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On sectorial block operator matrices

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The paper deals with linear operators in the Hilbert space $H_1 \oplus H_2$ defined by matrices with, in general, unbounded entries. Criterions for such operators to be sectorial with the vertex at the origin are obtained, parametrization of all its m-accretive and m-sectorial extensions and a description of root subspaces of such extensions by means of the transfer function (Schur complement) and its derivatives are given. Analytical properties of the Friedrichs extensions of the transfer function of a sectorial block operator matrix are established.

1. Introduction

Let H be the complex Hilbert space. As is well known [15], a linear operator S in H is called:

- *accretive* if its numerical range is contained in the closed right half-plane: $\mathrm{Re}\,\big(Sf,f\big) \ge 0$ for every f from the domain $\mathcal{D}(S)$ of S,
- *sectorial with the vertex at the complex point a and a semiangle* $\alpha \in [0,\pi/2)$ if its numerical range is contained in the closed sector $\Theta_a(\alpha) = \big\{z \in \mathbb{C} : |\arg(z-a)| \le \alpha\big\}$.

Below such an operator will be called sectorial. Clearly, an operator S is *sectorial with the vertex at the origin and a semiangle* α if and only if it satisfies one of the following equivalent conditions:

$$\big|\mathrm{Im}\,\big(Sf,f\big)\big| \le \tan\alpha\,\mathrm{Re}\,\big(Sf,f\big),\ f \in \mathcal{D}(S),$$

$$\big|\big(Sf,f\big)\big| \le \frac{1}{\cos\alpha}\mathrm{Re}\,\big(Sf,f\big),\ f \in \mathcal{D}(S).$$

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An operator in H is called maximal accretive (m-accretive) if it is accretive and has no accretive extensions in H. The definition of a maximal sectorial (m-sectorial) operator is analogous. An important class of sectorial operators forms the set of non-negative Hermitian operators ($a = 0,\ \alpha = 0$).

One of the problem considered in the present paper is the following: let H_1 and H_2 be the Hilbert spaces. What conditions on not necessarily unbounded linear operators $A : H_1 \to H_1,\ B : H_2 \to H_1,\ C : H_1 \to H_2,\ D : H_2 \to H_2$ guarantee that the operator T given by the matrix $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ in the Hilbert space $H = H_1 \oplus H_2$ is accretive or sectorial with the vertex at the origin?

The following statement is well known [14, 17] (the generalized Silvester criterion): the matrix operator T of such a form is bounded selfadjoint and non-negative in $H = H_1 \oplus H_2$ if and only if $A,\ B,\ C,\ D$ are bounded operatos and the following conditions are fulfilled:

$$a)\ A = A^* \geq 0,\ b)\ C = B^*,\ \ c)\ \mathcal{R}(B) \subseteq \mathcal{R}(A^{1/2}),\ d)\ D \geq B^*A^{-1}B,$$

where $B^*A^{-1}B$ is equal to the operator $(A^{-1/2}B)^*A^{-1/2}B$ and $A^{-1/2}$ is the inverse to the operator $A^{1/2}\big|\overline{\mathcal{R}(A)}$.

Conditions a)–d) are equivalent to the relation $C^* = B = A^{1/2}ZD^{1/2}$, where $Z : \overline{\mathcal{R}(D)} \to \overline{\mathcal{R}(A)}$ is a contraction.

Spectral properties of block operator matrices with in general unbounded entries and their applications to systems of differential operators were studied in [1, 10–13, 16, 18, 19]. The case of symmetric matrices T in the Hilbert and the Kreĭn spaces was considered in [1, 3, 19]. In particular, in [3] under certain conditions on entries all selfadjoint extensions of T were parametrized. Accretive block matrices of the form $T = \begin{bmatrix} A & B \\ -B^* & D \end{bmatrix}$ which arise in hydrodinamic problems were considered in [11, 12].

In this paper under some assumptions we give a solution of the above mentioned problem and give a description of m-accretive and m-sectorial extensions of a sectorial block operator matrix, the case of bounded entries is also considered. The paper is organized as follows: in Section 2 we develop the approach of the paper [7] for the parametrization of all m-accretive and m-sectorial extensions with the vertex at the origin of a nondensely defined closed and coercive sectorial operator S, the parametrization is given in the form of product of two matrices, in Section 3 we give a description of root subspaces of m-accretive extensions of S by means of corresponding transfer function (Schur complement) and its derivatives, in Section 4 using results of Section 2 necessary and sufficient conditions on entries of a matrix T in order to be sectorial with the vertex at the origin are given and it is proved that the Friedrichs extensions of the corresponding transfer function form a holomorphic family of types (A) and (B) [15], in Section 5 the case of bounded entries of block operator matrix is considered and an analog of Silvester's criterion for sectoriality in terms of Schur complements of the matrix and its real part is obtained and in the last Section 6 we consider examples of sectorial systems of differential operators and give applications of results of Section 4.

We use the following notations: $\mathcal{L}(\mathfrak{H},\mathfrak{K})$ denotes the Banach space of all continuous linear operators acting from the Hilbert space $\mathfrak{H}$ into the Hilbert space $\mathfrak{K}$ and $\mathcal{L}(\mathfrak{H}) = \mathcal{L}(\mathfrak{H},\mathfrak{H})$. Moreover, the domain, the range and the null-space of a linear operator T we denote by $\mathcal{D}(T)$, $\mathcal{R}(T)$ and $\operatorname{Ker} T$, respectively. As is well known [15] if S is a sectorial operator then the sesquilinear form $(S\cdot,\cdot)$ has the closure which we will denote by $S[\cdot,\cdot]$ and by $\mathcal{D}[S]$ its the domain of definition, let $S[u] = S[u,u]$ be the corresponding quadratic form. A linear operator S (a sesquilinear form $s[u,v]$) is said to be coercive if the quadratic form $\operatorname{Re}(Su,u)$ ($\operatorname{Re} s[u]$) is positive definite. We will use the well known representation of an m-sectorial operator with the vertex at the origin and the semiangle α and the corresponding closed associated form (see [15]):

$$T = T_R^{1/2}(I+iG)T_R^{1/2},\ \ T[u,v] = ((I+iG)T_R^{1/2}u, T_R^{1/2}v), \quad u,v\in\mathcal{D}[T], \quad (1.1)$$

where T_R is the so-called "real part" of T, i.e., the non-negative selfadjoint operator associated with the closed form $T_R[u,v] = (T[u,v] + \overline{T[v,u]})/2$ and $G = G^* \in \mathcal{L}(\overline{\mathcal{R}(T)})$, $||G|| \le \tan\alpha$. According to the Second Representation Theorem [15] we have

$$\mathcal{D}[T] = \mathcal{D}(T_R^{1/2}).$$

2. Maximal accretive extensions of nondensely defined sectorial operators

Let S be a closed sectorial operator defined on $\mathcal{D}(S)$ in a Hilbert space H. Then if $\mathcal{D}(S)$ is dense in H, the m-sectorial operator S_F in $\mathfrak{H}$ which is associated with $S[u,v]$ is called the Friedrichs extension of S [15]. If S is nondensely defined then the Friedrichs extension $\mathbf{S}_F$ is an m-sectorial linear relation and its operator part is an m-sectorial operator in the subspace $\overline{\mathcal{D}(S)}$ associated with the form $S[u,v]$, and the singular part $\mathbf{S}_F(0)$ coincides with the subspace $\mathfrak{H}\ominus\overline{\mathcal{D}(S)}$ [20]. Note that if α is the semiangle of S then using its Friedrichs extension and (1.1) we obtain the inequality

$$\left|S[u,v]\right|^2 \le \frac{1}{\cos^2\alpha}\operatorname{Re} S[u]\operatorname{Re} S[v],\ u,v\in\mathcal{D}[S]. \quad (2.1)$$

Among other m-sectorial extensions (operators or linear relations) with the vertex at the origin of a given sectorial operator with the vertex at the origin, there is a unique m-sectorial extension $\mathbf{S}_N$ which is called the von Neumann–Kreĭn m-sectorial extension and which has the following properties [4, 5]:

1) $\mathcal{D}[\widetilde{\mathbf{S}}] \subseteq \mathcal{D}[\mathbf{S}_N]$ for every sectorial extension $\widetilde{\mathbf{S}}$ with the vertex at the origin,

2) for every vector $u \in \mathcal{D}[\mathbf{S}_N]$ the relation holds

$$\inf\Big\{\big|\mathbf{S}_N[u-f]\big|,\ f \in \mathcal{D}(S)\Big\} = 0. \tag{2.2}$$

The domain $\mathcal{D}[\mathbf{S}_N]$ can be described by the equality:

$$\mathcal{D}[\mathbf{S}_N] = \left\{u \in H : \sup_{f \in \mathcal{D}(S)} \frac{\big|(u, Sf)\big|^2}{\mathrm{Re}\,(Sf, f)} < \infty\right\}. \tag{2.3}$$

If S is a closed coercive sectorial operator, i.e., $\mathrm{Re}\,\big(Sf, f\big) \geq m||f||^2$ for all $f \in \mathcal{D}(S)$, where $m > 0$, then the von Neumann–Kreĭn extension is a densely defined operator and

$$\mathcal{D}(S_N) = \mathcal{D}(S)\dot{+}\mathfrak{N}_0,\ \mathcal{D}[S_N] = \mathcal{D}[S]\dot{+}\mathfrak{N}_0,\ S_N\big|\mathfrak{N}_0 = 0,$$

where $\mathfrak{N}_0 = H \ominus \mathcal{R}(S)$. Note that all closed sectorial forms associated with m-sectorial extensions with the vertex at the origin of sectorial operators or sectorial linear relations were parametrized in [5, 6]. Abstract boundary conditions for m-accretive and m-sectorial extensions of densely defined sectorial operator were described in [7, 8]. Here we give a description of all m-accretive and m-sectorial extensions of a nondensely defined coercive sectorial operator in the form which is convenient for applying to operator block matrices.

Let S be a closed coercive sectorial operator in the Hilbert space H and let

$$\overline{\mathcal{D}(S)} = H_1 \subset H,\ H_2 = H \ominus H_1\,.$$

Denote by P_1 and P_2 the orthogonal projections in H onto H_1 and H_2, respectively. Suppose that

(s) the operator $A \stackrel{def}{=} P_1 S$ is m-sectorial in H_1.

Clearly, the operator A is a coercive. Let

$$A = A_R^{1/2}(I + iM)A_R^{1/2} \tag{2.4}$$

be the representation of A by means of (1.1). Then

$$\mathcal{D}[S] = \mathcal{D}[A] = \mathcal{D}(A_R^{1/2}),$$
$$S[u,v] = A[u,v] = \Big((I+iM)A_R^{1/2}u, A_R^{1/2}v\Big)\,,\ u, v \in \mathcal{D}[A].$$

The Friedrichs extension of S is the linear relation $\mathbf{S}_F = Gr(A) \oplus \langle 0, H_2 \rangle$. Since S is a closed operator, by closed graph theorem the operators $S(A - zI)^{-1}$, $z \in \rho(A)$ are bounded. Therefore the operators $P_2 S(A - zI)^{-1}$ are bounded. Put

$$C = P_2 S, \ Q(z) = C(A - zI)^{-1}, \ Q = Q(0) = CA^{-1} \tag{2.5}$$

and let $Q^* = \left(CA^{-1}\right)^* : H_2 \to H_1$ be the adjoint to Q. Since

$$Q(z) - Q(\xi) = (z - \xi) Q(z)(A - \xi I)^{-1},$$

the operator-function $Q(z)$ is a holomorphic function on the domain $\rho(A)$.

Proposition 2.1. *Let S be a closed sectorial coercive operator and let the condition* (s) *be fullfilled. Then a vector $u \in H$ belongs to $\mathcal{D}[S_N]$ if and only if the vector $P_1 u + Q^* P_2 u$ belongs to $\mathcal{D}[A]$.*

P r o o f. For every $u \in H$ and every $f \in \mathcal{D}(S) = \mathcal{D}(A)$ we have

$$(u, Sf) = (u, Af + Cf) = (u, Af + CA^{-1}Af) = ((P_1 + Q^* P_2)u, Af).$$

Taking into account that A is m-sectorial and $(Sf, f) = (Af, f)$, from (2.3) we obtain that $u \in \mathcal{D}[S_N] \iff (P_1 + Q^* P_2)u \in \mathcal{D}[A]$. ■

In particular, Proposition 2.1 yields the equivalence:

$$h \in \mathcal{D}[S_N] \cap H_2 \iff (CA^{-1})^* h \in \mathcal{D}[A]. \tag{2.6}$$

The next Proposition 2.2 was proved in [7, 9] for the case of densely defined closed sectorial operator S satisfying the condition $\mathcal{D}(S^*) \subset \mathcal{D}[S_N]$ but it remains true with the same proof for the situation under our consideration.

Proposition 2.2. *Let S be a closed sectorial coercive operator and let the condition* (s) *be fullfilled. If T is an accretive extension of S then $\mathcal{D}(T) \subseteq \mathcal{D}[S_N]$.*

Let us define on $\mathcal{D}[A]$ the quadratic functional

$$\mu[\varphi] = \sup\Big\{ \mathrm{Re}\ A[2\varphi - f, f], \ f \in \mathcal{D}(A) \Big\}. \tag{2.7}$$

From (2.4) it follows that

$$\mathrm{Re}\, A[2\varphi - f, f] = ||(I + iM) A_R^{1/2} \varphi||^2 - ||A_R^{1/2} f - (I + iM) A_R^{1/2} \varphi||^2.$$

Hence

$$\mu[\varphi] = ||(I + iM) A_R^{1/2} \varphi||^2. \tag{2.8}$$

Note that (2.4) and (2.8) yield for all $\varphi \in \mathcal{D}[A]$ the relation

$$\sup_{f \in \mathcal{D}(S)} \frac{|(\varphi, Sf)|^2}{\mathrm{Re}\,(Sf, f)} = ||(I - iM)A_R^{1/2}\varphi||^2 = \mu[\varphi].$$

From (2.4) it follows also that $\mathcal{D}(((A^{-1})_R)^{-1/2}) = \mathcal{D}[A]$,

$$\mu[\varphi] = \left\| \left(\frac{A^{-1} + A^{*-1}}{2} \right)^{-1/2} \varphi \right\|^2, \quad \varphi \in \mathcal{D}[A],$$

and if $A = A^*$ then $\mu[\varphi] = A[\varphi] = ||A^{1/2}\varphi||^2$ for all $\varphi \in \mathcal{D}[A]$.

As is well known [15] if τ is a closed sectorial form with the vertex at the origin in the Hilbert space then its domain $\mathcal{D}[\tau]$ is also the Hilbert space with the norm $\mathrm{Re}\,\tau[u] + ||u||^2$.

Proposition 2.3. *Let W be a sectorial operator with the vertex at the origin in the Hilbert space $\mathfrak{H}$ and let Y be a linear operator defined on $\mathcal{D}(W)$ with values in $\mathcal{D}[A]$ satisfying the condition*

$$\mu[Yh] \le c\,\mathrm{Re}\ (Wh, h) \quad \textit{for all} \quad h \in \mathcal{D}(W).$$

Then the operator Y has the continuation $\widehat{Y}$ on $\mathcal{D}[W]$ with values in $\mathcal{D}[A]$ and

$$\mu[\widehat{Y}h] \le c\,\mathrm{Re}\,W[h] \quad \textit{for all} \quad h \in \mathcal{D}[W].$$

P r o o f. Let $h \in \mathcal{D}[W]$. Then there exists a sequence $\{h_n\} \subset \mathcal{D}(W)$ such that

$$\lim_{n\to\infty} h_n = h \quad \text{in} \quad \mathfrak{H} \quad \text{and} \quad \lim_{m,n\to\infty} \mathrm{Re}\ (W(h_n - h_m), h_n - h_m) = 0.$$

Moreover,

$$\lim_{n\to\infty} (Wh_n, h_n) = W[h], \quad \lim_{n\to\infty} W[h_n - h] = 0.$$

Taking into account the subordination condition for Y and the expression (2.8), we get that $\{A_R^{1/2} Yh_n\}$ is the Cauchy sequence in H_1. Since A_R is the positive definite operator, the sequence $\{Yh_n\}$ is also the Cauchy sequence. Let $y = \lim_{n\to\infty} Yh_n$ in H_1. Then

$$y \in \mathcal{D}[A], \ \lim_{n\to\infty} A_R^{1/2} Yh_n = A^{1/2}y, \ \lim_{n\to\infty} \mu[Yh_n] = \mu[y].$$

Putting $\widehat{Y}h = y$, we get that $\mu[\widehat{Y}h] \le c\,\mathrm{Re}\,W[h]$. ∎

Theorem 2.4. *Let S be a closed sectorial coercive operator and let the condition* (s) *be fullfilled. Let T be an accretive extension of S. Then operators*

$$WP_2u \overset{def}{=} (P_2 - CA^{-1}P_1)Tu : P_2\mathcal{D}(T) \to H_2 \tag{2.9}$$

and

$$YP_2u \overset{def}{=} \frac{1}{2}\big(A^{-1}P_1Tu - (P_1 + Q^*P_2)u\big) : P_2\mathcal{D}(T) \to \mathcal{D}[A] \tag{2.10}$$

are well defined and hold the relations:

$$\begin{gathered}(Tu, v) = A[P_1u + Q^*P_2u + 2YP_2u, P_1v + Q^*P_2v] + (WP_2u, P_2v),\\ u \in \mathcal{D}(T),\ v \in \mathcal{D}[S_N],\end{gathered} \tag{2.11}$$

$$\begin{gathered}\mathcal{D}(T) = \Big\{u \in H : P_2u \in \mathcal{D}(W),\ P_1u + (Q^* + 2Y)P_2u \in \mathcal{D}(S)\Big\},\\ Tu = S(P_1u + (Q^* + 2Y)P_2u) + WP_2u.\end{gathered} \tag{2.12}$$

Moreover, the linear operator W is an accretive in H_2 and for all $u \in \mathcal{D}(T)$ the inequality

$$\mu[YP_2u] \le \operatorname{Re}(WP_2u, P_2u),\ u \in \mathcal{D}(T) \tag{2.13}$$

is fulfilled. If T is a sectorial extension with the vertex at the origin and a semiangle α then W is also a sectorial operator with the vertex a at the origin and the semiangle α and in addition

$$\mu[YP_2u] \le \sin^2\alpha \operatorname{Re}(WP_2u, P_2u),\ u \in \mathcal{D}(T). \tag{2.14}$$

If an extension T is m-accretive then the operator W is m-accretive in H_2.

P r o o f. Let us define operators Z and V by equalities

$$Zu = (P_2 - CA^{-1}P_1)Tu,\ Vu = \frac{1}{2}\big(A^{-1}P_1Tu - (P_1 + Q^*P_2)u\big),\ u \in \mathcal{D}(T). \tag{2.15}$$

Since T is an extension of S, by definitions we have $\operatorname{Ker} Z \supseteq \mathcal{D}(S)$ and $\operatorname{Ker} V \supseteq \mathcal{D}(S)$, and in view of Propositions 2.1 and 2.2 for all $u \in \mathcal{D}(T)$ the vector $P_1u + Q^*P_2u$ belongs to $\mathcal{D}[A]$. Therefore, $\mathcal{R}(V) \subseteq \mathcal{D}[A]$. Further for all $u \in \mathcal{D}(T)$ and all $v \in \mathcal{D}[S_N]$ from (2.15) we have

$$\begin{aligned}&A[P_1u + Q^*P_2u + 2Vu, P_1v + Q^*P_2v] + (Zu, P_2v)\\ &= A[A^{-1}P_1Tu, (P_1 + Q^*P_2)v] + \big(P_2Tu, P_2v\big) - \big(P_1Tu, Q^*P_2v\big)\\ &= \big(P_1Tu, (P_1 + Q^*P_2)v\big) + \big(P_2Tu, P_2v\big) - \big(P_1Tu, Q^*P_2v\big) = \big(Tu, v\big).\end{aligned}$$

It follows that

$$\begin{aligned}\mathrm{Re}\,(Tu,u) = \|A_R^{1/2}(P_1+Q^*P_2)u\|^2 + 2\mathrm{Re}\,\big((I+iM)A_R^{1/2}Vu, A_R^{1/2}(P_1+Q^*P_2)u\big)\\ + \mathrm{Re}\,(Zu,P_2u\big) = \|A_R^{1/2}(P_1+Q^*P_2)u + (I+iM)A_R^{1/2}Vu\|^2\\ + \mathrm{Re}\,(Zu,P_2u\big) - \|(I+iM)A_R^{1/2}Vu\|^2.\end{aligned}$$

Since T is an accretive operator, we have $\mathrm{Re}\,(T(u-g),u-g)\geq 0$ for all $u\in\mathcal{D}(T)$ and all $g\in\mathcal{D}(S)=\mathcal{D}(A)$. Hence

$$\begin{aligned}\|A_R^{1/2}(P_1u-g)+A_R^{1/2}Q^*P_2u+(I+iM)A_R^{1/2}Vu\|^2\\ +\mathrm{Re}\,(Zu,P_2u\big)-\|(I+iM)A_R^{1/2}Vu\|^2\geq 0\end{aligned}$$

for all $u\in\mathcal{D}(T),\ g\in\mathcal{D}(A)$. Choosing a sequence $\{g_n\}\subset\mathcal{D}(A)$ such that

$$\lim_{n\to\infty}A_R^{1/2}g_n = A_R^{1/2}(P_1+Q^*P_2)u+(I+iM)A_R^{1/2}Vu,$$

and using (2.8), we obtain

$$\mu[Vu]\leq \mathrm{Re}\,(Zu,P_2u),\ u\in\mathcal{D}(T).$$

This inequality yields that if $u\in\mathcal{D}(T)\cap H_1$, then $Vu=0$. Hence, using the definition of V, we obtain that $\mathcal{D}(T)\cap H_1=\mathcal{D}(S)$ and the operators

$$WP_2u\overset{def}{=}Zu,\ YP_2u\overset{def}{=}Vu,\ u\in\mathcal{D}(T)$$

are well defined on the linear manifold $P_2\mathcal{D}(T)\subseteq H_2$. So, we've proved equality (2.11) and inequality (2.13). Equalities (2.12) follow from (2.10) and (2.9).

Suppose that T is a sectorial extension of S with the vertex at the origin and the semiangle α, then from (2.1) it follows the inequality

$$\big|\big(Tu,v\big)\big|^2\leq\frac{1}{\cos^2\alpha}\mathrm{Re}\,\big(Tu,u\big)\,\mathrm{Re}\,\big(Tv,v\big),\ u,v\in\mathcal{D}(T).$$

Consequently, from (2.11) for $u,v\in\mathcal{D}(T)$ and $f,\varphi\in\mathcal{D}(S)$, using the operator $X=(I+iM)A_R^{1/2}Y$, we obtain

$$\begin{aligned}&\cos^2\alpha\Big|\Big((I+iM)A_R^{1/2}(P_1u-f+Q^*P_2u)+2XP_2u,A_R^{1/2}(P_1v-\varphi+Q^*P_2v)\Big)+\Big(WP_2u,P_2v\Big)\Big|^2\\ &\leq\mathrm{Re}\,\Big(\Big((I+iM)A_R^{1/2}(P_1u-f+Q^*P_2u)+2XP_2u,A_R^{1/2}(P_1u-f+Q^*P_2u)\Big)+\Big(WP_2u,P_2u\Big)\Big)\\ &\times\mathrm{Re}\,\Big(\Big((I+iM)A_R^{1/2}(P_1v-\varphi+Q^*P_2v)+2XP_2v,A_R^{1/2}(P_1v-\varphi+Q^*P_2v)\Big)+\Big(WP_2v,P_2v\Big)\Big).\end{aligned}$$

Choose sequences $\{f_n\}$ and $\{\varphi_n\}$ from $\mathcal{D}(A)$ such that

$$\lim_{n\to\infty}A_R^{1/2}f_n = XP_2u+A_R^{1/2}(P_1+Q^*P_2)u,\ \lim_{n\to\infty}A_R^{1/2}\varphi_n = A_R^{1/2}(P_1+Q^*P_2)v.$$

Then we get the inequality

$$\big|(WP_2u, P_2v)\big|^2 \le \frac{1}{\cos^2\alpha}\Big(\mathrm{Re}\,\big(WP_2u, P_2u\big) - \big\|XP_2u\big\|^2\Big)\mathrm{Re}\,\big(WP_2v, P_2v\big). \tag{2.16}$$

Putting $u = v$, we get that the linear operator W is sectorial with the vertex at the origin and the semiangle α. Moreover, from (2.16) follows the inequality

$$\big\|XP_2u\big\|^2 \le \sin^2\alpha\,\mathrm{Re}\,\big(WP_2u, P_2u\big),\ u \in \mathcal{D}(T).$$

Taking into account the equality $Y = A_R^{-1/2}(I + iM)^{-1}X$ and (2.8), we obtain (2.14).

Suppose now that T is an m-accretive extension of S. Let's show that W is an m-accretive operator in H_2. Let

$$G = W + I + Q(-1)(Q^* + 2Y) : H_2 \to H_2.$$

This operator is defined on $\mathcal{D}(W) = P_2\mathcal{D}(T)$ and in view of density of $\mathcal{D}(T)$ in H, the domain $\mathcal{D}(G)$ is dense in H_2. Since -1 is a regular point for T, the equation $(T + I) = h$ has a unique solution $u \in \mathcal{D}(T)$ for any $h \in H$. Using (2.12), we obtain that the system

$$\begin{cases} A(x_1 + Q^*x_2 + 2Yx_2) + x_1 = & h_1 \\ C(x_1 + Q^*x_2 + 2Yx_2) + Wx_2 + x_2 = & h_2 \end{cases}$$

has a unique solution $u = x_1 + x_2 \in \mathcal{D}(T)$ for any $h_1 \in H_1,\ h_2 \in H_2$. By the direct calculation we get that $\mathcal{R}(G) = H_2$ and $\mathrm{Ker}\,G = \{0\}$. Using (2.8) and (2.13), we have for all $h \in \mathcal{D}(G)$

$$\begin{aligned} \mathrm{Re}\,\big(Gh, h\big) &= \mathrm{Re}\,\big(Wh, h\big) + \big\|h\big\|^2 + 2\mathrm{Re}\,\big(Yh, Q^*(-1)h\big) + \mathrm{Re}\,\big(Q^*h, Q^*(-1)h\big) \\ &= \mathrm{Re}\,\big(Wh, h\big) - \big\|(I + iM)A_R^{1/2}Yh\big\|^2 \\ &\quad + \big\|(I + iM)A_R^{1/2}Yh + (I - iM)^{-1}A_R^{-1/2}Q^*(-1)h\big\|^2 \\ &\quad - \big\|(I - iM)^{-1}A_R^{-1/2}Q^*(-1)h\big\|^2 + \big\|h\big\|^2 + \mathrm{Re}\,\big(Q^*h, Q^*(-1)h\big) \\ &\ge \mathrm{Re}\,\big(Q^*h, Q^*(-1)h\big) - \mathrm{Re}\,\big(A^{-1}Q^*(-1)h, Q^*(-1)h\big) + \big\|h\big\|^2. \end{aligned}$$

Since $Q - Q(-1) = Q(-1)A^{-1}$, we obtain

$$\begin{gathered} \mathrm{Re}\,\big(Q^*h, Q^*(-1)h\big) - \mathrm{Re}\,\big(A^{-1}Q^*(-1)h, Q^*(-1)h\big) \\ = \mathrm{Re}\,\big(Q^*h - Q^*(-1)h, Q^*(-1)h\big) - \mathrm{Re}\,\big(A^{-1}Q^*(-1)h, Q^*(-1)h\big) + \big\|Q^*(-1)h\big\|^2 \\ = \big\|Q^*(-1)h\big\|^2. \end{gathered}$$

Thus $\mathrm{Re}\,\big(Gh,h\big)\geq\big\|h\big\|^2,\ h\in\mathcal{D}(G)$. This inequality means that G is an accretive operator in H_2 and in view of $\mathcal{R}(G)=H_2,\ G$ is m-accretive. Since W is an accretive operator and densely defined in H_2 it has a closure $\overline{W}$. If $\{h_n\}\subset\mathcal{D}(W)$ is the Cauchy sequence such that $\{Wh_n\}$ is also the Cauchy sequence then $h=\lim\limits_{n\to\infty}h_n\in\mathcal{D}(\overline{W})$ and $\overline{W}h=\lim\limits_{n\to\infty}Wh_n$. In view of (2.8) and (2.13), the sequence $\{Xh_n\}$, where $X=(I+iM)A_R^{1/2}$, is also the Cauchy sequence in H_1. Put $\overline{X}h=\lim\limits_{n\to\infty}Xh_n$. We get the operator $\overline{X}:\mathcal{D}(\overline{W})\to H_1$ which is an extension of X and which satisfies the condition $\big\|\overline{X}h\big\|^2\leq\mathrm{Re}\,\big(\overline{W}h,h\big)$ for all $h\in\mathcal{D}(\overline{W})$.

Let $\mathfrak{A}=(I-\overline{W})(I+\overline{W})^{-1}$ be the Cayley transform of $\overline{W}$. The operator $\mathfrak{A}$ is a contraction in H_2 defined on a subspace $\mathcal{D}(\mathfrak{A})$ in H_2. Clearly, for every vector $h=(I+\mathfrak{A})g\in\mathcal{D}(\overline{W})$ the condition

$$\big\|\overline{X}_0h\big\|^2\leq\mathrm{Re}\,\big(\overline{W}h,h\big)=\big\|g\big\|^2-\big\|\mathfrak{A}g\big\|^2$$

is fulfilled. Let $\widetilde{P}$ be the orthogonal projection in H_2 onto the subspace $\mathcal{D}(\mathfrak{A})$ and let $\widetilde{\mathfrak{A}}=\mathfrak{A}\widetilde{P}$. Then the operator $\widetilde{\mathfrak{A}}$ is a contractive extension of $\mathfrak{A}$ in H_2, and the operator $\widetilde{W}=(I-\widetilde{\mathfrak{A}})(I+\widetilde{\mathfrak{A}})^{-1}$ is an m-accretive extension of $\overline{W}$ in H_2. Let's define the operator $\widetilde{X}$ on $\mathcal{D}(\widetilde{W})$ by the equality

$$\widetilde{X}(I+\widetilde{\mathfrak{A}})\widetilde{g}=\overline{X}(I+\mathfrak{A})\widetilde{P}\widetilde{g}.$$

Then $\widetilde{X}$ is an extension of $\overline{X}$ on $\mathcal{D}(\widetilde{W})$ and for all $\widetilde{h}=(I+\widetilde{\mathfrak{A}})\widetilde{g}$ we have

$$\big\|\widetilde{X}\widetilde{h}\big\|^2=\big\|\overline{X}(I+\mathfrak{A})\widetilde{P}\widetilde{g}\big\|^2\leq\big\|\widetilde{P}\widetilde{g}\big\|^2-\big\|\mathfrak{A}\widetilde{P}\widetilde{g}\big\|^2\leq\big\|\widetilde{g}\big\|^2-\big\|\widetilde{\mathfrak{A}}\widetilde{g}\big\|^2=\mathrm{Re}\,\big(\widetilde{W}\widetilde{h},\widetilde{h}\big).$$

Thus, the operator $\widetilde{X}:\mathcal{D}(\widetilde{W})\to H_1$ satisfies the condition

$$\big\|\widetilde{X}_0\widetilde{h}\big\|^2\leq\mathrm{Re}\,\big(\widetilde{W}\widetilde{h},\widetilde{h}\big)\qquad\forall\widetilde{h}\in\mathcal{D}(\widetilde{W}).$$

Let $\widetilde{Y}=A_R^{-1/2}(I+iM)^{-1}\widetilde{X}$. Then $\mu[\widetilde{Y}\widetilde{h}]\leq\mathrm{Re}\,\big(\widetilde{W}\widetilde{h},\widetilde{h}\big)$ for all $\widetilde{h}\in\mathcal{D}(\widetilde{W})$.

It follows that the operator $\widetilde{G}=\widetilde{W}+I+Q(-1)(2\widetilde{Y}+Q^*)$ is an accretive extension of G. Since G is m-accretive, we get that $\widetilde{G}=G$ and therefore, $\widetilde{W}=W$. Thus, the operator W is m-accretive. ∎

Now we give a parametrization of all m-accretive and m-sectorial extensions.

Theorem 2.5. *1) Let S be a closed sectorial coercive operator and let the condition (***s***) be fullfilled. Then formulas*

$$\mathcal{D}(T)=\Big\{u=x_1+x_2:x_1\in H_1,\ x_2\in\mathcal{D}(W),\ x_1+(Q^*+2Y)x_2\in\mathcal{D}(S)\Big\},$$
$$Tu=S(x_1+(Q^*+2Y)x_2)+Wx_2\tag{2.17}$$

establish a bijective correspondence between all m-accretive extensions T of S and all pairs $\langle W, Y\rangle$, where W is an m-accretive operator in H_2, $Y : \mathcal{D}(W) \to \mathcal{D}[A]$ is a linear operator such that

$$\mu[Yh] \le \mathrm{Re}\,(Wh, h),\ h \in \mathcal{D}(W). \tag{2.18}$$

The relations (2.17) *are equivalent to the representation of the operator T by the product of two block matrices with respect to the decomposition $H = H_1 \oplus H_2$:*

$$T = \begin{bmatrix} A & 0 \\ C & W \end{bmatrix} \begin{bmatrix} I & Q^* + 2Y \\ 0 & I \end{bmatrix}. \tag{2.19}$$

The operator T is an m-sectorial extension of S wit h the vertex at the origin if and only if the operator W is a sectorial operator with the vertex at the origin and for some $\delta \in [0, 1)$ holds the inequality

$$\mu[Yh] \le \delta^2 \mathrm{Re}\,(Wh, h),\ h \in \mathcal{D}(W). \tag{2.20}$$

In this case the associated closed sectorial form is given by

$$\begin{aligned} &\mathcal{D}[T] = \Big\{u = x_1 + x_2 : x_1 \in H_1,\ x_2 \in \mathcal{D}[W],\ x_1 + Q^* x_2 \in \mathcal{D}[S]\Big\}, \\ &T[x_1 + x_2, y_1 + y_2] = S[x_1 + (Q^* + 2\widehat{Y})x_2, y_1 + Q^* y_2] + W[x_2, y_2], \end{aligned} \tag{2.21}$$

where $\widehat{Y}$ is the continuation of Y on the Hilbert space $\mathcal{D}[W]$.

2) If A is positive definite selfadjoint operator in H_1 then all non-negative selfadjoint extensions of S and its associated closed forms are given by (2.17) *and* (2.21) *when W is a non-negative selfadjoint operator in H_2 and $Y = 0$.*

P r o o f. Let $\langle W, Y\rangle$ be a pair consisting of an m-accretive operator W in the space H_2 and an operator $Y : \mathcal{D}(W) \to \mathcal{D}[A]$ which satisfies the condition (2.18) and let T be an operator defined by (2.17). Clearly, the operator T is an extension of S. Let us show that T is an m-accretive operator. Put $X = (I - iM)A_R^{1/2} Y$. Then by (2.18), the operator X satisfies the condition $||Xh||^2 \le \mathrm{Re}\,(Wh, h)$ for all $h \in \mathcal{D}(W)$. Let $u = x_1 + x_2 \in \mathcal{D}(T),\ e = x_1 + (Q^* + Y)x_2$ then

$$\begin{aligned} \mathrm{Re}\,\big(Tu, u\big) &= \mathrm{Re}\,\big((A + C)e, u\big) + \mathrm{Re}\,\big(Wx_2, x_2\big) \\ &= \mathrm{Re}\,\big((I + CA^{-1})Ae, u\big) + \mathrm{Re}\,\big(Wx_2, x_2\big) \\ &= \mathrm{Re}\,\big(Ae, e\big) - 2\mathrm{Re}\,\big(Ae, Yx_2\big) + \mathrm{Re}\,\big(Wx_2, x_2\big) \\ &= \big\|A_R^{1/2} e\big\|^2 - 2\mathrm{Re}\,\big(A_R^{1/2} e, (I - iM)A_R^{1/2} Yx_2\big) + \mathrm{Re}\,\big(Wx_2, x_2\big) \\ &= \big\|A_R^{1/2} e - Xx_2\big\|^2 + \mathrm{Re}\,\big(Wx_2, x_2\big) - \big\|Xx_2\big\|^2 \ge 0. \end{aligned}$$

Thus, T is an accretive operator. As was shown above, the operator

$$G = W + I + Q(-1)(Q^* + 2Y) : H_2 \to H_2$$

is also an accretive and coercive operator in H_2. Let us show that G is m-accretive. Since an arbitrary number $y < 0$ is a regular point of W and $\|Yg\|^2 \le c\,\mu[Yg]$ for every $g \in \mathcal{D}(W)$ and some c we have for arbitrary $h \in H_2$

$$\begin{aligned} \|Y(W-yI)^{-1}h\|^2 &\le c\,\mu\big[Y(W-yI)^{-1}h\big] \\ &\le c\,\mathrm{Re}\,\big(W(W-yI)^{-1}h,(W-yI)^{-1}h\big) \\ &= c\,\mathrm{Re}\,\big(h,(W-yI)^{-1}h\big) + c\,y\|(W+yI)^{-1}h\|^2 \le c|y|^{-1}\|h\|^2. \end{aligned}$$

It follows that for a sufficiently large $a > 0$ the operator $Q(-1)(2Y + Q^*)(W + (1+a)I)^{-1}$ has the norm less then 1 and therefore the operator $I + Q(-1)(2Y + Q^*)\big(W + (1+a)I\big)^{-1}$ has a bounded inverse defined on H_2. From the equality

$$\begin{aligned} & W + I + aI + Q(-1)(2Y + Q^*) \\ &= \Big(I + Q(-1)(2Y+Q^*)\big(W+(1+a)I\big)^{-1}\Big)(W+(1+a)I) \end{aligned}$$

it follows that $\mathcal{R}\big(W + (1+a)I + Q^*(2Y_0 + Q^*(-1))\big) = H_2$. Hence, the point $-a$ is regular for the operator $G = W + I + Q^*(2Y + Q^*(-1))$. Consequently, the operator G is m-accretive. Therefore, the system

$$\begin{cases} A(x_1 + Q^*x_2 + 2Yx_2) + x_1 = & h_1 \\ C(x_1 + Q^*x_2 + 2Yx_2) + Wx_2 + x_2 = & h_2 \end{cases}$$

has a unique solution for arbitrary $h_1 \in H_1,\ h_2 \in H_2$. By (2.17) it follows that -1 is a regular point for T. Consequently, T is an m-accretive operator.

Let W be an m-sectorial operator with the vertex at the origin and let the condition (2.20) be fulfilled. Let's show that T is an m-sectorial extension of S with the vertex at the origin. Let again $u = x_1 + x_2 \in \mathcal{D}(T),\ e = x_1 + (Q^* + Y)x_2$, then using (2.17) and (2.20), we have

$$\begin{aligned} \mathrm{Re}\,\big(Tu,u\big) &\le \|A_R^{1/2}e\|^2 + 2\big|\big((I-iM)A_R^{1/2}Yx_2, A_R^{1/2}e)\big)\big| + \mathrm{Re}\,\big(Wx_2,x_2\big)e \\ &\le \|A_R^{1/2}e\|^2 + \mathrm{Re}\,\big(Wx_2,x_2\big) + \delta\big(\mathrm{Re}\,\big(Wx_2,x_2\big) + \|A_R^{1/2}e\|^2\big) \\ &\le (1+\delta)\Big(\|A_R^{1/2}e\|^2 + \mathrm{Re}\,\big(Wx_2,x_2\big)\Big). \end{aligned}$$

Analogously, holds the inequality

$$\mathrm{Re}\,\big(Tu,u\big) \ge (1-\delta)\Big(\|A_R^{1/2}e\|^2 + \mathrm{Re}\,\big(Wx_2,x_2\big)\Big).$$

If α is the semiangle for A and β is the semiangle for W, then for the imaginary part of the quadratic form $\big(Tu,u\big)$ we obtain

$$\begin{aligned}&\big|\mathrm{Im}\,\big(Tu,u\big)\big|\\ &\le \tan\alpha\,\|A_R^{1/2}x|^2+2\big|\big((I-iM)A_R^{1/2}Yx_2,A_R^{1/2}e\big)\big|+\tan\beta\,\mathrm{Re}\,\big(Wx_2,x_2\big)\\ &\le \tan\alpha\,\|A_R^{1/2}e\|^2+\tan\beta\,\mathrm{Re}\,\big(Wx_2,x_2\big)+\delta\big(\mathrm{Re}\,\big(Wx_2,x_2\big)+\|A_R^{1/2}e\|^2\big)\\ &\le \big(max\{\tan\alpha,\ \tan\beta\}+\delta\big)\Big(\|A_R^{1/2}e\|^2+\mathrm{Re}\,\big(Wx_2,x_2\big)\Big)\\ &\le \big(max\{\tan\alpha,\ \tan\beta\}+\delta\big)(1-\delta)^{-1}\mathrm{Re}\,\big(Tu,u\big).\end{aligned}$$

Thus, T is a sectorial operator with the vertex at the origin and its semiangle γ has an estimate

$$\tan\gamma\le\big(max\{\tan\alpha,\ \tan\beta\}+\delta\big)(1-\delta)^{-1}.$$

Since W is m-sectorial, the operator T is an m-sectorial extension of S.

Let T given by (2.17) be the m-sectorial extension of S. Now we shall proof relations (2.21). As was proved above, we have the estimate for $u=x_1+x_2\in\mathcal{D}(T)$:

$$\begin{aligned}(1-\delta)\Big(\big\|A_R^{1/2}(x_1+Q^*x_2)\big\|^2+\mathrm{Re}\,\big(Wx_2,x_2\big)\Big)&\le\mathrm{Re}\,\big(Tu,u\big)\\ \le(1+\delta)\Big(\big\|A_R^{1/2}(x_1+Q^*x_2)\big\|^2&+\mathrm{Re}\,\big(Wx_2,x_2\big)\Big).\end{aligned}$$

Let $u\in\mathcal{D}[T]$, then $u=\lim\limits_{n\to\infty}u_n$, $\lim\limits_{n,m\to\infty}\big(T(u_n-u_m),u_n-u_m\big)=0$, where $u_n=x_1^{(n)}+x_2^{(n)}\in\mathcal{D}(T)$, $n=1,2,\dots$. Consequently,

$$\begin{aligned}&\lim_{n\to\infty}x_1^{(n)}=x_1^0\in H_1,\ \lim_{n\to\infty}x_2^{(n)}=x_2^0\in H_2,\ u=x_1^0+x_2^0,\\ &\lim_{n,m\to\infty}\big\|A_R^{1/2}(x_1^{(n)}-x_1^{(m)}+Q^*(x_2^{(n)}-x_2^{(m)}))\big\|=0,\\ &\lim_{n,m\to\infty}\mathrm{Re}\,\big(W(x_2^{(n)}-x_2^{(m)}),x_2^{(n)}-x_2^{(m)}\big)=0.\end{aligned}$$

It follows that $x_2^0\in\mathcal{D}[W]$, $x_1^0+Q^*x_2^0\in\mathcal{D}[A]$, and from (2.17) we obtain

$$T[u]=\lim_{n\to\infty}\big(Tu_n,u_n\big)=A[x_1^0+Q^*x_2^0+2\widehat{Y}x_2^0,x_1^0+Q^*x_2^0]+W[x_2^0],$$

where $\widehat{Y}:\mathcal{D}[W]\to\mathcal{D}[A]$ is the continuation of Y on the Hilbert space $\mathcal{D}[W]$.

Let $\varphi\in\mathcal{D}[A]$ and $x_2\in\mathcal{D}[W]$. Since $x_2\in\mathcal{D}[W]$, then there exists a sequence $\{x_2^{(n)}\}\subset\mathcal{D}(W)$ such that

$$\lim_{n\to\infty}W[x_2^{(n)}-x_2]=0,\ \lim_{n\to\infty}x_2^{(n)}=x_2.$$

In view of (2.20) and (2.8), we have $\lim\limits_{n\to\infty} Yx_2^{(n)} = \widehat{Y}x_2$ in $\mathcal{D}[A]$. Since $\varphi \in \mathcal{D}[A]$, there exists a sequence $\varphi_n \in \mathcal{D}(A)$ with the property

$$\lim_{n\to\infty} A[\varphi_n - \varphi] = \lim_{n\to\infty} A_R^{1/2}(\varphi_n - \varphi) = 0.$$

Consider the sequence $\{\varphi_n - (Q^* + 2Y)x_2^{(n)} + x_2^{(n)}\}$. According to (2.17), for every n the vector $u_n = \varphi_n - (Q^* + 2Y)x_2^{(n)} + x_2^{(n)}$ belongs to $\mathcal{D}(T)$ and $\lim\limits_{n\to\infty} u_n = \varphi - (Q^* + 2\widehat{Y})x_2 + x_2$. From (2.17) it follows

$$\begin{aligned}\big(T(u_n - u_m), u_n - u_m\big) =& \big(A(\varphi_n - \varphi_m), \varphi_n - \varphi_m - (Q^* + 2Y)(x_2^{(n)} - x_2^{(m)})\big)\\ &+ \big(CA^{-1}A(\varphi_n - \varphi_m), x_2^{(n)} - x_2^{(m)}\big) + \big(W(x_2^{(n)} - x_2^{(m)}), x_2^{(n)} - x_2^{(m)}\big)\\ =& \big(A(\varphi_n - \varphi_m), \varphi_n - \varphi_m - 2Y(x_2^{(n)} - x_2^{(m)})\big) + \big(W(x_2^{(n)} - x_2^{(m)}), x_2^{(n)} - x_2^{(m)}\big)\\ =& \big(A(\varphi_n - \varphi_m), \varphi_n - \varphi_m\big) - 2\big((I + iM)A_R^{1/2}(\varphi_n - \varphi_m), A_R^{1/2}Y(x_2^{(n)} - x_2^{(m)})\big)\\ &+ \big(W(x_2^{(n)} - x_2^{(m)}), x_2^{(n)} - x_2^{(m)}\big).\end{aligned}$$

Consequently,

$$\lim_{n\to\infty} \big(T(u_n - u_m), u_n - u_m\big) = 0.$$

This means that the vector $u = \varphi - (Q^* + 2Y)x_2 + x_2$ belongs to $\mathcal{D}[T]$ and moreover,

$$T[u] = \lim_{n\to\infty} \big(Tu_n, u_n\big) = A[\varphi, \varphi - 2\widehat{Y}x_2] + W[x_2, x_2].$$

So, the relations (2.21) are proved.

If A is a positive definite selfadjoint operator in H_1 then S is a positive definite Hermitian operator. Every non-negative selfadjoint extension T of S is an m-sectorial operator with vertex at the origin and the semiangle $\alpha = 0$. According to Theorem 2.4 and (2.14), we obtain that the corresponding operator W is a non-negative selfadjoint in H_2 and $Y = 0$. ∎

As is well known, a densely defined accretive operator has the closure. It is easy to see from (2.9)–(2.13) that the closure $\overline{T}$ of an accretive extension T of S takes the form

$$\begin{aligned}\mathcal{D}(\overline{T}) &= \Big\{u \in H : P_2u \in \mathcal{D}(\overline{W}),\ P_1u + (Q^* + 2\overline{Y})P_2u \in \mathcal{D}(S)\Big\},\\ \overline{T}u &= S(P_1u + (Q^* + 2\overline{Y})P_2u) + \overline{W}P_2u,\end{aligned}$$

where $\overline{W}$ is the closure of a densely defined accretive operator W and $\overline{Y}$ is a continuation of Y on $\mathcal{D}(\overline{W})$.

Let W be a closed densely defined sectorial operator in H_2 with the vertex at the origin and suppose that $Y : \mathcal{D}(W) \to \mathcal{D}[A]$ satisfies the condition (2.20).

Let W_F and W_N be the Friedrichs and von Neumann–Kreĭn extensions of W. In accordance with Proposition 2.3 the operator Y has the continuation $\widehat{Y}_F$ on the Hilbert space $\mathcal{D}[W] = \mathcal{D}[W_F]$ preserving the estimate (2.20). In view of the relation (2.2) we have for W_N:

$$\inf\Big\{\operatorname{Re} W_N[h-f],\ f \in \mathcal{D}(W)\Big\} = 0,\ h \in \mathcal{D}[W_N].$$

It follows that the operator Y has the continuation $\widehat{Y}_N$ on $\mathcal{D}[W_N]$ with values in $\mathcal{D}[A]$ and this continuation preserves the estimate

$$\mu[\widehat{Y}_N h] \le \delta^2 \operatorname{Re} W_N[h],\ h \in \mathcal{D}[W_N].$$

Theorem 2.6. *Let S be a closed sectorial coercive operator satisfying the condition* (s) *and let T be a densely defined accretive extension of S. Then there is a bijective correspondence given by* (2.17) *between all m-accretive extensions $\widetilde{T}$ of T and all pairs $\langle \widetilde{W}, \widetilde{Y}\rangle$, where $\widetilde{W}$ is an m-accretive extension of the closure $\overline{W}$ of the operator W defined by* (2.9) *and $\widetilde{Y}$ is an extension of the continuation $\overline{Y}$ on $\mathcal{D}(\overline{W})$ of the operator Y defined by* (2.10) *such that the condition* (2.18) *is fulfilled.*

If T is a closed densely defined sectorial extension of S with the vertex at the origin then its m-accretive extension $\widetilde{T}$ is m-sectorial with the vertex at the origin if and only if $\widetilde{W}$ is an m-sectorial extension of W with the vertex at the origin and for some $\delta \in [0,1)$ the condition (2.20) *for $\widetilde{Y}$ is fulfilled. In particular, the pairs $\langle W_F, \widehat{Y}_F\rangle$ and $\langle W_N, \widehat{Y}_N\rangle$ correspond to the Friedrichs and von Neumann–Kreĭn extensions T_F and T_N, respectively.*

P r o o f. All statements except one are immediate consequences of Theorems 2.4 and 2.5. We have to prove only that the pair $\langle W_N, \widehat{Y}_N\rangle$ defines by (2.17) the von Neumann–Kreĭn extension of T. Let the operator $\widetilde{T}$ be defined by the relations (2.17) with $\langle W_N, \widehat{Y}_N\rangle$ and let $u \in \mathcal{D}[\widetilde{T}]$. Then

$$u = \varphi - (Q^* + 2\widehat{Y}_N)x_2 + x_2,\ \varphi \in \mathcal{D}[A],\ x_2 \in \mathcal{D}[W_N]$$

and according to (2.21) for all $h = \psi - (Q^* + 2\widehat{Y}_N)g \in \mathcal{D}(T)$, where $\psi \in \mathcal{D}(A),\ g \in \mathcal{D}(\overline{W})$, we have

$$\widetilde{T}[u-h] = A[\varphi - \psi, \varphi - \psi - 2\widehat{Y}_N(x_2 - g)] + W_N[x_2 - g].$$

Since W_N is the von Neumann–Kreĭn extension of $\overline{W}$, there exists a sequence $\{g_n\} \subset \mathcal{D}(\overline{W})$ such that $\lim\limits_{n\to\infty} \operatorname{Re} W_N[x_2 - g_n] = 0$. It follows that $\lim\limits_{n\to\infty} \mu[\widehat{Y}_N(x_2 - g_n)] = 0$. Also we can find a sequence $\{\psi_n\} \subset \mathcal{D}(A)$ such that

$\lim\limits_{n\to\infty} \operatorname{Re} A[\varphi - \psi_n] = 0$. Hence, we obtain that $\lim\limits_{n\to\infty} \operatorname{Re} \widetilde{T}[u - h_n] = 0$, where $h_n = \psi_n - (Q^* + 2\widehat{Y}_N)g_n$. This means that

$$\inf\left\{\operatorname{Re} \widetilde{T}[u-h],\ h \in \mathcal{D}(T)\right\} = 0,\ u \in \mathcal{D}[\widetilde{T}].$$

Therefore, the operator $\widetilde{T}$ is an extremal m-sectorial extension of T [4]. From the relation

$$\mathcal{D}[\widetilde{T}] = \left\{u = \varphi - (Q^* + 2\widehat{Y}_N)x_2 + x_2 :\ \varphi \in \mathcal{D}[A],\ x_2 \in \mathcal{D}[W_N]\right\}$$

it follows that $\mathcal{D}[\widetilde{T}]$ is the largest domain among all domains of closed forms associated with extremal sectorial extensions. Consequently [4], the extension $\widetilde{T}$ coincides with the von Neumann–Kreĭn extension T_N of T. ■

3. On the resolvents and the spectrum of m-accretive extensions

Let T be an m-accretive extension of S. According to Theorem 2.5 the operator T is defined by the pair $\langle W, Y\rangle$ and by formulas (2.17). In this section we give a description of the resolvent and the spectrum of T. Let

$$V = Q^* + 2Y$$

and

$$W(z) = W - zI - zC(A - zI)^{-1}V,\ \ z \in \rho(A). \tag{3.1}$$

The operator function $W(z)$ is holomorphic on the domain $\rho(A)$ and can be considered as a generalization of Schur complement.

Proposition 3.1. *Let S be a closed sectorial coercive operator satisfying the condition* (s) *and let T be an m-accretive extension of S given by* (2.17). *Then*

$$\rho(T) \cap \rho(A) = \left\{z \in \rho(A) : W^{-1}(z) \in \mathcal{L}(H_2)\right\}$$

and the resolvent $(T - zI)^{-1}$ with respect to the decomposition $H = H_1 \oplus H_2$ has the following matrix representation

$$(T - zI)^{-1} = \begin{bmatrix} \left(I + A(A - zI)^{-1}VW^{-1}(z)C\right)(A - zI)^{-1} & -A(A - zI)^{-1}VW^{-1}(z) \\ -W^{-1}(z)C(A - zI)^{-1} & W^{-1}(z) \end{bmatrix}. \tag{3.2}$$

A number $z \in \rho(A)$ is an eigenvalue of T if and only if $\operatorname{Ker} W(z) \neq \{0\}$ *and in this case*

$$\operatorname{Ker}(T - zI) = \begin{bmatrix} -A(A-zI)^{-1}Vh \\ h \end{bmatrix}, \quad h \in \operatorname{Ker} W(z). \tag{3.3}$$

P r o o f. From (2.17) it follows that the equality $(T-zI)u = f$ is equivalent to the system

$$\begin{cases} A(x_1 + Vx_2) - zx_1 = & f_1 \\ C(x_1 + Vx_2) + Wx_2 - zx_2 = & f_2, \end{cases}$$

where $f_1 = P_1 f,\ f_2 = P_2 f,\ x_1 = P_1 u,\ x_2 = P_2 u$. Hence,

$$x_1 + Vx_2 = (A - zI)^{-1}(f_1 - zVx_2),\ W(z)x_2 = f_2 - C(A-zI)^{-1}f_1.$$

These equalities yield the assertions of the proposition and relations (3.2) and (3.3). ■

Further we give a description of root subspaces of T.

Proposition 3.2. *Let S be a closed sectorial coercive operator satisfying the condition* (s) *and let T be an m-accretive extension of S given by* (2.17). *Then for every natural $j \geq 2$, every $u \in \mathcal{D}(T^j)$ and every $z \in \rho(A)$ the equality holds*

$$\sum_{k=0}^{j-1} \frac{1}{k!} W^{(k)}(z) P_2 (T - zI)^k u = -C(A - zI)^{-j} P_1 (T - zI)^j u. \tag{3.4}$$

P r o o f. From (3.1) follow the equalities

$$W'(z)u = \Big(-I - C(A-zI)^{-1}A(A-zI)^{-1}V\Big)u,$$

$$W^{(k)}(z)u = -k!C(A-zI)^{-1}A(A-zI)^{-k}Vu,\ k \geq 2,\ u \in \mathcal{D}(W).$$

From (2.17) it follows that

$$\begin{gathered} P_1(T - zI)u = A(P_1 + VP_2)u - zP_1 u, \\ P_2(T - zI)u = C(P_1 + VP_2)u + (W - zI)P_2 u. \end{gathered}$$

We shall prove (3.4) by induction. Let $j = 2$ and $u \in \mathcal{D}(T^2)$. Then

$$\begin{gathered} P_2(T - zI)u = C(P_1 + VP_2)u + (W - zI)P_2 u \in \mathcal{D}(W), \\ P_1(T - zI)u + VP_2(T - zI)u = A(P_1 + VP_2)u - zP_1 u \\ +V\Big(C(P_1 + VP_2)u + (W - zI)P_2 u\Big) \in \mathcal{D}(A). \end{gathered}$$

We obtain following equalities:

$$\begin{aligned}
&W(z)P_2u + W'(z)P_2(T - zI)u = (W - zI - zC(A - zI)^{-1}V)P_2u\\
&\quad + \Big(-I - C(A - zI)^{-1}A(A - zI)^{-1}V\Big)\Big(C(P_1 + VP_2) + (W - zI)P_2u\Big)\\
&= -zC(A - zI)^{-1}VP_2u - C(P_1 + VP_2)\\
&\quad -C(A - zI)^{-1}A(A - zI)^{-1}V\Big(C(P_1 + VP_2)u + (W - zI)P_2u\Big)\\
&= -C(A - zI)^{-1}\Big(zVP_2u + (A - zI)(P_1 + VP_2)u\Big)\\
&\quad -C(A - zI)^{-1}A(A - zI)^{-1}VP_2(T - zI)u\\
&= -C(A - zI)^{-1}\Big(A(P_1 + VP_2)u - zP_1u\Big)\\
&\quad -C(A - zI)^{-1}A(A - zI)^{-1}VP_2(T - zI)u =\\
&\quad -C(A - zI)^{-1}P_1(T - zI)u - C(A - zI)^{-1}A(A - zI)^{-1}VP_2(T - zI)u\\
&= -C(A - zI)^{-1}A(A - zI)^{-1}\Big((A - zI)A^{-1}P_1 + VP_2\Big)(T - zI)u\\
&= -C(A - zI)^{-1}A(A - zI)^{-1}\Big(P_1 - zA^{-1}P_1 + VP_2\Big)(T - zI)u\\
&= -C(A - zI)^{-1}A(A - zI)^{-1}A^{-1}\Big(A(P_1 + VP_2) - zP_1\Big)(T - zI)u\\
&= -C(A - zI)^{-2}P_1(T - zI)^2u.
\end{aligned}$$

Suppose that (3.4) holds for fixed $j \geq 2$ and let $u \in \mathcal{D}(T^{j+1})$. Then

$$\begin{aligned}
&\sum_{k=0}^{j}\frac{1}{k!}W^{(k)}(z)P_2(T - zI)^ku = -C(A - zI)^{-j}P_1(T - zI)^ju\\
&\quad = -C(A - zI)^{-1}A(A - zI)^{-j}VP_2(T - zI)^ju\\
&\quad = -C(A - zI)^{-j}A(A - zI)^{-1}\Big((A - zI)A^{-1}P_1 + VP_2\Big)(T - zI)^ju\\
&\quad = -C(A - zI)^{-j-1}A(P_1 + VP_2 - zA^{-1}P_1)(T - zI)^ju\\
&\quad = -C(A - zI)^{-j-1}\Big(A(P_1 + VP_2) - zP_1\Big)(T - zI)^ju\\
&\quad = -C(A - zI)^{-j-1}P_1(T - zI)^{j+1}u.
\end{aligned}$$

■

Proposition 3.3. *Let $H_2{}^n = \sum_{k=1}^{n} \oplus H_2$ and let $\mathbf{W}_n(z)$ be the operator function in $H_2{}^n$ defined by the equality*

$$\mathbf{W}_n(z) = \begin{bmatrix}
W(z) & 0 & & \dots & 0\\
\frac{1}{1!}W'(z) & W(z) & & \dots & 0\\
\vdots & \vdots & & \ddots & \vdots\\
\frac{1}{(j-1)!}W^{(j-1)}(z) & \dots & W(z) & \dots & 0\\
\vdots & \vdots & & \ddots & \vdots\\
\frac{1}{(n-1)!}W^{(n-1)}(z) & \frac{1}{(n-2)!}W^{(n-2)}(z) & & \dots & W(z)
\end{bmatrix}.$$

Let

$$K_n(z)\,u \overset{def}{=} \begin{bmatrix} P_2(T-zI)^{n-1}u \\ P_2(T-zI)^{n-2}u \\ \vdots \\ P_2 u \end{bmatrix}, \quad u \in \mathcal{D}(T^n).$$

Then $\mathbf{W}_n(z)K_n(z)\,u = 0$ *for every* $u \in \mathcal{D}(T^n)$ *such that* $(T-zI)^n u = 0$.

P r o o f. Let $(T-zI)^n u = 0$ and $j \geq 2$. Then according to Proposition 3.2 we have

$$\sum_{k=0}^{j-1} \frac{1}{k!} W^{(k)}(z) P_2 (T-zI)^{n-j+k} u = \Big(\sum_{k=0}^{j-1} \frac{1}{k!} W^{(k)}(z) P_2 (T-zI)^k\Big)(T-zI)^{n-j} u$$
$$= -C(A-zI)^{-j} P_1 (T-zI)^j (T-zI)^{n-j} u = 0.$$

Since $(T-zI)(T-zI)^{n-1}u = 0$, according to Proposition 3.1 we obtain $P_2(T-zI)^{n-1}u \in \operatorname{Ker} W(z)$. ■

Thus, the operator $K_n(z)$ maps $\operatorname{Ker}(T-zI)^n$ into $\operatorname{Ker}\mathbf{W}_n(z)$.

Proposition 3.4. *Let the operator function* $\mathbf{V}_n(z) : {H_2}^n \to H$ *be defined by the equality:*

$$\mathbf{V}_n(z) \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_{n-1} \end{bmatrix} = -\sum_{k=1}^{n-1} A(A-zI)^{-k} V h_{n-k} + h_{n-1},\ h_k \in \mathcal{D}(W),\ k = 0, \dots, n-1.$$

Then $\mathbf{V}_n(z)\vec{h} \in \operatorname{Ker}(T-zI)^n$ *for every* $\vec{h} = \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_{n-1} \end{bmatrix} \in \operatorname{Ker}\mathbf{W}_n(z)$.

P r o o f. By the definition of $\mathbf{W}_n(z)$ the condition $\vec{h} \in \operatorname{Ker}\mathbf{W}_n(z)$ is equivalent to the system

$$-\sum_{k=1}^{r} C(A-zI)^{-k} A(A-zI)^{-1} V h_{r-k} - h_{r-1} + W(z) h_r = 0, \qquad (3.5)$$
$$r = 0, 1, \dots, n-1.$$

For the vector $\mathbf{V}_n(z)\vec{h} = -\sum_{k=1}^{n-1} A(A-zI)^{-k}Vh_{n-k} + h_{n-1}$ we have

$$\begin{aligned}(P_1+VP_2)\mathbf{V}_n(z)\vec{h} &= -\sum_{k=1}^{n} A(A-zI)^{-k}Vh_{n-k} + Vh_{n-1}\\ &= -\sum_{k=2}^{n} A(A-zI)^{-k}Vh_{n-k} + Vh_{n-1} - A(A-zI)^{-1}Vh_{n-1}\\ &= -\sum_{k=2}^{n} A(A-zI)^{-k}Vh_{n-k} - z(A-zI)^{-1}Vh_{n-1} \in \mathcal{D}(A).\end{aligned}$$

By Theorem 2.5 we obtain that $u = \mathbf{V}_n(z)\vec{h} \in \mathcal{D}(T)$ and

$$\begin{aligned}&(T-zI)u = (A-zI)(P_1+VP_2)u + zVP_2u + F(P_1+VP_2)u + (W-zI)P_2u\\ &= -(A-zI)\sum_{k=2}^{n} A(A-zI)^{-k}Vh_{n-k} - z(A-zI)(A-zI)^{-1}Vh_{n-1} + zVh_{n-1}\\ &\quad -\sum_{k=2}^{n} FA(A-zI)^{-k}Vh_{n-k} - zC(A-zI)^{-1}Vh_{n-1} + (W-zI)h_{n-1}\\ &= -\sum_{k=2}^{n} A(A-zI)^{-k+1}Vh_{n-k}\\ &\quad -\sum_{k=2}^{n} FA(A-zI)^{-k+1}A(A-zI)^{-1}Vh_{n-k} + W(z)h_{n-1}.\end{aligned}$$

Since from (3.5) for $r = n-1$ we have

$$h_{n-2} = -\sum_{k=2}^{n} C(A-zI)^{-k+1}A(A-zI)^{-1}Vh_{n-k} + W(z)h_{n-1},$$

then

$$(T-zI)\mathbf{V}_n(z)\vec{h} = h_{n-2} - \sum_{k=2}^{n} A(A-zI)^{-k+1}Vh_{n-k}.$$

Analogously, by induction one can prove the equalities

$$(T-zI)^j\mathbf{V}_n(z)\vec{h} = h_{n-j-1} - \sum_{k=j+1}^{n} A(A-zI)^{-k+j}Vh_{n-k},\ j = 1,2\ldots,n-1.$$

It follows if $j = n-2$:

$$\begin{aligned}
&(T-zI)^{n-2}\mathbf{V}_n(z)\vec{h} = -A(A-zI)^{-1}Vh_1 + h_1 - A(A-zI)^{-2}Vh_0,\\
&(P_1+VP_2)(T-zI)^{n-2}\mathbf{V}_n(z)\vec{h}\\
&= -(A-zI)^{-1}A(A-zI)^{-}Vh_0 - z(A-zI)^{-1}Vh_1 \in \mathcal{D}(A),\\
&P_2(T-zI)^{n-2}\mathbf{V}_n(z)\vec{h} = h_1,\\
&P_1(T-zI)^{n-1}\mathbf{V}_n(z)\vec{h} = P_1(T-zI)(T-zI)^{n-2}\mathbf{V}_n(z)\vec{h}\\
&= \big(A(P_1+VP_2) - zP_1\big)(T-zI)^{n-2}\mathbf{V}_n(z)\vec{h} = -A(A-zI)^{-1}Vh_0,\\
&P_2(T-zI)^{n-1}\mathbf{V}_n(z)\vec{h} = P_2(T-zI)(T-zI)^{n-2}\mathbf{V}_n(z)\vec{h}\\
&= \big(F(P_1+VP_2) + (W-zI)P_2\big)(T-zI)^{n-2}\mathbf{V}_n(z)\vec{h}\\
&= -C(A-zI)^{-2}Vh_0 - zC(A-zI)^{-1}Vh_1 + (W-zI)h_1.
\end{aligned}$$

From (3.5) for $r = 1$ we obtain

$$h_0 = -C(A-zI)^{-2}Vh_0 - zC(A-zI)^{-1}Vh_1 + (W-zI)h_1.$$

Therefore,

$$(T-zI)^{n-1}\mathbf{V}_n(z)\vec{h} = h_0 - A(A-zI)^{-1}Vh_0.$$

Hence

$$(P_1+VP_2)(T-zI)^{n-1}\mathbf{V}_n(z)\vec{h} = -z(A-zI)^{-1}Vh_0$$

and in view of $W(z)h_0 = 0$, we get $(T-zI)^n\mathbf{V}_n(z)\vec{h} = 0$. ■

From Propositions 3.3 and 3.4 it follows that

$$\dim \operatorname{Ker}(T-zI)^n = \dim \operatorname{Ker} \mathbf{W}_n(z).$$

4. Block operator matrices

In the sequel we suppose that the following conditions on operators are fulfilled:

(a) A is a closed m-sectorial and coercive operator in H_1 with the domain $\mathcal{D}(A)$,

(b) C is a closed operator from H_1 into H_2 and $\mathcal{D}(C) \supseteq \mathcal{D}(A)$,

(c) D is a linear operator in H_2,

(d) B is a linear operator from H_2 into H_1,

(e) the linear manifold $\mathcal{D}_0 = \mathcal{D}(B) \cap \mathcal{D}(D)$ is dense in H_2.

Consider in the Hilbert space $H = H_1 \oplus H_2$ the operator T defined by the block operator matrix

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{4.1}$$

with the domain $\mathcal{D}(T) = \mathcal{D}(A) \oplus \mathcal{D}_0$. Note that with T can be associated the system of differential equations

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + T\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix} \iff \begin{cases} x_1'(t) + Ax_1(t) + Bx_2(t) = & f(t) \\ x_2'(t) + Cx_1(t) + Dx_2(t) = & g(t) \end{cases}$$

and the linear stationary dynamical system

$$\begin{cases} x'(t) + Ax(t) + Bu(t) = & 0, \\ v(t) + Cx(t) + Du(t) = & 0. \end{cases}$$

Note that the operator T given by (4.1) one can consider as an extension of the operator $S = A + C$, $\mathcal{D}(S) = \mathcal{D}(A)$. In view of the conditions **(a)** and **(b)**, the operator S is closed sectorial with the vertex at the origin, coercive and the condition **(s)** is fulfilled. Here we give necessary and sufficient conditions for the operator T given by (4.1) to be an accretive or sectorial operator with the vertex at the origin.

Proposition 4.1. *Let the conditions* (**a**) *and* (**b**) *be fulfilled. Then the following assertions are equivalent:*
1) $(CA^{-1})^* H_2 \subseteq \mathcal{D}[A]$*; 2)* $\mathcal{D}(C) \supseteq \mathcal{D}[A]$.

P r o o f. 1) $\Rightarrow$ 2) Let $(CA^{-1})^* H_2 \subseteq \mathcal{D}[A] = \mathcal{D}(A_R^{1/2})$, then from the equality $\big((CA^{-1})^*h, Ax\big) = \big(h, Cx\big)$ for every $h \in H_2$ and $x \in \mathcal{D}(A)$ and (2.3) follows $H_2 \subset \mathcal{D}[S_N]$, where S_N is the von Neumann–Krein extension of the operator $S = A + C$ and by closed graph theorem we obtain that the operator $S_{NR}^{1/2}\restriction H_2$ is bounded. Since for every $u \in \mathcal{D}[S_N]$ holds the equality (see [4, 5]):

$$\sup_{f\in\mathcal{D}(S)} \frac{\big|(u, Sf)\big|^2}{\operatorname{Re}(Sf, f)} = \big\|(I + iM_N)S_{NR}^{1/2}u\big\|^2,$$

where $S_N = S_{NR}^{1/2}(I + iM_N)S_{NR}^{1/2}$, we obtain $\big|(h, Cx)\big| \le c\big\|h\big\|\,\big\|A_R^{1/2}x\big\|$ for all $h \in H_2$, $\forall x \in \mathcal{D}(A)$. Consequently, $\big\|Cx\big\| \le c\big\|A_R^{1/2}x\big\|$ for all $x \in \mathcal{D}(A)$. Since C is closed, it follows that the domain of C contains $\mathcal{D}(A_R^{1/2})$.

2)$\Rightarrow$ 1) Let $\mathcal{D}(C) \supseteq \mathcal{D}(A_R^{1/2})$, then the operator $CA_R^{-1/2}$ is bounded and $(CA^{-1})^* = A_R^{-1/2}(I - iM)^{-1}(CA_R^{-1/2})^*$. Therefore, $(CA^{-1})^* H_2 \subseteq \mathcal{D}(A_R^{1/2})$. ∎

Proposition 4.2. *Let the conditions* **(a)**–**(e)** *be fulfilled. Then*
1) the following statements are equivalent:
i) the operator T *given by* (4.1) *is accretive in* H*,*

ii) holds the inclusion $(CA^{-1})^*\mathcal{D}_0 \subseteq \mathcal{D}[A]$ *and for every vector* $h \in \mathcal{D}_0$ *the condition*

$$\frac{1}{4}\mu\big[\big(A^{*-1}B + (CA^{-1})^*\big)h\big] \le \mathrm{Re}\,\big(Dh, h\big) \tag{4.2}$$

is fulfilled,

iii) the operator $D - CA^{-1}B$ *defined on the domain* $\mathcal{D}_0$ *is accretive in* H_2*, the inclusion* $(CA^{-1})^*\mathcal{D}_0 \subseteq \mathcal{D}[A]$ *holds and for every vector* $h \in \mathcal{D}_0$ *the condition*

$$\frac{1}{4}\mu\big[\big(A^{-1}B - (CA^{-1})^*\big)h\big] \le \mathrm{Re}\,\big((D - CA^{-1}B)h, h\big) \tag{4.3}$$

is fullfilled;

2) the operator T *is sectorial with the vertex at the origin if and only if:* $(CA^{-1})^*\mathcal{D}_0 \subseteq \mathcal{D}[A]$*, the operator* $D - CA^{-1}B$ *defined on* $\mathcal{D}_0$ *is sectorial with the vertex at the origin in* H_2 *and*

$$\frac{1}{4}\mu\big[\big(A^{-1}B - (CA^{-1})^*\big)h\big] \le \delta^2 \mathrm{Re}\,\big((D - CA^{-1}B)h, h\big) \;\; \forall h \in \mathcal{D}_0, \tag{4.4}$$

where $\delta \in [0, 1)$.

3) the operator T *is m-accretive if and only if:* $(CA^{-1})^*\mathcal{D}_0 \subseteq \mathcal{D}[A]$*, the operator* $D - CA^{-1}B$ *defined on* $\mathcal{D}_0$ *is an m-accretive in* H_2 *and holds* (4.3).

P r o o f. i)$\Rightarrow$ ii). Let T be an accretive operator. Then T is an accretive extension of the operator S. From Proposition 2.2 and (2.6) it follows that $(CA^{-1})^*\mathcal{D}_0 \subseteq \mathcal{D}[A]$. For this case the operators W and Y defined by (2.9) and (2.10) take the form

$$W = D - CA^{-1}B,\; Y = \frac{1}{2}\big(A^{-1}B - (CA^{-1})^*\big),\; \mathcal{D}(W) = \mathcal{D}(Y) = \mathcal{D}_0. \tag{4.5}$$

According to Theorem 2.4, the operator W is accretive in H_2 and (4.3) holds. If T is sectorial with the vertex at the origin then W is also sectorial with the vertex at the origin in H_2 and (4.4) holds.

The equivalence ii) $\Longleftrightarrow$ iii) follows from the equality which can be easily checked:

$$\begin{aligned}&\mathrm{Re}\,\big((D - CA^{-1}B)h, h\big) - \frac{1}{4}\mu\big[\big(A^{-1}B - (CA^{-1})^*\big)h\big]\\ &= \mathrm{Re}\,\big(Dh, h\big) - \frac{1}{4}\mu\big[\big(A^{*-1}B + (CA^{-1})^*\big)h\big],\; h \in \mathcal{D}_0.\end{aligned}$$

Evidently, for the operator T we have

$$\mathcal{D}(T) = \Big\{u = x_1 + x_2 : x_1 \in H_1, \quad x_2 \in \mathcal{D}_0, \quad x_1 + \Big((CA^{-1})^* + A^{-1}B - (CA^{-1})^*\Big)x_2 \in \mathcal{D}(S)\Big\},$$
$$Tu = S(x_1 + ((CA^{-1})^* + A^{-1}B - (CA^{-1})^*)x_2) + (D - CA^{-1}B)x_2.$$

If $W = D - CA^{-1}B$ is accretive (sectorial) in H_2 and $Y = \frac{1}{2}\big(A^{-1}B - (CA^{-1})^*\big)$ satisfies the condition (4.3) ((4.4)) then by Theorem 2.5 the operator T is accretive (sectorial). Thus, iii)⇒ i) and assertions 2) and 3) are proved. ■

Remark 4.3. *Proposition 4.2 is a generalization of Silvester's criterion. If A is a positive definite selfadjoint operator in H_1, $B \subseteq C^*$ and D is a symmetric in H_2, then $(CA^{-1})^*\mathcal{D}_0 = A^{-1}B\mathcal{D}_0 \subseteq \mathcal{D}(A^{1/2})$, the operator $D - CA^{-1}B$ is a symmetric and $\big(A^{-1}B - (CA^{-1})^*\big)h = 0$ for all $h \in \mathcal{D}_0$. In this case we obtain from Proposition 4.2 the following necessary and sufficient condition of non-negativity of T:*

$$\big((D - CA^{-1}B)h, h\big) \geq 0 \quad \textit{for all} \quad h \in \mathcal{D}_0.$$

E x a m p l e. Let $H^2 = H \oplus H$ and let A be an m-sectorial coercive operator in H, D be a bounded m-accretive operator in H. On the domain $\mathcal{D}(T) = \mathcal{D}(A) \oplus \mathcal{D}(A^*) \subseteq H^2$ consider the operator defined by the matrix

$$T = \begin{bmatrix} A & \xi A^* \\ -\bar{\xi} A & D \end{bmatrix},$$

where $\xi \neq 0$ is a complex number. Here $B = \xi A^*,\ C = -\bar{\xi} A$.

Let us check the conditions of Proposition 4.2. Since D is bounded and accretive and A^* is m-accretive, the operator $D - CA^{-1}B = D + |\xi|^2 A^* = D + |\xi|^2 A^*$ is also m-accretive in H. Note that $D + |\xi|^2 A^*$ is also m-sectorial. In fact, for every $h \in \mathcal{D}(A^*)$ from sectoriality and coercivity A^* we have

$$\begin{aligned} \big|\mathrm{Im}\,\big(Dh + |\xi|^2 A^* h, h\big)\big| &\leq ||D||\,||h||^2 + k|\xi|^2\,\mathrm{Re}\,\big(A^*h, h\big) \\ &\leq c\,\mathrm{Re}\,\big(A^*h, h\big) \leq c\,\mathrm{Re}\,\big(Dh + |\xi|^2 A^* h, h\big) \end{aligned}$$

for some $c > 0$. In addition, $(CA^{-1})^*\mathcal{D}(B) = \mathcal{D}(A^*) \subseteq \mathcal{D}(A_R^{1/2})$,

$$\begin{aligned} \frac{1}{4}\mu\big[(A^{-1}B - (CA^{-1})^*)h\big] &= \frac{1}{4}\mu\big[\xi A^{-1}A^*h + \xi h\big] \\ &= \frac{|\xi|^2}{4}\big\|(I + iM)A_R^{1/2}(A^{-1}A^* + I)h\big\|^2 \\ &= \frac{|\xi|^2}{4}\big\|(I - iM)A_R^{1/2}h + (I + iM)A_R^{1/2}h\big\|^2 \\ &= |\xi|^2\big\|A_R^{1/2}h\big\|^2 = |\xi|^2\,\mathrm{Re}\,\big(A^*h, h\big),\ h \in \mathcal{D}(A^*). \end{aligned}$$

Finally,

$$\begin{aligned} \frac{1}{4}\mu\big[(A^{-1}B - (CA^{-1})^*)h\big] &= |\xi|^2\,\mathrm{Re}\,\big(A^*h, h\big) \\ \leq \mathrm{Re}\,\big(Dh, h\big) + |\xi|^2\,\mathrm{Re}\,\big(A^*h, h\big) &= \mathrm{Re}\,\big((D - CA^{-1}B)h, h\big),\ h \in \mathcal{D}(A^*). \end{aligned}$$

In accordance with Proposition 4.2 we get that T is m-accretive. Let's show that T is m-sectorial with the vertex at the origin if and only if A is a bounded operator and D is a coercive operator. In fact, if the condition (4.4) is fulfilled with $\delta \in [0,1)$ then

$$|\xi|^2 \operatorname{Re}\big(A^*h,h\big) \le \delta^2 \Big\{\operatorname{Re}\big(Dh,h\big) + |\xi|^2 \operatorname{Re}\big(A^*h,h\big)\Big\}.$$

It follows that $\operatorname{Re}\big(A^*h,h\big) \le k \operatorname{Re}\big(Dh,h\big) \le m\,||h||^2$ for all $h \in \mathcal{D}(A^*)$. Therefore, A is a bounded operator and D is a coercive.

Remark 4.4. *In [11] there were considered operators in the Hilbert space $H = H_1 \oplus H_2$ given by the matrix*

$$T = \begin{bmatrix} A & B^* \\ -B & D \end{bmatrix},$$

where A is a positive definite selfadjoint operator in H_1, $B : H_2 \to H_1$ is an unbounded closed operator such that $\mathcal{D}(B^*) \supseteq \mathcal{D}(A^{1/2})$ and D is a bounded positive definite selfadjoint operator in H_2.

Under these conditions there was shown in [11] that T is an unclosed, essentially m-accretive and coercive operator $(\operatorname{Re}(Th,h) \ge a\big\|h\big\|^2$ *for all $h \in \mathcal{D}(T) = \mathcal{D}(A) \oplus \mathcal{D}(G)$ for some $a > 0$). This result follows also from the fact that $D + B^*A^{-1}B = D + B^*A^{-1/2}A^{-1/2}B$ is an essentially selfadjoint and positive definite operator on $\mathcal{D}(B)$ in view of $B^*A^{-1/2}(B^*A^{-1/2})^*$ is a bounded operator and $B^*A^{-1/2}A^{-1/2}B \subset B^*A^{-1/2}(B^*A^{-1/2})^*$.*

Remind [15] that a family of closed operators $V(z)$ is called a holomorpic of the type (A) in a domain Ω of the complex plane $\mathbb{C}$ if $\mathcal{D}(V(z)) = \mathcal{D} = const$ (does not depend on $z \in \Omega$) and a vector function $V(z)u$ is holomorphic in Ω for every $u \in \mathcal{D}$, and a family of m-sectorial operators $V(z)$ forms a holomorphic family of the type (B) in Ω if $\mathcal{D}[V(z)] = \mathcal{D} = const$ and a function $V(z)[x,y]$ is holomorphic in Ω for every $x, y \in \mathcal{D}$.

Theorem 4.5. *Let the conditions* **(a)**–**(e)** *be fulfilled and $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a sectorial operator with the vertex at the origin. Then in $\rho(A)$ operators $D - C(A - zI)^{-1}B$ are sectorial on $\mathcal{D}_0$ and its Friedrichs extensions $\big(D - C(A - zI)^{-1}B\big)_F$ form a holomorphic family of the types (A) and (B).*

P r o o f. Note that if U is a linear isomorphism of H and $L = U^*TU$, then operators T and L are sectorial simultaneously and in this case the equalities hold

$$U\mathcal{D}[L] = \mathcal{D}[T], \quad L[u,v] = T[Uu,Uv], \quad u,v \in \mathcal{D}[L],$$
$$U\mathcal{D}(L_F) = \mathcal{D}(T_F), \quad L_F = U^*T_FU.$$

Let $z \in \rho(A)$, $Q(z) = C(A - zI)^{-1}$ and let $U_z = \begin{bmatrix} I & -Q^*(z) \\ 0 & I \end{bmatrix}$. Then U_z is a linear isomorphism of $H = H_1 \oplus H_2$ and $U_z^* = \begin{bmatrix} I & 0 \\ -Q(z) & I \end{bmatrix}$, $U_z^{-1} = \begin{bmatrix} I & Q^*(z) \\ 0 & I \end{bmatrix}$. Consider the operator

$$L_z = U_z^*(T - zI)U_z = \begin{bmatrix} I & 0 \\ -Q(z) & I \end{bmatrix} \begin{bmatrix} A - zI & B \\ C & D - zI \end{bmatrix} \begin{bmatrix} I & -Q^*(z) \\ 0 & I \end{bmatrix}.$$

Clearly,

$$\mathcal{D}(L_z) = \left\{ \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} : \begin{array}{c} e_1 - Q^*(z)e_2 \in \mathcal{D}(A), \\ e_2 \in \mathcal{D}_0 \end{array} \right\},$$

$$L_z \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} (A - zI)(e_1 - Q^*(z)e_2) + Be_2 \\ (D - Q(z)B - zI)e_2 \end{bmatrix}.$$

Let T be a sectorial operator. Then L_z is also sectorial. Therefore, there exists a positive number c such that the operator $L_z + cI$ is sectorial and coercive. Let

$$\begin{aligned} &A_z = A - zI,\ Y_z = \left(A_z^{-1}B - Q^*(z)\right)/2, \\ &Y_c = Y_z - c(A_z + cI)^{-1}Y_z,\ W_z = D - CA_z^{-1}B - zI, \\ &\mu_c[\varphi] = \sup\left\{ \operatorname{Re}(A_z + cI)[2\varphi - f, f],\ f \in \mathcal{D}(A) \right\}. \end{aligned}$$

Then for $u = e_1 + e_2 \in \mathcal{D}(L_z)$ we have

$$\left((L_z + cI)u, u\right) = \left((A_z + cI)(e_1 + 2Y_c e_2), e_1\right) + \left((W_z + cI)e_2, e_2\right).$$

The operator $L_z + cI$ is a sectorial extension in the space $H_1 \oplus H_2$ of the operator $A_z + cI$. Therefore, from Theorems 2.4 and 2.5 it follows that the operator $W_z + cI$ is sectorial with the vertex at the origin and $\mu_c[Y_c e_2] \le \delta^2 \operatorname{Re}\left((W_z + cI)e_2, e_2\right)$ for all $e_2 \in \mathcal{D}(W_z)$ with some $\delta \in [0, 1)$. Moreover, replacing T by $L_z + cI$ and S by $A_z + cI$ in (2.21), we obtain that

$$\mathcal{D}[L_z] = \mathcal{D}[L_z + cI] = \mathcal{D}[A] \oplus \mathcal{D}[W_z].$$

Since

$$P_2\mathcal{D}[T] = \mathcal{D}[D - CA^{-1}B],\ P_2\mathcal{D}(T_F) = \mathcal{D}\left((D - CA^{-1}B)_F\right),$$

by definition of the operator U_z we obtain

$$\mathcal{D}[W_z] = P_2\mathcal{D}[L_z] = P_2\mathcal{D}[T - zI] = P_2\mathcal{D}[T] = \mathcal{D}[W] = const,$$

where

$$W = D - CA^{-1}B = D - QB.$$

Clearly, the operator Y_c is bounded from the Hilbert space $\mathcal{D}[W_z + cI]$ into the Hilbert space $\mathcal{D}[A_z + cI] = \mathcal{D}[A]$ and hence, it has a continuation $\overline{Y_c}$ on $\mathcal{D}[W_z]$ with values in $\mathcal{D}[A]$. Applying Theorems 2.5 and 2.6 and formulas (2.21), (2.17) for the operators $A_z + cI \subset L_z + cI$, we obtain equalities

$$\begin{aligned}
&(L_z + cI)[u,v] = (A_z + cI)[e_1 + \overline{Y_c}e_2, h_1] + (W_z + cI)[e_2, h_2],\\
&u = e_1 + e_2, \quad v = h_1 + h_2 \in \mathcal{D}[L_z],\\
&\mathcal{D}\big((L_z)_F\big) = \left\{ u = e_1 + e_2 : \begin{array}{l} e_1 + \overline{Y_c}e_2 \in \mathcal{D}(A), \\ e_2 \in \mathcal{D}\big((W_z)_F\big) \end{array} \right\},\\
&\big((L_z)_F + cI\big)u = (A_z + cI)(e_1 + \overline{Y_c}e_2) + \big((W_z)_F + cI\big)e_2.
\end{aligned}$$

It follows that

$$\mathcal{D}\big((W_z)_F\big) = P_2\mathcal{D}\big((L_z)_F\big) = P_2\mathcal{D}(T_F) = W_F = const.$$

Let us prove that the function $W_z[h,g]$ is holomorphic on the domain $\rho(A)$ for all $h, g \in \mathcal{D}[W]$. For the operator Y_z we have the equality

$$Y_z = (I + cA_z^{-1})Y_c = \big(I + c(A - zI)^{-1}\big)Y_c.$$

Since for every $f \in H_1$ hold inequalities

$$\begin{aligned}
\big\|A_R^{1/2}(A - zI)^{-1}f\big\|^2 &= \operatorname{Re}\big(A(A - zI)^{-1}f, (A - zI)^{-1}f\big)\\
&\le \big|\big(f, (A - zI)^{-1}f\big)\big| + |z|\big\|(A - zI)^{-1}f\big\|^2\\
&\le k(z)\big\|f\big\|^2 \le b(z)\big\|A_R^{1/2}f\big\|^2,
\end{aligned}$$

the operator Y_z is bounded from the Hilbert space $\mathcal{D}[W]$ into the Hilbert space $\mathcal{D}[A]$. Its domain is dense in $\mathcal{D}[W]$. It follows that Y_z has a continuation $\overline{Y}_z$ on $\mathcal{D}[W]$ and

$$\overline{Y}_z = \big(I + c(A - zI)^{-1}\big)\overline{Y}_c.$$

In addition for $u = e_1 + e_2,\ v = g_1 + g_2 \in \mathcal{D}[L_z] = \mathcal{D}[A] \oplus \mathcal{D}[W]$ we have

$$L_z[u,v] = (A - zI)[e_1 + 2\overline{Y}_z e_2, g_1] + W_z[e_1, g_2].$$

Taking into account the relation $T - zI = U_z^{*-1}L_zU_z^{-1}$, we obtain

$$\begin{aligned}
&\mathcal{D}[T] = \left\{ u = x_1 + x_2 : \begin{array}{l} x_1 + Q^*(z)x_2 \in \mathcal{D}[A], \\ x_2 \in \mathcal{D}[W] \end{array} \right\},\\
&(T - zI)[x,y] = (A - zI)[x_1 + Q^*(z)x_2 + 2\overline{Y}_z x_2, y_1 + Q^*(z)y_2] + W_z[x_2, y_2]
\end{aligned}$$

for $x = x_1 + x_2,\ y = y_1 + y_2 \in \mathcal{D}[T]$. Besides, from (2.21) we have

$$\begin{aligned}
&\mathcal{D}[T] = \left\{ u = x_1 + x_2 : \begin{array}{l} x_1 + Q^*x_2 \in \mathcal{D}[A], \\ x_2 \in \mathcal{D}[W] \end{array} \right\},\\
&T[u,v] = A[x_1 + Q^*x_2 + 2\overline{Y}x_2, y_1 + Q^*(z)y_2] + W[x_2, y_2]
\end{aligned}$$

for $u = x_1 + x_2,\ v = y_1 + y_2 \in \mathcal{D}[T]$, where $Y = \frac{1}{2}\big(A^{-1}B - Q^*\big)$ (see (4.5)).
Let $x = -Q^*x_2 + x_2,\ y = -Q^*(z)y_2 + y_2$, then $-Q^*x_2 + Q^*(z)x_2 = \overline{z}(A^* - \overline{z}I)^{-1}Q^*x_2 \in \mathcal{D}[A]$ and

$$W_z[x_2, y_2] = (T - zI)[x, -Q^*(z)y_2 + y_2]. \tag{4.6}$$

The function $z(x, Q^*(z)y_2 - y_2)$ is holomorphic on z in $\rho(A)$. Since $Q^*(\xi) - Q^*(z) = (\overline{\xi} - \overline{z})(A^* - \overline{\xi}I)^{-1}Q^*(z)$, we obtain $\mathcal{R}\big(Q^*(\xi) - Q^*(z)\big) \subset \mathcal{D}[A]$. In addition

$$A_R^{1/2}\big((A^* - \overline{\xi}I)^{-1} - (A^* - \overline{z}I)^{-1}\big) = (\overline{\xi} - \overline{z})A_R^{1/2}(A^* - \overline{z}I)^{-1}(A^* - \overline{\xi}I)^{-1},$$

and the operator $A_R^{1/2}(A^* - \overline{z}I)^{-1}$ is bounded in H_1. From (4.6) we have

$$\begin{aligned} &W_z[x_2, y_2] - W_\xi[x_2, y_2] \\ &= T[x, (Q^*(\xi) - Q^*(z)y_2] + z(x, Q^*(z)y_2 - y_2) - \xi(x, Q^*(\xi)y_2 - y_2). \end{aligned}$$

Let $T_F = T_{FR}^{1/2}(I + iM_F)T_{FR}^{1/2}$ be the representation of the type (1.1) of the Friedrichs extension T_F, where T_{FR} is the "real" part of T_F. From the relations

$$\begin{aligned} T[\varphi] &= T[\varphi, \varphi] = \big((I + iM_F)T_{FR}^{1/2}\varphi, T_{FR}^{1/2}\varphi\big) \\ &= A[\varphi] = \big((I + iM)A_R^{1/2}\varphi, A_R^{1/2}\varphi\big),\ \varphi \in \mathcal{D}[A] \end{aligned}$$

it follows that $T_{FR}^{1/2}\varphi = \mathfrak{U}A_R^{1/2}\varphi$, where $\mathfrak{U}$ is an isometry from H_1 onto $T_{FR}^{1/2}\mathcal{D}[A]$. Hence,

$$\begin{aligned} T[x, (Q^*(\xi) - Q^*(z))y_2] &= \big((I + iM_F)T_{FR}^{1/2}x, \mathfrak{U}A_R^{1/2}(Q^*(\xi) - Q^*(z))y_2\big) \\ &= (\xi - z)\big((I + iM_F)T_{FR}^{1/2}x, \mathfrak{U}A_R^{1/2}(A^* - \overline{z}I)^{-1}(A^* - \overline{\xi}I)^{-1}y_2\big). \end{aligned}$$

This yields that the function $W_z[x_2, y_2]$ is holomorphic on z in $\rho(A)$.

Let us show that operators $(W_z)_F = \big(D - C(A - zI)^{-1}B - zI\big)_F$ forms in $\rho(A)$ a holomorphic family of the type (A). At first, we note that the equality

$$(W_z)_F = W_F - zI - zC(A - zI)^{-1}(Q^* + 2\overline{Y}) \tag{4.7}$$

holds. In fact by the definitions of operators Q^* and Y for every $g \in \mathcal{D}(W) = \mathcal{D}_0$ we have

$$\begin{aligned} &\big(W - zI - zC(A - zI)^{-1}(Q^* + 2\overline{Y}_0\big)g \\ &= \big(B - CA^{-1}B - zI - zC(A - zI)^{-1}A^{-1}B\big)g \\ &= \big(D - C(A - zI)^{-1} - zI\big)g = W_z g. \end{aligned}$$

It follows that (4.7) holds on $\mathcal{D}(W) = \mathcal{D}_0$. Let $g \in \mathcal{D}(W_F)$. Then there exists a sequence $\{g_n\} \subset \mathcal{D}(W)$ such that

$$\lim_{n\to\infty} g_n = g,\ \lim_{n,m\to\infty}\big(W(g_n - g_m), g_n - g_m\big) = 0.$$

For every $h \in \mathcal{D}[W]$ we obtain

$$\lim_{n\to\infty}\big(Wg_n, h\big) = \big(W_F g, h\big),\ \lim_{n\to\infty}\big(W_z g_n, h\big) = \big((W_z)_F g, h\big).$$

For the sequence $\{g_n\}$ we also have $\lim_{n\to\infty} Yg_n = \overline{Y}g$. Therefore,

$$\begin{aligned}\big((W_z)_F g, h\big) &= \lim_{n\to\infty}\big(W_z g_n, h\big) = \lim_{n\to\infty}\big((W - zI - zQ(z)(Q^* + 2Y))g_n, h\big)\\ &= \big((W_F - zI - zC(A - zI)^{-1}(Q^* + 2\overline{Y}))g, h\big).\end{aligned}$$

Thus, (4.7) is true. Let $g \in \mathcal{D}(W_F)$ and $\xi, z \in \rho(A)$. Then from (4.7) we obtain

$$\begin{aligned}&(W_\xi)_F g - (W_z)_F g = (z-\xi)g + \big(zQ(z) - \xi Q(\xi)\big)(Q^* + 2\overline{Y})g\\ &= (z-\xi)g + (z-\xi)Q(z)(Q^* + 2\overline{Y})g + z(Q(z) - Q(\xi))(Q^* + 2\overline{Y})g\\ &= (z-\xi)g + (z-\xi)Q(z)(Q^* + 2\overline{Y})g + z(z-\xi)Q(z)(A-\xi)^{-1}(Q^* + 2\overline{Y})g.\end{aligned}$$

It follows that the vector function $(W_z)_F g$ is holomorphic on z in $\rho(A)$. ■

5. Bounded sectorial block matrices

Let $A = A_R^{1/2}(I + iM)A_R^{1/2} \in \mathcal{L}(H_1)$ be a bounded sectorial operator with the vertex at the origin, $C \in \mathcal{L}(H_1, H_2)$ and let $S = A + C : H_1 \to H$. In this case we have $A_R = (A + A^*)/2$. In accordance with [4], the operator S has bounded m-sectorial extensions in H with the vertex at the origin if and only if

$$\mathcal{R}(C^*) \subseteq \mathcal{R}(A_R^{1/2}). \tag{5.1}$$

Let (5.1) be satisfied. Then the von Neumann–Kreĭn extension of S is a bounded operator and takes the form [4]:

$$S_N = SP_1 + \big(A_R^{1/2}(I + iM) + X_0^*\big)(I - iM)^{-1}X_0 P_2, \tag{5.2}$$

where $X_0 \in \mathcal{L}\big(H_2, \overline{\mathcal{R}(A)}\big)$ is an operator connected with C by the relation $C^* = A_R^{1/2}X_0$. Relation (5.2) can be rewritten in the form

$$S_N = SP_1 + \big(AA^{*-1}C^* + CA^{*-1}C^*\big)P_2$$

or in the matrix form with respect to the orthogonal decomposition $H = H_1 \oplus H_2$:

$$S_N = \begin{bmatrix} A & AA^{*-1}C^* \\ C & CA^{*-1}C^* \end{bmatrix}, \tag{5.3}$$

where by definition

$$AA^{*-1}C^* = A_R^{1/2}(I + iM)(I - iM)^{-1}X_0,\ CA^{*-1}C^* = X_0^*(I - iM)^{-1}X_0.$$

From (5.2) it follows the equality

$$\left\|S_{NR}^{1/2}f\right\|^2 = \left\|A_R^{1/2}P_1 f + A_R^{1/2}A^{*-1}C^*P_2 f\right\|^2,\ f \in H,$$

where $S_{NR} = \bigl(S_N + S_N^*\bigr)/2$ is the real part of S_N and $A_R^{1/2}A^{*-1}C^* = (I - iM)^{-1}X_0$. Therefore,

$$S_{NR}^{1/2} = U_0\bigl(A_R^{1/2}P_1 + A_R^{1/2}A^{*-1}C^*P_2\bigr),$$

where U_0 is an isometry from $\overline{\mathcal{R}(A)}$ onto $\overline{\mathcal{R}(S_N)}$. Hence, we obtain

$$S_{NR}^{1/2}f = \bigl(A_R^{1/2} + CA^{-1}A_R^{1/2}\bigr)U_0^*f,\ f \in \overline{\mathcal{R}(S_N)}. \tag{5.4}$$

As was shown in [6], the relation

$$T = S_N + \bigl(W + 2S_{NR}^{1/2}X\bigr)P_2 \tag{5.5}$$

establishes a one-to-one correspondence between all m-sectorial extensions of S with the vertex at the origin and all pairs $\langle W, X\rangle$, where W is an m-sectorial operator with the vertex at the origin in the subspace H_2 and $X : \mathcal{D}(W) \to \overline{\mathcal{R}(S_N)}$ is a linear operator satisfying the condition $\left\|Xh\right\|^2 \le \delta^2\,\mathrm{Re}\,\bigl(Wh, h\bigr),\ h \in \mathcal{D}(W)$ with some $\delta \in [0, 1)$. Moreover, T is bounded if and only if W is bounded in H_2.

Using (5.4), we can rewrite the relation (5.5) in the form

$$T = S_N + WP_2 + 2\bigl(A_R^{1/2} + CA^{-1}A_R^{1/2}\bigr)YP_2$$

or in the matrix form

$$T = \begin{bmatrix} A & AA^{*-1}C^* + 2A_R^{1/2}Y \\ C & CA^{*-1}C^* + 2CA^{-1}A_R^{1/2}Y + W \end{bmatrix}, \tag{5.6}$$

where $Y : \mathcal{D}(W) \to \overline{\mathcal{R}(A)}$ satisfies the condition $\left\|Yh\right\|^2 \le \delta^2\,\mathrm{Re}\,\bigl(Wh, h\bigr),\ h \in \mathcal{D}(W)$ with $\delta \in [0, 1)$ and $CA^{-1}A_R^{1/2} = X_0^*(I + iM)^{-1}$. From (5.6) it follows that $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is a bounded sectorial operator with the vertex at the origin if and only if the following conditions are fulfilled:

1. $\mathcal{R}(C^*) \subseteq \mathcal{R}(A_R^{1/2}),\ \mathcal{R}\bigl(B\bigr) \subseteq \mathcal{R}(A_R^{1/2})$;
2. $D - CA^{-1}B$ is a bounded m-sectorial operator in H_2 with the vertex at the origin;
3. $\left\|A_R^{-1/2}(B - AA^{*-1}C^*)h\right\|^2 \le 4\,\delta^2\,\mathrm{Re}\,\bigl((D - CA^{-1}B)h, h\bigr)$ for all $h \in H_2$ and for some $\delta \in [0, 1)$, where $CA^{-1}B := \left(A_R^{-1/2}C^*\right)^*(I + iM)^{-1}A_R^{-1/2}B$.

Let $D_R = (D + D^*)/2$ be the real part of D. The next theorem gives a criterion of sectoriality by means of Schur complements of the block operator matrix T and its real part T_R.

Theorem 5.1. *Let $A \in \mathcal{L}(H_1)$ be a bounded sectorial operator with the vertex at the origin. Then the following conditions are equivalent:*

(A) $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *is a bounded sectorial operator with the vertex at the origin in* $H = H_1 \oplus H_2$;

(B) $\mathcal{R}(C^*), \mathcal{R}(B) \subseteq \mathcal{R}(A_R^{1/2}),\ 4D_R - \left(A_R^{-1/2}(C^* + B)\right)^* \left(A_R^{-1/2}(C^* + B)\right) \geq 0$ *and*

$$\sup\left\{ \frac{\left|\big((D - CA^{-1}B)f, f\big)\right|}{4\big(D_R f, f\big) - \left\|A_R^{-1/2}(C^* + B)f\right\|^2},\ f \in H_2 \right\} < \infty. \tag{5.7}$$

P r o o f. **(A)** $\Rightarrow$ **(B)**. Let T be a bounded sectorial operator with the vertex at the origin. Then $\mathcal{R}(C^*) \subseteq \mathcal{R}(A_R^{1/2})$, $\mathcal{R}\big(B\big) \subseteq \mathcal{R}(A_R^{1/2})$, its real part $T_R = (T + T^*)/2$ is a non-negative bounded selfadjoint operator and has the matrix representation

$$T_R = \begin{bmatrix} A_R & (C^* + B)/2 \\ (C + B^*)/2 & D_R \end{bmatrix}.$$

According to the generalized Silvester criterion [17], it follows that

$$4D_R - (C + B^*)A_R^{-1}(C^* + B) \geq 0.$$

In addition, the operator $D - CA^{-1}B$ is a bounded sectorial operator in H_2 and $\left\|A_R^{-1/2}(B - AA^{*-1}C^*)h\right\|^2 \leq 4\delta^2 \operatorname{Re}\big((D - CA^{-1}B)h, h\big)$ for all $h \in H_2$ and some $\delta \in [0, 1)$.

Let $X = (C^* + B)/2$. Then

$$\begin{aligned} B - AA^{*-1}C^* &= B - (2A_R - A^*)A^{*-1}C^* = 2X - 2A_R A^{*-1}C^*, \\ D - CA^{-1}B &= D - CA^{-1}(2X - C^*) = D + CA^{-1}C^* - 2CA^{-1}X. \end{aligned}$$

Hence, for all $h \in H_2$ we obtain

$$\begin{aligned} &\operatorname{Re}\big((D - CA^{-1}B)h, h\big) \\ &= \big(D_R h, h\big) - \left\|A_R^{-1/2}Xh\right\|^2 + \left\|A_R^{1/2}A^{*-1}C^*h - A_R^{-1/2}Xh\right\|^2. \end{aligned} \tag{5.8}$$

From the inequality

$$\begin{aligned} &\left\|A_R^{-1/2}Xh - A_R^{1/2}A^{*-1}C^*h\right\|^2 \\ &\leq \delta^2\left(\left\|D_R^{1/2}h\right\|^2 - \left\|A_R^{-1/2}Xh\right\|^2 + \left\|A_R^{1/2}A^{*-1}C^*h - A_R^{-1/2}Xh\right\|^2\right) \end{aligned}$$

and in view of $\delta^2 < 1$ we have

$$\left\|A_R^{-1/2}Xh - A_R^{1/2}A^{*-1}C^*h\right\|^2 \le k^2\left(\left\|D_R^{1/2}h\right\|^2 - \left\|A_R^{-1/2}Xh\right\|^2\right),$$

where $k^2 = \delta^2/(1-\delta^2)$. It follows

$$\begin{aligned}&\left\|D_R^{1/2}h\right\|^2 - \left\|A_R^{-1/2}Xh\right\|^2\\ &\le \operatorname{Re}\big((D - CA^{-1}B)h, h\big) \le (1+k^2)\left(\left\|D_R^{1/2}h\right\|^2 - \left\|A_R^{-1/2}Xh\right\|^2\right).\end{aligned}$$

Thus the condition (5.7) is fulfilled.

(B) $\Rightarrow$ **(A)**. From (5.7) it follows that

$$\left|\big((D - CA^{-1}B)h, h\big)\right| \le m\left((D_R h, h) - \left\|A_R^{-1/2}Xh\right\|^2\right)$$

for all $h \in H_2$. Since $\operatorname{Re}\big((D - CA^{-1}B)h, h\big) \le \left|\big((D - CA^{-1}B)h, h\big)\right|$, from (5.8) we obtain $m \ge 1$ and

$$\begin{aligned}\left\|A_R^{-1/2}(B - AA^{*-1}C^*)h\right\|^2 &= 4\left\|A_R^{1/2}A^{*-1}C^*h - A_R^{-1/2}Xh\right\|^2\\ &= 4\operatorname{Re}\big((D - CA^{-1}B)h, h\big) - 4\left((D_R h, h) - \left\|A_R^{-1/2}Xh\right\|^2\right)\\ &\le 4\,\frac{m-1}{m}\operatorname{Re}\big((D - CA^{-1}B)h, h\big).\end{aligned}$$

Consequently, the operator T is sectorial.

Note that from (5.8) and (5.7) we have

$$\left|\big((D-CA^{-1}B)h, h\big)\right| \le m\left((D_R h, h) - \left\|A_R^{-1/2}Xh\right\|^2\right) \le m\operatorname{Re}\big((D-CA^{-1}B)h, h\big).$$

This means that the operator $W = D - CA^{-1}B$ is sectorial with the vertex at the origin in H_2. ■

6. Examples

In this section we consider applications of previous results to the systems of differential operators.

6.1. Let $H_1 = H_2 = L^2[0,1]$ and let $W_2^1[0,1]$, $\overset{0}{W}{}_2^1[0,1]$, $W_2^2[0,1]$ be the standart Sobolev spaces. Put

$$\begin{aligned}&A = -\frac{d^2}{dx^2},\ \mathcal{D}(A) = \overset{0}{W}{}_2^1[0,1] \cap W_2^2[0,1] = \left\{f(x) \in W_2^2[0,1] : f(0) = f(1) = 0\right\},\\ &B = C = \frac{d}{dx},\ \mathcal{D}(B) = \mathcal{D}(C) = W_2^1[0,1],\\ &(Dg)(x) = b(x)g(x),\ g(x) \in L^2[0,1],\end{aligned}$$

where $b(x) \in L^\infty[0,1]$ and $\operatorname{Re} b(x) \geq m|\operatorname{Im} b(x)|$ a.e. with $m > 0$. Thus, for $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ we have

$$T \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} -f''(x) + g'(x) \\ f'(x) + b(x)g(x) \end{bmatrix}, \quad f(0) = f(1) = 0.$$

The operator A is selfadjoint positive definite and

$$A[u,v] = \int_0^1 u'(x)\overline{v'(x)}dx, \ u,v \in \mathcal{D}[A] = \overset{0}{W}{}_2^1[0,1].$$

The Green function of the operator A takes the form

$$G(x,s) = \begin{cases} s(1-x), \ s < x, \\ (1-s)x, \ s > x. \end{cases}$$

It follows that

$$\begin{aligned}
&CA^{-1}f = \int_x^1 f(s)ds - \int_0^1 sf(s)dx, \ f \in L^2[0,1], \\
&Q^*g = (CA^{-1})^*g = \int_0^x g(s)ds - x\int_0^1 g(s)dx, \ g \in L^2[0,1], \\
&A^{-1}Bg = x\int_0^1 g(s)ds - \int_0^x g(s)ds, \ g \in W_2^1[0,1], \\
&Yg = \frac{1}{2}\big(A^{-1}B - (CA^{-1})^*\big)g = x\int_0^1 g(s)dx - \int_0^x g(s)ds, \ g \in W_2^1[0,1], \\
&CA^{-1}Bg = -g(x) + \int_0^1 g(s)ds, \ g \in W_2^1[0,1], \\
&Wg = \big(D - CA^{-1}B\big)g = (b(x)+1)g(x) - \int_0^1 g(s)ds, \ g \in W_2^1[0,1], \\
&\operatorname{Re}(Wg,g) = \int_0^1 (\operatorname{Re} b(s) + 1)|g(s)|^2 ds - \left|\int_0^1 g(s)ds\right|^2.
\end{aligned}$$

Hence, the operator W is a densely defined bounded sectorial operator with the vertex at the origin. From (2.8) we obtain

$$\mu[Yg] = \int_0^1 \left|\int_0^1 g(s)ds - g(x)\right|^2 dx = \int_0^1 |g(s)|^2 ds - \left|\int_0^1 g(s)ds\right|^2.$$

It follows that

$$\mu[Yg] \leq \operatorname{Re}(Wg,g) \quad \text{for all} \quad g(x) \in W_2^1[0,1].$$

Since $\mathcal{R}(Q^*) \subset \overset{0}{W}{}_2^1[0,1] = \mathcal{D}[A]$, in accordance with Proposition 4.2 we obtain that the operator T is accretive and is sectorial if and only if the following condition is fulfilled:

$$\int_0^1 |g(s)|^2 ds - \left|\int_0^1 g(s)ds\right|^2 \le k\mathrm{Re}\int_0^1 b(s)|g(s)|^2 ds$$

for some $k > 0$ and all $g \in L^2[0,1]$. In particular, it will be true if $\mathrm{Re}\, b(x) \ge c > 0$ a.e. Since W is densely defined and bounded, the operator T has a unique m-accretive extension which coincides with its closure $\overline{T}$:

$$\mathcal{D}(\overline{T}) =$$
$$\left\{\begin{bmatrix}\varphi(x) - x\int_0^1 g(s)dx - \int_0^x g(s)ds\\ g(x)\end{bmatrix}, \varphi(x) \in \overset{0}{W}{}_2^1[0,1]\cap W_2^2[0,1], g(x) \in L_2[0,1]\right\},$$
$$\overline{T}\begin{bmatrix}\varphi(x) - x\int_0^1 g(s)dx - \int_0^x g(s)ds\\ g(x)\end{bmatrix} = \begin{bmatrix}-\varphi''(x)\\ \varphi'(x) + (b(x)+1)g(x) - \int_0^1 g(x)dx\end{bmatrix}.$$

6.2. Let again $H_1 = H_2 = L^2[0,1]$ and

$$\begin{aligned}
&A = -\frac{d^2}{dx^2},\\
&\mathcal{D}(A) = \left\{f(x) \in W_2^2[0,1] : f(0) = \beta f'(0),\ f(1) = 0\right\},\ \mathrm{Re}\,\beta > 0,\\
&B = \frac{d}{dx},\ \mathcal{D}(B) = \left\{f(x) \in W_2^1[0,1] : f(0) = 0\right\},\ C = \frac{d}{dx},\ \mathcal{D}(C) = W_2^1[0,1],\\
&D = -\frac{d^2}{dx^2},\ \mathcal{D}(D) = \overset{0}{W}{}_2^2[0,1].
\end{aligned}$$

As was shown in [18], the operator A is coercive and m-sectorial. Clearly, for the associated closed form and its domain we have

$$\begin{aligned}
\mathcal{D}[A] &= \left\{f(x) \in W_2^1[0,1] :\ f(1) = 0\right\},\\
A[f,g] &= \int_0^1 f'(x)\overline{g'(x)}dx + \frac{1}{\beta}f(0)\overline{g(0)}, \quad f,g \in \mathcal{D}[A].
\end{aligned}$$

The Green function of A is

$$G(x,s) = \begin{cases} (\beta+s)(1-x)(1+\beta)^{-1},\ s < x,\\ (\beta+x)(1-s)(1+\beta)^{-1},\ s > x.\end{cases}$$

Therefore, for $g \in \mathcal{D}(B)$ we obtain

$$A^{-1}Bg = \int_0^1 G(x,s)g'(s)ds = \frac{1}{1+\beta}\left(\beta\int_x^1 g(s)ds - \int_0^x g(s)ds + x\int_0^1 g(s)ds\right)$$

and

$$CA^{-1}f = \frac{1}{1+\beta}\left(\int_x^1 f(s)ds - \beta\int_0^x f(s)ds - \int_0^1 sf(s)ds\right),\ f(x)\in L^2[0,1],$$

$$(CA^{-1})^*g = \frac{1}{1+\overline{\beta}}\left(\int_0^x g(s)ds - \overline{\beta}\int_x^1 g(s)ds - x\int_0^1 g(s)ds\right),\ g(s)\in L^2[0,1],$$

$$CA^{-1}Bg = -g(x) + \frac{1}{1+\beta}\int_0^1 g(s)ds,\ g(x)\in W_2^1[0,1],\ \ g(0)=0,$$

$$Yg = \frac{1}{2}\big(A^{-1}B - (CA^{-1})^*\big)g =$$

$$\frac{1}{|1+\beta|^2}\left((|\beta|^2+\beta_R)\int_x^1 g(s)ds - (1+\beta_R)\int_0^x g(s)ds + (1+\beta_R)x\int_0^1 g(s)ds\right),$$
$$g\in\mathcal{D}(B),$$

where $\beta_R = \operatorname{Re}\beta$,

$$Wg = (D - CA^{-1}B)g = -g''(x) + g(x) - \frac{1}{1+\beta}\int_0^1 g(s)ds,$$

$$\big(Wg,g\big) = \int_0^1 |g'(s)|^2ds + \int_0^1 |g(s)|^2ds - \frac{1}{1+\beta}\left|\int_0^1 g(s)ds\right|^2,\ g\in \overset{0}{W}{}_2^2[0,1].$$

It follows that

$$\begin{aligned}
\operatorname{Re}\big(Wg,g\big) &= \int_0^1 |g'(s)|^2ds + \int_0^1 |g(s)|^2ds - \frac{1+\beta_R}{|1+\beta|^2}\left|\int_0^1 g(s)ds\right|^2\\
&= \int_0^1 |g'(s)|^2ds + \frac{1+\beta_R}{|1+\beta|^2}\left(\int_0^1 |g(s)|^2ds - \left|\int_0^1 g(s)ds\right|^2\right)\\
&\quad + \frac{|\beta|^2+\beta_R}{|1+\beta|^2}\int_0^1 |g(s)|^2ds,\\
\operatorname{Im}\big(Wg,g\big) &= \frac{\operatorname{Im}\beta}{|1+\beta|^2}\left|\int_0^1 g(s)ds\right|^2.
\end{aligned}$$

Hence we obtain that the operator W defined on $\overset{0}{W}{}_2^2[0,1]$ is closed coercive and sectorial and the semiangle of W is smaller than $\pi/4$.

Let us calculate the quadratic functional $\mu[\varphi]$ defined by (2.7). For $\varphi,\ f\in\mathcal{D}[A]$ we have

$$A[\varphi,f] = \int_0^1 \varphi'(x)\overline{f'(x)}dx - \varphi(0)\overline{f(0)} + \frac{1+\beta}{\beta}\varphi(0)\overline{f(0)}.$$

Let $c = (1+\beta)\beta^{-1}$. Then $c_R = \operatorname{Re} c = (|\beta|^2+\beta_R)|\beta|^{-2}$. Represent c in the form $c = c_R(1+ib)$, where $\operatorname{Im} b = 0$, and note that

$$|1+ib|^2c_R - 1 = \frac{1+\beta_R}{|\beta|^2+\beta_R}.$$

Further we have

$$\operatorname{Re} A[2\varphi - f, f] = \int_0^1 |\varphi'(x)|^2 dx - |\varphi(0)|^2 - \int_0^1 |\varphi'(x) - f'(x)|^2 dx$$
$$+ |\varphi(0) - f(0)|^2 + |1+ib|^2 c_R |\varphi(0)|^2 - c_R |f(0) - (1+ib)\varphi(0)|^2.$$

It follows that

$$\operatorname{Re} A[2\varphi - f, f] \le \int_0^1 |\varphi'(x)|^2 dx + \frac{1+\beta_R}{|\beta|^2 + \beta_R} |\varphi(0)|^2 \quad \text{for all} \quad f(x) \in \mathcal{D}[A].$$

Choosing $f_0(x) = \varphi(x) - ib\,\varphi(0)(x-1)$, we get for $f(x) = f_0(x)$ the equality

$$\operatorname{Re} A[2\varphi - f_0, f_0] = \int_0^1 |\varphi'(x)|^2 dx + \frac{1+\beta_R}{|\beta|^2 + \beta_R} |\varphi(0)|^2.$$

Hence for all $\varphi \in \mathcal{D}[A]$, we obtain

$$\mu[\varphi] = \sup\Big\{\operatorname{Re} A[2\varphi - f, f],\ f \in \mathcal{D}[A]\Big\} = \int_0^1 |\varphi'(x)|^2 dx + \frac{1+\beta_R}{|\beta|^2 + \beta_R} |\varphi(0)|^2.$$

Using the expression for the operator Y, we obtain for $g(x) \in W_2^1[0,1],\ g(0) = 0$:

$$(Yg)'(x) = -g(x) + \frac{1+\beta_R}{|1+\beta|^2} \int_0^1 g(s)ds,\ (Yg)(0) = \frac{|\beta|^2 + \beta_R}{|1+\beta|^2} \int_0^1 g(s)ds,$$

$$\mu[Yg] = \int_0^1 |g(s)|^2 ds + \frac{(1+\beta_R)^2}{|1+\beta|^4} \left|\int_0^1 g(s)ds\right|^2 - 2\frac{1+\beta_R}{|1+\beta|^2} \left|\int_0^1 g(s)ds\right|^2$$
$$+ \frac{1+\beta_R}{|\beta|^2 + \beta_R} \frac{(|\beta|^2 + \beta_R)^2}{|1+\beta|^4} \left|\int_0^1 g(s)ds\right|^2 = \int_0^1 |g(s)|^2 ds - \frac{1+\beta_R}{|1+\beta|^2} \left|\int_0^1 g(s)ds\right|^2.$$

Now we note that the operator $-\frac{d^2}{dx^2}$ defined on $\overset{0}{W}{}_2^2[0,1]$ is positive definite (with the lower bound π^2). It implies an estimate

$$\int_0^1 |g'(s)|^2 ds \ge m \left(\int_0^1 |g(s)|^2 ds - \frac{1+\beta_R}{|1+\beta|^2} \left|\int_0^1 g(s)ds\right|^2 \right)$$

for some $m > 0$ and all $g \in \overset{0}{W}{}_2^2[0,1]$. This yields

$$\mu[Yg] \le \delta^2 \operatorname{Re}\big(Wg, g\big), \quad g \in \overset{0}{W}{}_2^2[0,1]$$

for some $\delta \in [0,1)$. In accordance with Proposition 4.2 we get that the operator T given by the block matrix $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ on the domain

$$\mathcal{D}(T) = \left\{ \begin{bmatrix} f \\ g \end{bmatrix} : f(x) \in W_2^2[0,1] \cap W_2^1[0,1], f(0) = \beta f'(0), f(1) = 0, g(x) \in \overset{0}{W}{}_2^2[0,1] \right\}$$

is sectorial in $H = L^2[0,1] \oplus L^2[0,1]$ with the vertex at the origin. Its the Friedrichs extension is defined by the pair $\langle W_F, \widehat{Y}_F \rangle$, where W_F is the Friedrichs extension of the operator

$$Wg = (D - CA^{-1}B)g = -g''(x) + g(x) - \frac{1}{1+\beta} \int_0^1 g(s)ds, \ g \in \mathcal{D}(W) = \overset{0}{W}{}_2^2[0,1]$$

and $\widehat{Y}_F$ is the continuation of $Y \restriction \mathcal{D}(W)$ on $\mathcal{D}(W_F)$ preserving the condition $\mu[Yg] \leq \delta^2 \mathrm{Re}\,(Wg, g)$. It is easy to see that

$$W_F g = -g''(x) + g(x) - \frac{1}{1+\beta} \int_0^1 g(s)ds, \ g \in \mathcal{D}(W_F) = \overset{0}{W}{}_2^1[0,1] \cap W_2^2[0,1],$$
$$\widehat{Y}_F g = \frac{1}{|1+\beta|^2}$$
$$\times \left((|\beta|^2 + \beta_R) \int_x^1 g(s)ds - (1+\beta_R) \int_0^x g(s)ds + (1+\beta_R)x \int_0^1 g(s)ds \right).$$

Since the operator W is sectorial coercive, the von Neumann–Kreĭn extension W_N of W is defined as follows [4–9]:

$$\mathcal{D}(W_N) = \mathcal{D}(W) \dot{+} \mathrm{Ker}\, W^* = \overset{0}{W}{}_2^2[0,1] \dot{+} \mathrm{Ker}\, W^*,$$
$$W_N(g + \varphi) = Wg, \ g \in \mathcal{D}(W), \ \varphi \in \mathrm{Ker}\, W^*.$$

Since

$$W^* h = -h'' + h - \frac{1}{1+\overline{\beta}} \int_0^1 h(s)ds, \ h \in \mathcal{D}(W^*) = W_2^2[0,1],$$

we get

$$\mathrm{Ker}\, W^* = \left\{ c_1 \left(e^x + \frac{e-1}{\overline{\beta}} \right) + c_2 \left(e^{-x} + \frac{e-1}{e\overline{\beta}} \right), \ c_1, \ c_2 \in \mathbb{C} \right\}.$$

The corresponding extension $\widehat{Y}_N$ of $Y \restriction \mathcal{D}(W)$ takes the form

$$\mathcal{D}(\widehat{Y}_N) = \mathcal{D}(W_N), \ \widehat{Y}_N(g + \varphi) = Yg, \ g \in \overset{0}{W}{}_2^2[0,1], \ \varphi \in \mathrm{Ker}\, W^*.$$

According to Theorem 2.6 the pair $\langle W_N, \widehat{Y}_N \rangle$ defines the von Neumann–Kreĭn m-sectorial extension of the block operator matrix T. Using the approach suggested in [7–9] all m-accretive and m-sectorial extensions of W can be parametrized.

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