $u_0 \mapsto u(t,u_0),\ t \ge 0$, defines a semigroup $S_\varepsilon(t)$ on $H^1(\Omega_\varepsilon)$. $S_\varepsilon(t)$ is turned out to be compact for $t > 0$ and thus the ω-limit set of every orbit $\{S_\varepsilon(t)u : t \ge 0\}$ is not empty. Our main result is as follows.

Theorem 1. *For any sufficiently small* a_1, $a_2 > 0$ *there exists a positive number* ε_0 *such that the* ω*-limit set of every orbit of* $S_\varepsilon(t)$ *for* $0 < \varepsilon < \varepsilon_0$ *is a single fixed point.*

P r o o f. If we make the change of variables (see [1])

$$\begin{cases} X = x, \\ Y = g_i(x,\varepsilon), \quad (X,Y) \in \Omega_{i,\varepsilon},\ i = 1,2, \end{cases} \tag{8}$$

we obtain the transformed domain $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1 = (0,\, l) \times (0,\, 1)$, $\Omega_2 = (0,\, l) \times (-1,\, 0)$. The problem (3)–(5) will be converted into the system

$$\begin{cases} \partial_t u_i = \triangle_{i,\varepsilon} u_i - \alpha u_i + f(u_i), \quad t > 0, \\ u_i|_{t=0} = u_{i0}(x,y), \quad (x,y) \in \Omega_i,\ i = 1,\ 2, \end{cases} \tag{9}$$

$$\left(\frac{\partial u_i}{\partial n_{i,\varepsilon}} - (-1)^i k(x)(u_1 - u_2) \right)\Bigg|_\Gamma = 0, \qquad i = 1,\ 2, \tag{10}$$

$$\frac{\partial u}{\partial n_\varepsilon}\Bigg|_{\partial\Omega\setminus\Gamma} = 0, \tag{11}$$

where $\triangle_{i,\varepsilon} = \frac{1}{g_i} div B_{i,\varepsilon}$, $\frac{\partial}{\partial n_{i,\varepsilon}} = B_{i,\varepsilon} \cdot n_i$, $B_{i,\varepsilon}$ is a formal differential operator of the first order

$$B_{i,\varepsilon} u_i(x,y) = \begin{bmatrix} g_i \partial_x u_i - (\partial_x g_i y) \partial_y u_i \\ \\ -(\partial_x g_i y)\partial_x u_i + \frac{1}{g_i}(1 + (\partial_x g_i y)^2)\partial_y u_i \end{bmatrix}$$

$$((x,y) \in \Omega_i, \quad \varepsilon \in [0, \varepsilon_0]),$$

n_i is the unit outward normal to $\partial\Omega_i$.

The transformation of coordinates (8) induces a linear continuous isomorphism

$$E_\varepsilon : H^1(\Omega_\varepsilon) \to V = H^1(\Omega_1) \oplus H^1(\Omega_2).$$

It is clear that if $T_\varepsilon(t)$ is the semigroup defined by the problem (9)–(11) in the space V, then

$$E_\varepsilon S_\varepsilon(t) = T_\varepsilon(t) E_\varepsilon, \quad t \geq 0.$$

The latter identity implies that the convergence of every orbit $\{T_\varepsilon(t)u : t \geq 0\}$ to a single equilibrium point is equivalent to the same property of $S_\varepsilon(t)$.

Let H_ε ($\varepsilon \in [0, \varepsilon_0]$) be the space $H = L^2(\Omega)$ endowed with the inner product

$$(u,v)_{H_\varepsilon} = \sum_{i=1}^{2} \int\limits_{\Omega_i} (u_i v_i g_i)(x,y)\, dx\, dy\,.$$

From (1) one derives that for $\|u\|^2_{H_\varepsilon} = (u,u)_{H_\varepsilon}$, $\|u\| = \|u\|_{L_2(\Omega)}$

$$C_1\|u\| \le \|u\|_{H_\varepsilon} \le C_2\|u\| \tag{12}$$

with some constants $C_1,\ C_2$ which do not depend on ε. Thanks to the continuity of $g_i,\ i = 1,\, 2$, one also obtains

$$\lim_{\varepsilon\to 0} |(u,v)_{H_\varepsilon} - (u,v)_{H_0}| = 0 \tag{13}$$

uniformly for bounded $u, v \in H$.

There is a bilinear form associated with the operator $(-\triangle + \alpha)$ on Ω_ε with the boundary conditions (4), (5)

$$a(u,v) = \int\limits_{\Omega_\varepsilon} ((\nabla u, \nabla v) + \alpha uv)\, dX\, dY + \sum_{i,j=1}^{2} (-1)^{i+j} \int\limits_{\Gamma} (k\, \gamma_{j,\varepsilon} u_j\, \gamma_{i,\varepsilon} v_i)(X)\, dX,$$

where $\gamma_{i,\varepsilon} : H^1(\Omega_{i,\varepsilon}) \to H^{1/2}(\partial\Omega_{i,\varepsilon})$ is the trace mapping. By the change of variables we transform $a(u,v)$ into the form

$$a_\varepsilon = (L_\varepsilon u, L_\varepsilon v)_{H_\varepsilon} + \alpha(u,v)_{H_\varepsilon} + \sum_{i,j=1}^{2} (-1)^{i+j} \int\limits_{\Gamma} (k\, \gamma_j u_j\, \gamma_i v_i)(x)\, dx,$$

where

$$L_\varepsilon u(x,y) = (\partial_x u_i - \frac{\partial_x g_i y}{g_i} \partial_y u_i,\ \frac{1}{g_i} \partial_y u_i)$$

for $(x,y) \in \Omega_i$. The space V serves as the natural domain of definition for $a_\varepsilon(u,v)$.

The Friedrichs inequality implies that the symmetric form $a_\varepsilon(u,v)$ is closed and sectorial in H_ε [4]. Hence, the triple $\{V,\ H_\varepsilon,\ a_\varepsilon\}$ defines a unique unbounded self-adjoint operator A_ε on H_ε with the domain $D(A_\varepsilon)$, such that

$$a_\varepsilon(u,v) = (A_\varepsilon u, v)_{H_\varepsilon}$$

for any $u \in D(A_\varepsilon)$, $v \in V$. The operator A_ε is positive due to the conditions (1) and (7). Note also that $D(A_\varepsilon^{1/2}) = V$ and

$$\lim_{\varepsilon\to 0} \sup_{u\in V,\ \|u\|_V \le 1} \|(A_\varepsilon^{1/2} - A_0^{1/2})u\| = 0. \tag{14}$$

The latter relation is a direct consequence of (13) and the inequality (see (2))

$$\|(L_\varepsilon - L_0)u\| \le M\varepsilon\|u\|_V, \qquad u \in V.$$

Using the standard arguments of the theory of elliptic boundary value problems, one can show that A_ε^{-1} is compact,

$$D(A_\varepsilon) = \{u: \ u_i \in H^2(\Omega_i), \ u \ \text{ satisfies } (10), (11)\},$$

and for any $u \in D(A_\varepsilon)$

$$(A_\varepsilon u)(x,y) = -\triangle_{i,\varepsilon} u_i(x,y) + \alpha u_i(x,y), \quad (x,y) \in \Omega_i.$$

Let us now consider the Nemitskii operator

$$(F(u))(x,y) = f(u(x,y)), \quad (x,y) \in \Omega.$$

The hypothesis (6) implies that F is well defined on V with values in H and for every $R > 0$

$$\begin{aligned} &\|F(u^1) - F(u^2)\| \le C_R \|u^1 - u^2\|_V, \\ &\|F(u^1)\| \le C_R, \quad u^1, \, u^2 \in V, \ \|u^i\|_V \le R, \end{aligned} \tag{15}$$

with some constant $C_R > 0$. This operator is of class C^1 and its Frechét derivative satisfies for any $R > 0$ and some $D_R > 0$ the following estimates

$$\|F'(u^1)v\| \le D_R \|v\|_V, \tag{16}$$

$$\|(F'(u^1) - F'(u^2))v\| \le D_R \|u^1 - u^2\|_V^\beta \|v\|_V, \tag{17}$$

where $u^1, \ u^2, \ v \in V, \ \|u^1\|_V, \ \|u^2\|_V, \ \|v\|_V \le R$.

Write (9)–(11) in the abstract operator form

$$\frac{du}{dt} + A_\varepsilon u = F(u), \quad t > 0, \quad u(0) = u_0. \tag{18}$$

Lemma 1. *The following statements hold true:*

1. *For any $u_0 \in V$ there is a unique strong solution (in the sense of Henry [3]) of (18) defined for all $t > 0$.*

2. *The semigroup $T_\varepsilon(t) : V \to V$ associated with (18) enjoys a V-compact absorbing set which does not depend on $\varepsilon \in [0, \varepsilon_0]$.*

3. *$T_\varepsilon(t)$ is gradient-like, i.e., the ω-limit set of every orbit $\{T_\varepsilon(t)u_0 : \ t \ge 0\}$ consists of the equilibrium points $u = T_\varepsilon(t)u, \ \forall t \ge 0$.*

4. *The corresponding family of global attractors $\mathcal{A}_\varepsilon, \ \varepsilon \in [0, \varepsilon_0]$, is upper semicontinuous in ε at $\varepsilon = 0$.*

The proof of the Lemma 1 seems to be rather standard and will be dropped here.

Let $V_x \subset V$ denote the subspace of y-homogeneous functions $V_x = \{u(x,y) \equiv v(x) \in V\}$. One easily shows that V_x is positively invariant under the semigroup $T_0(t)$. Moreover, for any sufficiently small parameters $a_1,\ a_2 > 0$ this subspace attracts exponentially every orbit of $T_0(t)$, that is

$$dist_V\{T_0(t)u, V_x\} \le Ke^{-\sigma t}\, dist_V\{u, V_x\}, \quad u \in V,\ t \ge 0,$$

with some constants σ, $K > 0$ (see [5]). In this case, the attractor $\mathcal{A}_0$ of $T_0(t)$ can be identified with the attractor $\mathcal{A}$ of the problem

$$\begin{cases} \partial_t u = \partial_{xx} u - \alpha u + f(u), & t > 0, \\ \frac{du}{dx}\Big|_{x=0} = \frac{du}{dx}\Big|_{x=l} = 0. \end{cases}$$

Due to statement 4 of Lemma 1 this implies that

$$\lim_{\varepsilon \to 0} dist_V\{\mathcal{A}_\varepsilon,\, \mathcal{A}\} = 0$$

for any sufficiently small $a_1,\ a_2 > 0$.

Hale and Raugel [1] have shown that if the abstract parabolic equation (18) defines a gradient-like semigroup and satisfies the property

$$\dim \ker(A_\varepsilon - F'(u)) \le 1 \tag{19}$$

for any fixed point u then every bounded solution of (18) which approaches a set of equilibria actually converges to a single fixed point. We are going to apply this statement to our particular problem.

Consider the linear operator

$$C_\varepsilon(v)u = A_\varepsilon u - F'(v)u,$$

where $v \in V$, $D(C_\varepsilon(v)) = D(A_\varepsilon)$. The perturbation operator $u \mapsto F'(v)u$ is well defined on V, symmetric, and satisfies the inequality

$$\|F'(v)u\| \le \delta\|A_\varepsilon u\| + N_\delta\|u\|, \quad u \in D(A_\varepsilon),$$

for any $\delta > 0$ with some $N_\delta > 0$. Since A_ε is a self-adjoint bounded from below operator in H_ε and its inverse is compact, the same properties hold true for the perturbed operator $C_\varepsilon(v)$. Hence, the spectrum of $C_\varepsilon(v)$ forms a denumerable sequence of real eigenvalues

$$\lambda_{1,\varepsilon}(v) \le \lambda_{2,\varepsilon}(v) \le \ldots \le \lambda_{n,\varepsilon}(v) \to +\infty.$$

Lemma 2. *For any sufficiently small $a_1,\ a_2$ every operator $C_0(v)$ with $v \in \mathcal{A}$ satisfies the condition*

$$\dim\ker C_0(v) \leq 1. \tag{20}$$

P r o o f. Since $\mathcal{A}$ is a bounded set in V, we have thanks to (16)

$$\|F'(v)u\| \leq M\|u\|_V, \quad u \in V, \quad v \in \mathcal{A},$$

with some $M > 0$. Let $u_0 \in \{u \in V :\ C_0(v)u = 0\}$ for some $v \in \mathcal{A}$. If $a_1,\ a_2$ are small then the subspace V_x attracts exponentially every solution of the equation

$$\frac{du}{dt} + A_0 u = F'(v)u, \quad t > 0,$$

(see [5]). In particular, we have

$$\lim_{t\to+\infty} dist_V\{\exp(-tC_0(v))u^0, V_x\} = 0.$$

On the other hand, by the choice of u_0

$$\exp(-tC_0(v))u^0 = u^0$$

for every $t \geq 0$. Therefore, u^0 belongs to V_x and satisfies the regular Sturm-Liouville problem

$$\begin{cases} \frac{d^2}{dx^2}u(x) - \alpha u(x) + f'(v(x))u(x) = 0, \quad 0 < x < l, \\ \frac{du}{dx}\Big|_{x=0} = \frac{du}{dx}\Big|_{x=l} = 0. \end{cases}$$

As is well known, this problem has at most a one-dimensional subspace of solutions (see [6], for instance). This completes the proof of Lemma 2.

Let $\mathbf{T}$ be the space $[0,\ \varepsilon_0] \times V$ endowed with the product topology. We have the following result.

Lemma 3. *If the parameters $a_1,\ a_2$ are as in Lemma 2 and $v^0 \in \mathcal{A}$, then there is a $\mathbf{T}$-neighborhood U of $(0, v^0) \in \mathbf{T}$ such that*

$$\dim\ker C_\varepsilon(v) \leq 1$$

for $(\varepsilon, v) \in U$.

P r o o f. Consider the family of bilinear forms on V generated by the operators $C_\varepsilon(v)$:

$$c_\varepsilon(v) : (u^1, u^2) \mapsto (A_\varepsilon^{1/2}u^1,\ A_\varepsilon^{1/2}u^2)_{H_\varepsilon} - (F'(v)u^1, u^2)_{H_\varepsilon}.$$

If the sequence $\{v^n\}_{n=1}^{\infty} \subset V$ converges to v^0, then from (13), (14), and (17) we deduce

$$\lim_{\varepsilon\to 0,\ n\to+\infty} (c_\varepsilon(v^n))(u^1, u^2) = (c_0(v^0))(u^1, u^2)$$

for any $u^1,\ u^2 \in V$. This relation together with (20) imply (see [4]) that the eigenvalues and associated projectors of $C_\varepsilon(v)$ are continuous in (ε, v) at $(\varepsilon, v) = (0, v^0)$.

Thanks to Lemma 2 there is either a number k such that

$$\lambda_{k,0}(v^0) = 0, \quad \lambda_{i,0}(v^0) \neq \lambda_{k,0}(v^0), \quad i \neq k$$

or $\lambda_{i,0}(v^0) \neq 0,\ i \in \mathbf{N}$. In any case there is a neighborhood $U \in \mathbf{T}$ of $(0, v^0)$ such that

$$\dim\ker C_\varepsilon(v) = \dim\ker C_0(v^0) \leq 1$$

for $(\varepsilon, v) \in U$ and the proof is complete.

Now we are able to complete the proof of the main theorem. Let the parameters $a_1,\ a_2$ be so small that all the statements of Lemmata 1– 3 hold true. By Lemma 3 for every $v \in \mathcal{A}$ there is a neighborhood U_v of $(0, v) \in \mathbf{T_0} = \{0\} \times \mathcal{A}$ such that

$$\dim\ker C_\varepsilon(u) \leq 1$$

for $(\varepsilon, u) \in U(v)$. Since the subset $\mathbf{T_0} = \{0\} \times \mathcal{A} \subset \mathbf{T}$ is compact we have $\mathbf{T_0} \subset \cup_{i=1}^{n} U_{v^i}$ for some elements $v^1,\ v^2, \ldots,\ v^n \in \mathcal{A}$. The statement 4 of Lemma 1 implies that for some $\varepsilon_1 > 0$ and every $\varepsilon \in [0,\ \varepsilon_1]$

$$\{\varepsilon\} \times \mathcal{A}_\varepsilon \subset \cup_{i=1}^{n} U_{v^i},$$

therefore, every equilibrium point u of (18) satisfies the condition (19). This proves the theorem.

Acknowledgements. I would like to thank my scientific supervisor Professor I.D. Chueshov for his guidance and support.

References

[1] *J.K Hale and G. Raugel*, Convergence in gradient-like systems with applications to PDE. — *Z. Angew. Math. Phys.* (1992), v. 43, p. 63–124.

[2] *P. Poláčik and K.R. Rybakowski*, Nonconvergent bounded trajectories in semilinear heat equations. — *J. Diff. Eq.* (1996), v. 124, p. 472–494.

[3] *D. Henry*, Geometric theory of semilinear parabolic equations. — *Lect. Notes Math.* Springer, Berlin (1981), v. 840.

[4] *T. Kato*, Perturbation theory for linear operators. Springer, Berlin, New York (1966).

[5] *A. Rekalo*, Asymptotic behavior of solutions of nonlinear parabolic equations on two-layer thin domains. (Submitted to *Nonlinear Analysis: TMA*).

[6] *P. Hartman*, Ordinary differential equations. Wiley, New York (1964).