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# Intersection number and eigenvectors of quasilinear Hilbert–Schmidt operators

Ya.M. Dymarskii

*Department of Mathematics, Lugansk State Pedagogical University*
*2 Oboronnaya Str., Lugansk, 91011, Ukraine*

E-mail:dymarsky@lep.lg.ua

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For the spesial class of quasilinear operators a topologocal construction is described with the help of which existence theorems of normalized eigenvectors may be obtained. The construction is based on utilization of an intersection number of two Hilbert submanifolds one of which is generated by the given operator and another is anchanged.

## Introduction

For the first time an existence of the countable set of *normalized* eigenvectors was proved for some *nonvariational* quasilinear boundary eigenvalue problem in L.A. Lusternik's paper [1] by the original method of parametric continuation. The Lusternik theorem and following results refer to the case, when an associated (in a certian sense) linear problem has simple eigenvalues only (Ch. Cosner's paper [2] is devoted to the investigation of this case). Our works [3–5] contain results analogous to [1] for quasilinear boundary eigenvalue problems, for which an appearance of "multiple" eigenvalues is possible. In this paper a topological construction is described with the help of which theorems of Lusternik's type may be obtained for the spesial class of nonvariational infinite-dimensional *abstract* operators. (The finite-dimensional case is described in [6–8].) The construction is based on utilization of an intersection number of two Hilbert submanifolds (the case when one of submanifolds is finite-dimensional is described in the survey [9]). In our situation we defined such intersection number by its "finite-dimensional invariance": a finite-dimensional approximation of one submanifold induces the natural finite-dimensional approximation of the other submanifold without changing of this number. We shall introduce the intersection number in a way analogous to that of M.A. Krasnoselskii [10] used in the degree theory of a completely continuous vector fields.

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## 1. The basic notions and notations

Let $\langle\cdot,\cdot\rangle$ be a scalar product in the real separable Hilbert space $H$; let $u \in H$ and $S = \{u : ||u||_H = 1\}$. Let $L$ be the Hilbert space of self-adjoint Hilbert–Schmidt operators $\mathbf{A} : H \to H$ supplied with the absolute operator norm: $||\mathbf{A}||^2 = \sum_i ||\mathbf{A}e_i||_H^2$, where $\{e_i\}$ is an arbitrary orthonormal basis. It is known [11] that the space $L$ is the completion by this norm of the subspace of self-adjoint operators with finite-dimensional image, characteristic values $\gamma$ of operator $\mathbf{A} \in L$ are real (a characteristic value is the inverse value to an eigenvalue), the set of all characteristic values of operator $\mathbf{A} \in L$ is at the most countable, characteristic values have finite multiplicities only (moreover, the algebraical multiplicity is equal to the geometrical one), zero may not be a characteristic value, the characteristic values of the same sign are arranged in ascending order of their absolute values (with calculation of multiplicities) and it is possible that they have the limit points plus (minus) infinity. By $L_n \subset L$ denote the open connected subset of operators having positive characteristic values with number $n$.

We consider the vector space $C = C(S, L)$ of all completely continuous mappings $A : S \to L$, i.e., $A$ is continuous mapping and its image $Im(A)$ is compact in $L$. The space $C$ is Banach space supplied with the norm $||A||_C = \sup_{u\in S} ||A(u)||$ [10]. By $C_n \subset C$ denote the open connected subset of operators $A$ such that $\overline{Im(A)} \subset L_n$. We note that if $A \in C_n$, then $A \in C_i$ for all $i = 1, \dots, n-1$.

We shall investigate a quasilinear *normalized eigenvectors* (n.e.v.) $u$ and *characteristic values* (c.v.) $\gamma$ problem

$$\gamma A(u)u = u, \quad u \in S, \quad \gamma \in \mathbf{R}. \tag{1}$$

E x a m p l e s. 1). A mapping $A(u) \equiv \mathbf{A}$ is a constant operator. In this case problem (1) is the self-adjoint eigenvector problem.

2). A mapping $A(u)$ is an integral operator with a symmetrical bounded kernel. In this case the mapping $B(u) = A(u)u$ is the special Urison operator [10]:

$$A(u)u = \int_0^1 K(x, u(x); z, u(z))u(z)dz,$$

where a function $u \in L_2(0,1) = H$, a function $K(x,a;z,b)$ is continuous on $(0,1) \times \mathbf{R} \times (0,1) \times \mathbf{R}$, $|K| < const$, and $K(x,a;z,b) = K(z,b;x,a)$.

In Item 5 we shall consider a quasilinear boundary eigenvalue problem, which is reduced to problem (1).

**Lemma 1.** *Let $A \in C$. Then the mapping*

$$B : S \to H, \quad B(u) := A(u)u \tag{2}$$

*is completely continuous.*

P r o o f. Let $\{u_i\} \subset S$ be an arbitrary sequence. By virtue of the complete continuity of the mapping $A$, we can choose from the sequence $\{A(u_i)\}$ a convergent subsequence (we shall retain for the latter the earlier notation): $A(u_i) \to \mathbf{A} \in L$ as $i \to \infty$. Since the operator $\mathbf{A} \in L$, $\mathbf{A}$ is compact. Therefore we can choose from sequence $\mathbf{A}u_i$ a convergent subsequence: $\mathbf{A}u_i \to u \in H$. We have $||B(u_i) - u||_H \leqslant ||A(u_i)u_i - \mathbf{A}u_i||_H + ||\mathbf{A}u_i - u||_H \leqslant ||A(u_i) - \mathbf{A}|| \cdot ||u_i||_H + ||\mathbf{A}u_i - u||_H \to 0$. Thus $B(u_i) \to u$ as $i \to \infty$. ■

Nonlinear eigenvector problems $\gamma B(u) = u$ with a completely continuous mapping $B$ are investigated from different points of view [10]. If an operator $B$ has form (2), it is reasonable to say that $B$ is the quasilinear Hilbert–Schmidt operator. Due to $B$ has a spesial form (2), we shall introduce a number and a multiplicity for c.v. of problem (1) analogous to that for the linear self-adjoint problem. We note that the condition of complete continuity of mapping $A$ (which we used in the proof of Lemma 1) cannot be discarded: there exist continuous mappings $A : S \to L$, for which the mapping $B(u) = A(u)u$ is not completely continuous. For instance, define for any $u \in S$ the mapping $A(u) = \pi_u$ as orthogonal projection onto $u$-axis, that is $\pi_u v = \langle u, v\rangle u$ for $v \in H$. Clearly, that the one-dimensional operator $\pi_u \in L$. But the mapping $B(u) = \pi_u u = \langle u, u\rangle u = u$ is an identical mapping on $S$, which isn't completely continuous as it is well known.

A pair $(\gamma, u)$ is called *a normalized solution* (n.s.) if it satisfies problem (1). If $(\gamma^0, u^0)$ is an n.s., then $\gamma^0$ is an c.v. of the linear problem

$$\gamma \mathbf{A} u = u, \quad u \in S, \tag{3}$$

where

$$\mathbf{A} = A(u^0) \in L. \tag{4}$$

We note that among of normalized eigenvectors of linear problem (3), (4) which correspond to the c.v. $\gamma^0$ the vector $u^0$ can be found. We interest of normalized solutions such that its characteristic values are (for determinancy) positive only.

**Definition 1.** *Let $(\gamma^0, u^0)$ be an n.s. of nonlinear problem (1) and let $\gamma^0 > 0$. The c.v. $\gamma^0$ receives some number and multiplicity as an c.v. of linear problem (3), (4). We assign the same number and multiplicity to the n.s. $(\gamma^0, u^0)$ and its elements. The n.s. $(\gamma^0, u^0)$ and its elements are called simple (m-multiple) if $\gamma^0$ is simple (m-multiple) as an c.v. of linear problem (3), (4).*

R e m a r k. Certainly, Definition 1 extends to an n.s. of linear problem (3). A pair $(\gamma^0, u^0)$ itself dosn't contain an information about its number and multiplicity. It receives these properties as the n.s. of the concrete problem (1).

Definition 1 is introduced into consideration in [12]. The analagous definition is given in [2].

Let us define the basic objects in our investigation: 1) the subset of pairs

$$P = \{p = (\mathbf{A}, u) \in L \times S : \text{there exists } \gamma > 0, \text{ such that } \gamma \mathbf{A} u = u\}; \qquad (5)$$

2) the mapping "graph $A$"

$$GrA : S \to L \times S, \quad GrA(u) = (A(u), u). \qquad (6)$$

We note that the set $P$ does not depend on a mapping $A$. Since an c.v. is finite-multiple, we assign the pair $(n, m)$ = (the number of $\gamma$, the multiplicity of $\gamma$) to each point $p \in P$. By $P(n, m) \subset P$ denote the subset of all points with number $n$ and multiplicity $m$. It is clear that $P = \cup_{n,m} P(n, m)$, where $n, m \geqslant 1$. If $m = 1$, then we say that the point $p \in P$ is *simple*. By $P_n^* = \cup_{m \geqslant 2} P(n, m)$ denote the subset of all multiple points with number $n$. We shall need the complement $R_n = (S \times L) \backslash P_n^*$ and the finite intersections $R_{(\leqslant n)} = \cap_{i=1}^{n} R_i$. By $\pi_1 : L \times S \to L$, where $\pi_1(\mathbf{A}, u) = \mathbf{A}$ denote the natural projection onto the first factor. Next assertian describes the properties of the introduced sets.

**Lemma 2** [13]. *The subset $P \subset L \times S$ is a connected $C^\infty$-submanifold modeled on the space $L$. Each of stratum $P(n, m)$ is $C^\infty$-submanifold of the manifold $P$, $codim P(n, m) = m(m-1)/2$. In particular, strata $P(n, 1)$, which consist of simple points are opened in $P$. The restriction $\pi_1 : P(n, 1) \to L$ of projection $\pi_1$ to the stratum of simple points is a local diffeomorphism.*

Now we formulate the obvious theorem, which explains the role of the manifold $P$ and the mapping $GrA$ in the finding of n.e.v.

**Theorem 1.** *A vector $u \in S$ is a n.e.v. with a positive c.v. of problem (1) iff $(GrA)(u) \in P$. In this case the number and the multiplicity of the n.e.v. are defined by the indices $(n, m)$ of the stratum $P(n, m)$ such that the point $(GrA)(u) \in P(n, m) \subset P$.* ■

Henceforth the following notions shall play a basic role.

**Definition 2.** *We say that a mapping $A \in C_n$ and corresponding problem (1) are $n$-typical, if $Im(GrA) \subset R_n$. We say that a mapping $A \in C_n$ and corresponding problem (1) are $(\leqslant n)$-typical, if $Im(GrA) \subset R_{(\leqslant n)}$.*

By $C_n^{tip}$ ($C_{(\leqslant n)}^{tip}$) we denote the set of all $n$-typical (($\leqslant n$)-typical) mappings.

**Definition 3.** *We say that two n-typical mappings $A_0, A_1 \in C_n^{tip}$ and corresponding problems (1) are n-homotopic, if there exists a continuous mapping (n-homotopy) $G : [0,1] \to C_n^{tip}$ such that $G(0) = A_0$ u $G(1) = A_1$. We say that two $(\leqslant n)$-typical mappings $A_0, A_1 \in C_{(\leqslant n)}^{tip}$ and corresponding problems (1) are $(\leqslant n)$-homotopic, if there exists a continuous mapping ($(\leqslant n)$-homotopy) $G : [0,1] \to C_{(\leqslant n)}^{tip}$ such that $G(0) = A_0$ u $G(1) = A_1$.*

## 2. The finite-dimensional case

Our reasonings are founded on the reduction to a finite-dimensional case, therefore we shall need some facts from the theory of finite-dimensional problems of the form (1) (the details can be found in [7, 8]). Introduce following notations: let $\langle \cdot, \cdot \rangle_k$ be a scalar product in the oriented space $\mathbf{R}^k$, $y = (y^1, \dots, y^k) \in \mathbf{R}^k$, $S^{k-1} = \{y : \langle y, y \rangle_k = 1\}$, $L^{(k)}$ be the oriented space of real self-adjoint operators $\mathbf{A}^{(k)}$ on $\mathbf{R}^k$ ($dimL^{(k)} = (k+1)k/2$), and $C^{(k)} = C(S^{k-1}, L^{(k)})$ be the Banach space of continuous mappings supplied with the usual norm of a uniform convergence. Let $A^{(k)} \in C^{(k)}$. We consider a finite-dimensional quasilinear eigenvector and characteristic value problem that is analogous to problem (1):

$$\gamma A^{(k)}(y)y = y, \quad y \in S^{k-1}, \quad \gamma \in \mathbf{R}. \tag{1,(k)}$$

The positive caracteristic values of any operator $\mathbf{A}^{(k)} \in L^{(k)}$ are arranged in ascending order with calculation of multiplicities. After that Definition 1 is carried over to an n.s. $(\gamma, y)$ of problem $(1,(k))$ unchanged. The set $P^{(k)}$ (later the subsets $P^{(k)}(n,m), P_n^{(k),*} \subset P^{(k)}$, the subsets $R_n^{(k)}, R_{(\leqslant n)}^{(k)} \subset L^{(k)} \times S^{k-1}$ and the projection $\pi_1^{(k)}$) and the mapping $GrA^{(k)}$ are defined analogous to (5) and (6), respectively. For these objects the statements of Lemma 2 and Theorem 1 remain true. Moreover, the finite-dimensional $P^{(k)}$ is orientable. The orientation of $P^{(k)}$ induce the orientation in the open subset $P^{(k)}(n,1)$ $(n = 1, \cdots, k)$, i.e., in the stratums consisting of simple points. On the stratum $P^{(k)}(n,1)$ the local diffeomorphism $\pi_1^{(k)}$ either retains the orientation at each point $p$ (that is the derivative operator $D\pi_1^{(k)}$ takes a right basis for the tangential space $T_pP^{(k)}(n,1)$ to a right basis for the space $L^{(k)}$) or interchanges one at each point. As has been shown in [8] the alternation takes place: if on the stratum $P^{(k)}(n,1)$ an orientation is retained (changed), then on the next stratum $P^{(k)}(n+1,1)$ it is changed (retained). We shall agree to orient the first stratum $P^{(k)}(1,1)$ (at the same time the whole manifold $P^{(k)}$) such that the local diffeomorfism $\pi_1^{(k)}$ retains the chosen orientation of the space $L^{(k)}$ at points $p \in P^{(k)}(1,1)$. Definitions 2 and 3 are carried over to the mapping $A^{(k)}$ and problem $(1,(k))$ unchanged. We note the coincidence of the subset: $C_{(\leqslant(k-1))}^{(k),tip} = C_{(\leqslant k)}^{(k),tip} \subset C^{(k)}$.

**Lemma 3** [8]. *For any $n = 1, \dots, k-1$ the subset $C^{(k),tip}_{(\leqslant n)} \subset C^{(k)}$ of $(\leqslant n)$-typical mappings is open and dence.*

By Lemma 3, it has sense to work with $(\leqslant n)$-typical mappings $A^{(k)}$ only, i.e., to seek a simple n.s. Since $dim P^{(k)}(i,1) = dim P^{(k)}$, in the product $L^{(k)} \times S^{k-1}$ the codimensional $codim P^{(k)}(i,1) = k-1 = dim S^{k-1}$ (Lemma 2). Due to this reason and by virtue of the compactness of sphere $S^{k-1}$, for any natural $i \leqslant n$ the oriented intersection number $\chi_i^{(k)} = \chi_i^{(k)}(A^{(k)}) = \chi(P^{(k)}(i,1), GrA^{(k)})$ of the stratum $P^{(k)}(i,1)$ with the mapping $GrA^{(k)}$ is defined. This intersection number we shall call the intersection number of the $(\leqslant n)$-typical mapping $A^{(k)}$ and at the same time the intersection number of the corresponding problem $(1,(k)))$. By Theorem 1 and properties of the intersection number, if $\chi_i^{(k)} \neq 0$, then problem $(1,(k))$ has at least one simple n.s. with number $i$. If a mapping $GrA^{(k)}$ is in transverse position with respect to the stratum $P^{(k)}(i,1)$ (this case is typical for smooth mappings $A^{(k)}$), then a number of n.s. is at least $|\chi_i^{(k)}|$.

E x a m p l e [8]. $A^{(k)}(y) \equiv \mathbf{A}^{(k)}$. In this case $\chi_i^{(k)} = (-1)^{i-1}2$.

Let a mapping $A^{(k)}$ be $(\leqslant (k-1))$-typical. Here all of intersection numbers $\chi_i^{(k)}$ $(i = 1, \dots, k)$ are defined. Between them and the Euler characteristic there exist a linear dependence:

**Lemma 4** [8]. *Let $A^{(k)} \in C^{(k),tip}_{(\leqslant (k-1))}$. Then the sum of all intersection numbers is equal to the Euler characteristic of the sphere $S^{k-1}$, i.e., the sum $\chi_1^{(k)} + \ldots + \chi_k^{(k)}$ is equal to 2 if $k$ is odd and one is equal to 0 if $k$ is even.*

**Definition 4.** *Let a finite-dimensional mapping $A^{(k)} \in C^{(k),tip}_{(\leqslant n)}$ and $n \leqslant k-1$. We shall call the intersection $n$-vector of the mapping $A^{(k)}$ (and at the same time of the corresponding problem $(1,(k))$) the integervalue $n$-dimensional vector $\overline{\chi}^{(k)}_{(\leqslant n)} = \overline{\chi}^{(k)}_{(\leqslant n)}(A^{(k)}) = (\chi_1^{(k)}, \dots, \chi_n^{(k)})$.*

With the help of introducing intersection numbers the set of typical problems can be homotopically classified.

**Theorem 2.** *1) The intersection $n$-vector $\overline{\chi}^{(k)}_{(\leqslant n)}$ of problem (1,(k)) may take on all values in the group $\mathbf{Z}^n$.*

*2) The intersection $n$-vector of a constant mapping $A^{(k)}(y) \equiv \mathbf{A}^{(k)} \in C^{(k),tip}_{(\leqslant n)}$ is equal to $(2, -2, \dots)$.*

*3) Two of the $(\leqslant n)$-typical problems (1,(k)) are $(\leqslant n)$-homotopic iff their intersection $n$-vectors coincide.*

*4) If an intersection number $\chi_i^{(k)}(A) \neq 0$, then the problem $(1,(k))$ generated by the mapping $A^{(k)}$ has at least one simple n.s. $(\gamma_i, y_i)$ with the number $i$.*

P r o o f. The case $n = k-1$ was considered in [8]. The proof of common case is by the same method. ■

For the passage from the finite-dimensional case to the infinite-dimensional one we need two lemmas.

**Lemma 5.** *The intersection n-vector* $\overline{\chi}^{(k)}_{(\leqslant n)}$ *does not depend on the choice of orientations in the factors of the product* $L^{(k)} \times S^{k-1}$.

P r o o f. Under the substitution of an orientation in a factor of the product $L^{(k)} \times S^{k-1}$ the substitution of an orientation in the manifolds $P^{(k)}(i,1)$ ($i = 1,\dots,n$) and $Im(GrA^{(k)})$ happens automatically. ■

Let $\mathbf{R}^k \subset \mathbf{R}^l$ and let $\{e_1,\dots,e_k,\dots,e_l\}$ be some orthonormal basis for $\mathbf{R}^l$ adapted to the embedding. By $\pi^k$ denote the orthogonal projection onto subspace $\mathbf{R}^k$. Consider a mapping $A^{(l)}: S^{l-1} \to L^{(l)}$, that the identity $A^{(l)}(y) \equiv \pi^k A^{(l)}(y)\pi^k$ takes place on $S^{l-1}$. I.e., the image $Im(A^{(l)}) \subset L^{(k)}$ and in the chosen basis the mapping $A^{(l)}$ has a block diagonal matrix:

$$A^{(l)}(y) = \begin{pmatrix} A^{(k)}(y) & 0 \\ 0 & 0 \end{pmatrix}, \tag{7}$$

where $A^{(k)}(y) = (\langle A^{(l)}(y)e_i, e_j\rangle_l)$ $(i,j = 1,\dots,k)$ is symmetric $k$-dimensional matrix. It is clear that the restriction of the mapping $A^{(l)}$ to $S^{k-1} = S^{l-1} \cap \mathbf{R}^k$ is the mapping $A^{(k)}: S^{k-1} \to L^{(k)}$. We shall call the subspace $\mathbf{R}^k$ and the shpere $S^{k-1}$ invariant for a mapping $A^{(l)}$ of the form (7).

**Lemma 6.** *Let a mapping* $A^{(l)}$ *have the form (7) and let* $A^{(k)}$ *be the restriction of a mapping* $A^{(l)}$ *to the invariant sphere* $S^{k-1}$. *If* $A^{(k)} \in C^{(k),tip}_{(\leqslant n)}$, *then* $A^{(l)} \in C^{(l),tip}_{(\leqslant n)}$ *and* $\overline{\chi}^{(l)}_{(\leqslant n)}(A^{(l)}) = \overline{\chi}^{(k)}_{(\leqslant n)}(A^{(k)})$.

P r o o f. The first assertion is obvious. To prove the second assertion, it suffices to check one for any component of intersection $n$-vector, for example, the first. With the help of the transversality theorem [14] we substitute the mapping $GrA^{(l)}$ for a near smooth mapping $(\widetilde{GrA^{(l)}}) : S^{l-1} \to S^{l-1} \times L^{(k)}$ that is in transverse position with respect to the stratum $P^{(k)}(1,1)$ (the mapping $(\widetilde{GrA^{(l)}})$ is not a graph of some mapping from $S^{l-1}$ into $L^{(k)}$). By $\pi^{(l)}_2 : L^{(l)} \times S^{l-1} \to S^{l-1}$ denote the natural projection onto the second factor. Then, at first, the mapping

$$\widetilde{A^{(l)}} : S^{l-1} \to L^{(k)} \subset L^{(l)}, \quad \widetilde{A^{(l)}} = \pi^{(l)}_1 \cdot (\widetilde{GrA^{(l)}}) \cdot (\pi^{(l)}_2 \cdot (\widetilde{GrA^{(l)}}))^{-1}$$

is near to the starting mapping $A^{(l)}$ (see [8]) and one is of the form (7). By this reason and by virtue of Lemma 3 for the mapping $\widetilde{A^{(l)}}$ all conditions of Lemma 6 are satisfied. Secondly, the graph of the mapping $\widetilde{A^{(l)}}$ coincides with the image under the mapping $(\widetilde{GrA^{(l)}})$ as a set, therefore one is in a transverse position with respect to the stratum $P^{(k)}(1,1)$ in the product $S^{l-1} \times L^{(k)}$. It is not difficult to check that this graph is in transverse position with respect to the stratum $P^{(l)}(1,1)$ in the product $S^{l-1} \times L^{(l)}$. Now, there exists the finite set of points $y_i \in S^{k-1} \subset S^{l-1}$ such that the conditions $p_i^{(l)} = (\widetilde{A^{(l)}}(y_i), y_i) = Gr(\widetilde{A^{(l)}})(y_i) \in P^{(l)}(1,1)$ are true, and on the sphere $S^{l-1}$ other points satisfying these conditions do not exist. Note that for the restriction to the invariant sphere the analogous conditions $p_i^{(k)} = (A_\delta^{(k)}(y_i), y_i) = Gr(A_\delta^{(k)})(y_i) \in P^{(k)}(1,1)$ are true too. We shall construct an orienting tangential basis for $P^{(l)}(1,1)$ at the point $p_i^{(l)}$ and an orienting tangential basis for the graph of the mapping $\widetilde{A^{(l)}}$ at the same point $p_i^{(l)}$. Coordinates of all basis vectors mention above form the $(dimL^{(l)} + l - 1)$-dimensinal matrix $M(p_i^{(l)})$. The sign of $detM(p_i^{(l)})$ defines the intersection number at the point $p_i^{(l)}$. (The matrix $M(p_i^{(k)})$ is defined analogously, it is formed by an orienting tangential basis for $P^{(k)}(1,1)$ at the point $p_i^{(k)}$ and an orienting tangential basis for the graph of the mapping $\widetilde{A^{(k)}}$.) It is known [14], that the sum up these intersection numbers over all the values $i$ gives the intersection number $\chi_1^{(l)}$. In order to construct an orienting tangential basis for $P^{(l)}(1,1)$ at the point $p_i^{(l)}$ make use of the fact that near a *simple* point $p_i^{(l)}$ the manifold $P^{(l)}(1,1)$ is the *graph* of the mapping $y = y(\mathbf{A}^{(l)})$, where $y$ is the normalized eigenvector corresponding to the first of c.v. of operator $\mathbf{A}^{(l)}$. We shall choose the basic vectors of the space $\mathbf{R}^l$ in a special way: $e_1 = y_i$, $e_2, \dots, e_l$ are orthonormal eigenvectors of the fixed operator $\widetilde{A^{(l)}}(y_i)$. Let us now make use of the wellknown formulae from the perturbation theory for the coordinates of eigenvectors $y = y(\mathbf{A}^{(l)})$ in the basis $\{y_i, e_2, \cdots\}$: $\partial y^1/\partial a_{1,1} = 0$; if $1 < j \leqslant l$, then $\partial y^j/\partial a_{1,j} = \gamma_1\gamma_j/(\gamma_j - \gamma_1)$; if $1 < q \leqslant l$, then $\partial y^j/\partial a_{q,j} = 0$. To write an orienting tangential basis for the graph of the mapping $\widetilde{A^{(l)}}$, it remains to note that by the form of the operator $\mathbf{A}^{(l)} = \widetilde{A^{(l)}}(y)$ (see (7)) there take place the equalities: $\partial a_{q,r}/\partial y^j = 0$, if $q > k$. When the matrix $M(p_i^{(l)})$ is constructed it is easy to make sure that $detM(p_i^{(l)}) = detM(p_i^{(k)})$. ■

## 3. Typical mappings and finite-dimensional apparoximation

By $\Gamma_n(A)$ denote the set of all c.v. of problem (1) that have number $n$, and by $U_n(A)$ denote the set of all n.e.v. of problem (1) that have number $n$. We shall prove the theorem about *a priori* estimate.

**Theorem 3.** *Let a subset $T \subset C_n$ be a compact subspace. Then: 1) the following lower and upper estimates take place*

$$0 < \inf_{A \in T} \Gamma_n(A), \ \sup_{A \in T} \Gamma_n(A) < \infty; \tag{8}$$

*2) the subset $\cup_{A \in T} U_n(A) \subset S$ is a compact subspace.*

P r o o f. The assertion is a direct consequence of the continuous dependence of characteristic values under perturbations of operators $\mathbf{A} \in L$ and the compactness of images of these operators. ■

Let $\mathbf{R}^k \subset H$ be an arbitrary subspace. As previously denote by $\pi^k$ the orthogonal projection onto this subspace. Let $\{e_i\}_{i=1}^{\infty}$ be some orthonormal basis for $H$, which accorded with the chosen subspace, i.e., $\{e_1, \dots, e_k\}$ is a basis for $\mathbf{R}^k$. A mapping $A$ is called *image-k-dimensional (image-finite-dimensional)* and is denoted by $A^{[k]}$ if the following condition is true: $A(u) \equiv \pi^k A(u) \pi^k$, i.e.,

$$A^{[k]}(u) = \begin{pmatrix} A^{(k)}(u) & 0 \\ 0 & 0 \end{pmatrix}, \tag{9}$$

where $A^{(k)}(u) = (\langle A(u)e_i, e_j \rangle)$ $(i, j = 1, \dots, k)$ is a symmetric $k$-dimensional matrix (compare with (7)). Similar to the case finite, we shall call the subspace $\mathbf{R}^k$ and the shpere $S^{k-1}$ are invariant for a mapping $A^{[k]}$. If an orthonormal basis is chosen, then by $A^{\{k\}}(u) := \pi^k A(u) \pi^k$ $(k = 0, 1, \dots)$ we denote *the image-k-dimensional approximation* of an arbitrary mapping $A$.

**Lemma 7.** *In the spase $C$ image-finite-dimensional mappings of the form (9) are dense. For any choice of an orthnormal basis $\{e_i\}_{i=1}^{\infty}$ the convergence of $k$-approximation takes place: $A^{\{k\}} \to A$ in $C$ as $k \to \infty$.*

P r o o f. It suffices to prove the second assertion. Let $\mathbf{A} \in L$. By the definition of absolute operator norm, $||\mathbf{A} - \pi^k \mathbf{A} \pi^k|| \to 0$ as $k \to \infty$. Now, the assertion of Lemma 7 is the consequence of the compactness of the image $Im(A(S)) \subset L$. ■

**Theorem 4.** *The subset $C^{tip}_{(\leqslant n)} \subset C_n$ is open and dence.*

P r o o f. By Definition 2, it suffices to prove the first assertion for the subset $C^{tip}_n \subset C_n$. Let the mapping $A_0$ be $n$-typical. Suppose the contrary. Then in $C_n$ there exists a convergent sequence $A_1, A_2, \dots \to A_0$ of mappings that are not $n$-typical. Hence there exists the sequence of pairs $\{(A_i(u_i), u_i)\} \in P^*_n$

$(i = 1, 2, \dots)$. By Theorem 3, from the sequence $\{u_i\}$ $(i = 1, 2, \dots)$ it may be chosen a convergent subsequence $u_i \to u_0$ (for which we shall retain the early notation). We obtain that $(A_0(u_0), u_0) \in P_n^*$. This contradicts the condition that the mapping $A_0$ is $n$-typical.

Let $A \in C_n$. By Lemma 7, for any $\varepsilon > 0$ there exists such $k_0$ that under the condition $k \geqslant k_0$ the inequality $||A - A^{\{k\}}||_C < \varepsilon$ is true. Since the set $C_n$ is open, we shall choose such small $\varepsilon$ that $k$-approximation $A^{\{k\}} \in C_n$. By $A^{(k)} : S^{k-1} \to L^{(k)}$ denote the restriction of the mapping $A^{\{k\}}$ to the invariant sphere. By Lemma 3, in the space $C^{(k)}$ of finite-dimensional mappings there exists such $\varepsilon$-perturbution $A_\varepsilon^{(k)} : S^{k-1} \to L^{(k)}$ that the mapping $A^{(k)} + A_\varepsilon^{(k)}$ is $(\leqslant n)$-typical. Now, we shall define $\varepsilon$-perturbation of the mapping $A^{\{k\}}$: $A_\varepsilon^{\{k\}}(u) \equiv 0$, if $||\pi^k u||_H < 1 - \varepsilon$; $A_\varepsilon^{\{k\}}(u) = (1/\varepsilon)(||\pi^k u||_H - 1 + \varepsilon) A_\varepsilon^{(k)}(\pi^k u/||\pi^k u||)$, if $1 - \varepsilon \leqslant ||\pi^k u|| \leqslant 1$ (it is obvious that $||A_\varepsilon^{\{k\}}||_C < \varepsilon$ ). Since the mapping $A^{\{k\}} + A_\varepsilon^{\{k\}}$ is image-$k$-dimensional having the form (9) and coincides with $A^{(k)} + A_\varepsilon^{(k)}$ on $S^{k-1}$, it is $(\leqslant n)$-typical. ■

**Corollary 1.** *Let $G$ be $(\leqslant n)$-homotopy. Then there exists such $\varepsilon > 0$ that $\varepsilon$-neihgbourhood of the image $G([0,1])$ belongs to $C_{(\leqslant n)}^{tip}$.* ■

Since $(\leqslant n)$-homotopy relation is equivalence one, it partitions the set $C_{(\leqslant n)}^{tip}$ into equivalence classes. We investigate the correlation between these classes and image-finite-dimensional approximations. First, we note that if $\mathbf{R}^l \subset \mathbf{R}^k$, then automatically any image-l-dimensional mapping $A^{[l]}$ of the form (9) is an image-k-dimensional mapping of the form (9) for any $k > l$. Therefore we can assume without loss of generality that two image-finite-dimensional mappings take into the same space $L^{(k)}$. Let two image-finite-dimensional mappings $A_i^{[l]}$ $(i = 0, 1)$ are joined by $(\leqslant n)$-homotopy $G(t)$, and there exists such subspace $\mathbf{R}^k \supset \mathbf{R}^l$ that for any $t \in [0, 1]$ the operator $G(t) = G^{[k]}(t)$ is an image-k-dimensional mapping. In this case we say that the operators $A_i^{[k]} \equiv A_i^{[l]}$ are *image-finite-dimensional (image-$k$-dimensional) $(\leqslant n)$-homotopic.*

**Lemma 8.** *Every homotopic class of $(\leqslant n)$-typical mappings contains image-finite-dimensional mappings $A^{[k]}$. If two image-finite-dimensional $(\leqslant n)$-typical mappings $A_i^{[k]}$ $(i = 0, 1)$ are $(\leqslant n)$-homotopic, then they are image-finite-dimensional $(\leqslant n)$-homotopic.*

P r o o f. The first assertion follows from Lemma 7 and Theorem 4.

Let the mappings $A_i^{[k]}$ $(i = 0, 1)$ be connected by $(\leqslant n)$-homotopy $G$. Consider image-finite-dimensional approximations of this homotopy (see (9)):

$G^{\{m\}}(t) = \pi^m G(t)\pi^m$. Once $m \geqslant k$ the equalities $G^{\{m\}}(0) = A_0^{[k]}$, $G^{\{m\}}(1) = A_1^{[k]}$ become true. Hence $G^{\{m\}}(t)$ is the image-finite-dimensional homotopy joining the mappings $A_i^{[k]}$ $(i = 0, 1)$. By compactness of the image $G([0, 1])$, it follows from Lemma 7 and Corollary 1 that for any sufficiently large $m$ the homotopy $G^{\{m\}}$ is $(\leqslant n)$-homotopy. ∎

**Lemma 9.** *Two image-k-dimensional mappings $A_i^{[k]}$ $(i = 0, 1)$ are image-k-dimensional $(\leqslant n)$-homotopic iff their restrictions $A_i^{(k)}$ to the invariant sphere $S^{k-1}$ are $(\leqslant n)$-homotopic.*

P r o o f. Let image-$k$-dimensional $(\leqslant n)$-homotopy $G^{[k]}$ join the mappings $A_i^{[k]}$. Then the restriction $G^{(k)}$ of this homotopy to the invariant sphere is the homotopy of the restrictions $A_i^{(k)}$ to be found.

Conversely. Let the finite-dimensional $(\leqslant n)$-homotopy $G^{(k)}$ join the restrictions $A_i^{(k)}$ of two image-$k$-dimensional mappings $A_i^{[k]}$ $(i = 0, 1)$. Let us extend this finite-dimensional homotopy to an image-$k$-dimensional homotopy joining the mappings $A_i^{[k]}$. Consider two image-$k$-dimensional mappings

$$D_i^{[k]}(u) = \begin{cases} ||\pi^k u|| A_i^{[k]}(\pi^k u/||\pi^k u||), & \pi^k u \neq 0; \\ 0 \in L, & \pi^k u = 0. \end{cases}$$

Obviously, the rectrictions of these mappings to the invariant sphere coincide with the restrictions of the given mappings: $D_i^{(k)} = A_i^{(k)}$. Let us join given mappings with the introduced ones by the linear homotopies: $G_i^{[k]}(u, t) = (1-t)A_i^{(k)} + tD_i^{(k)}$, $(t \in [0, 1])$. Since these linear homotopies are image-$k$-dimensional, they are $(\leqslant n)$-homotopies and do not change the restrictions $A_i^{(k)}$. Now, we join the introduced mappings $D_i^{(k)}$ by $(\leqslant n)$-homotopy, using given finite-dimensional $(\leqslant n)$-homotopy $G^{(k)}$:

$$G^{[k]}(u, t) = \begin{cases} ||\pi^k u|| G^{(k)}(\pi^k u/||\pi^k u||, t), & \pi^k u \neq 0; \\ 0 \in L, & \pi^k u = 0. \end{cases}$$

∎

It follows from Lemmas 8 and 9 that the investigation of classes of $(\leqslant n)$-homotopic completely continuous mappings is reduced in the end to the investigation of similar classes of finite-dimensional mappings.

## 4. An intersection number and its properties

At first, consider an image-finite-dimensional ($\leqslant n$)-typical mapping $A$ of the form (9). This yields that there exists some (not unique!) finite-dimensional subspace $\mathbf{R}^k \subset H$ for which the image $ImA \subset L^{(k)}$. As above denote by $A^{(k)}$ the restriction of the mappig $A = A^{[k]}$ to the invariant sphere $S^{k-1}$.

**Definition 5.** *We shall call the intersection n-vector* $\overline{\chi}_{(\leqslant n)}(A)$ *of an image-finite-dimensional mapping A (and at the same time of the corresponding problem (1)) the intersection n-vector* $\overline{\chi}^{(k)}_{(\leqslant n)}(A^{(k)})$ *of the mapping* $A^{(k)}$.

By Lemma 5, introducing vector does not depend on the choice of orientations in the factors of the product $L^{(k)} \times S^{k-1}$. Make sure that it does not depend on the choice of an invariant subspace $\mathbf{R}^k$ either. Let $\mathbf{R}^r$ be an other subspace such that $ImA \subset L^{(r)}$. We take any subspace $\mathbf{R}^l$, such that $\mathbf{R}^k, \mathbf{R}^r \subset \mathbf{R}^l$. From Lemma 6 follow next equalities: $\overline{\chi}^{(k)}_{(\leqslant n)}(A^{(k)}) = \overline{\chi}^{(l)}_{(\leqslant n)}(A^{(l)}) = \overline{\chi}^{(r)}_{(\leqslant n)}(A^{(r)})$.

Now we give the main definition. Let a mapping $A$ be ($\leqslant n$)-typical. By Lemma 8, in the homotopic class of ($\leqslant n$)-typical mappings that contains the mapping $A$ there exsist image-finite-dimensional mappings $A^{[k]}$.

**Definition 6.** *We shall call the intersection n-vector* $\overline{\chi}_{(\leqslant n)}(A)$ *of a mapping A (and at the same time of the corresponding problem (1)) the intersection n-vector* $\overline{\chi}^{(k)}_{(\leqslant n)}(A^{[k]})$ *of any image-finite-dimensional mapping* $A^{[k]}$ *that is homotopic to the given mapping A.*

We shall prove that the definition is correct if we show that the introducing intersection vector does not depend on the choice of an image-finite-dimensional mapping $A^{[k]}$. Let two image-finite-dimensional ($\leqslant n$)-typical mappings $A^{[k]}_i$ ($i = 0, 1$) be ($\leqslant n$)-homotopic to the given mapping $A$. Then the mappings $A^{[k]}_i$ are ($\leqslant n$)-homotopic to each other. By Lemma 8, they are image-finite-dimensional ($\leqslant n$)-homotopic; let $G^{[k]}(t)$ be an image-finite-dimensional homotopy joining the mappings $A^{[k]}_i$. By Definition 5, there takes place the equality: $\overline{\chi}^{(k)}_{(\leqslant n)}(A^{[k]}_i) = \overline{\chi}^{(k)}_{(\leqslant n)}(A^{(k)}_i)$. Still the restriction of image-finite-dimensional ($\leqslant n$)-homotopy $G^{[k]}(t)$ to the invariant sphere $S^{k-1}$ is a finite-dimensional homotopy of the finite-dimensional mappings $A^{(k)}_i$. Consequently $\overline{\chi}^{(k)}_{(\leqslant n)}(A^{(k)}_0) = \overline{\chi}^{(k)}_{(\leqslant n)}(A^{(k)}_1)$. The definition is correct. It follows from Lemma 7 and Theorem 4 that for any orthonormal basis there exists such natural $k_0$ that the $k$-approximation $A^{[k]} = A^{\{k\}}$, where $k \geqslant k_0$ can be used as an image-finite-dimensional mapping ($\leqslant n$)-homotoping to the given mapping $A$.

Let us describe the properties of the introduced intersection vector and its coordinates $\chi_i(A)$ (compare with Theorem 2).

**Theorem 5.** *For any natural $n$ the next assertions are true.*

*1) The intersection $n$-vector $\overline{\chi}_{(\leqslant n)}$ of problem (1) may take on all values in the group $\mathbf{Z}^n$.*

*2) The intersection $n$-vector of a constant mapping $A(y) \equiv \mathbf{A} \in C^{tip}_{(\leqslant n)}$ is equal to $(2, -2, \dots)$.*

*3) Two of the $(\leqslant n)$-typical problems (1) are $(\leqslant n)$-homotopic iff their intersection $n$-vectors coincide.*

*4) If an intersection number $\chi_i(A) \neq 0$, then problem (1) generated by the mapping $A$, has at least one simple n.s. $(\gamma_i, y_i)$ with a number $i$.*

P r o o f. 1) By Item 1 of Theorem 2, there exists a finite-dimensional mapping $A^{(k)}$ that has the intersection vector chosen in advance. Let us extend this mapping to an image-finite-dimensional mapping $A^{[k]}$ having the same intersection vector: $A^{[k]}(u) = ||\pi^k u|| A^{(k)}(\pi^k u/||\pi^k u||)$, if $\pi^k u \neq 0$; $A^{[k]}(u) = 0 \in L$, if $\pi^k u = 0$. Note that in neighborhood of the constructing image-finite-dimensional mapping $A^{[k]}$ there exist mappings of common form having the same intersection vector.

2) Let us take as a basis the set of orthonormal eigenvectors of operator $\mathbf{A}$. Then the second assertion follows from the Item 2 of Theorem 2, Lemma 7, and Definition 6.

3) Let two $(\leqslant n)$-typical mappings $A_i$ $(i = 0, 1)$ be $(\leqslant n)$-homotopic. Let $A_i^{[k]}$ be image-finite-dimensional mappings that are $(\leqslant n)$-homotopic to the given mappings. Then the mappings $A_i^{[k]}$ are $(\leqslant n)$-homotopic to each other and, by Lemma 8, they are image-finite-dimensional $(\leqslant n)$-homotopic. Reasoning as in the proof of correctitude of Definition 6, we obtain that the finite-dimensional mappings $A_i^{(k)}$ which are the restrictions of the mappings $A_i^{[k]}$ to the invariant sphere $S^{k-1}$ are finite-dimensional $(\leqslant n)$-homotopic to each other. It follows from Item 3 of Theorem 2 that the intersection $n$-vectors of the mappings $A_i^{(k)}$ coincide.

Conversely. Let the intersection $n$-vectors of the $(\leqslant n)$-typical mappings $A_i$ $(i = 0, 1)$ coincide. Let $A_i^{[k]}$ be image-finite-dimensional mappings that are $(\leqslant n)$-homotopic to the given mappings and $A_i^{(k)}$ are restrictions of the mappings $A_i^{[k]}$ to the invariant sphere $S^{k-1}$. Then, by the Definitions 5 and 6, intersection $n$-vectors of the finite-dimensional mappings $A_i^{(k)}$ coincide too. It is follows from Item 3 of Theorem 2 that there exists a finite-dimensional $(\leqslant n)$-homotopy $G^{(k)}$ joining the mappings $A_i^{(k)}$. It follows from Lemma 9 that the mappings $A_i^{[k]}$ are $(\leqslant n)$-homotopic too.

4) Consider image-$k$-dimensional approximations $A^{\{k\}}$ $(k = 1, 2, \dots)$ of a mapping $A$ and their restrictions $A^{(k)}$ to invariant spheres $S^{k-1}$. From the Definitions 5 and 6, and the Item 4 of Theorem 2 it is follows there exists such $k_0$ that under the condition $k \geqslant k_0$ problem (1,(k)) generated by the mapping $A^{(k)}$ has

at least one simple n.s. $(\gamma_i^{(k)}, y_i^{(k)})$ with a number $i$. But if we consider the n.e.v. $y_i^{(k)} \in \mathbf{R}^k \subset H$ as an element of the space $H$, then the same n.e.v. is of the n.e.v. of the problem $\gamma A^{\{k\}}(u)u = u$. By Lemma 7 and Theorem 3, the sequence $(\gamma_i^{(k)}, y_i^{(k)})$ has at least one limit point $(\gamma_i, u_i)$, which is a simple n.s. with a number $i$ of problem (1). ■

Note that there exist the distinctions between finite-dimensional problem $(1, (k))$ and infinite-dimensional problem (1). For the first problem, the intersection $n$-vector is defined under the condition $n \leqslant k-1$ (see Definition 4), and between of all intersection numbers including $\chi_k^{(k)}$ there exists a dependence (see Lemma 4). For an infinite problem, the intersection vector can have any dimension and its coordinates are independent.

## 5. A quasilinear boundary eigenvalue problem

Consider the following quasilinear boundary problem: find an eigenfunction $u \in C^2[0, 2\pi]$ and an eigenvalue $\lambda \in \mathbf{R}$ of

$$-u''(x) + p(u(x), u'(x), x)u(x) = \lambda u(x), \quad V(u) = 0, \quad \int_0^{2\pi} u^2 dx = 1, \tag{10}$$

where $V(u) = 0 \in \mathbf{R}^2$ are regular boundary conditions (for example, Dirichlet, Neuman, periodic, antiperiodic). Let a function $p(a, b, x)$ be continuous on the domain $\mathbf{R}^2 \times [0, 2\pi]$ and following lower and upper estimates take place:

$$0 < \sigma \leqslant p(a, b, x) \leqslant N(|a|^{4-\varepsilon} + |b|^{4/3-\varepsilon} + 1), \tag{11}$$

where $\sigma, N > 0$ and a small $\varepsilon > 0$ are some constants. We shall show that problem (10) may be investigated by *a sequence* of problems of the form (1) in the space $H = L_2(0, 2\pi)$.

At first, we shall formulate the lemma about *a priori* estimates. Introduce following notations: $\Phi(\sigma, N, \varepsilon)$ is the set of continuous functions that satisfy estimates (11); $\{(\lambda, u)\}_p^n$ is the set of normalised solutions of problem (10), which have the number $n$; $\{(\lambda, u)\}_\Phi^n = \cup_{i \leqslant n} \cup_{p \in \Phi} \{(\lambda, u)\}_p^i$, where $\Phi = \Phi(\sigma, N, \varepsilon)$.

**Lemma 10** [4]. *For any fixed $n$ the set $\{(\lambda, u)\}_\Phi^n$ is bounded in the space $\mathbf{R} \times C^2[0, 2\pi]$ by some constant $T = T(n, \Phi)$.*

**Corollary 2.** *Let us give $T = T(n, \Phi)$. Let $p_T = p_T(a, b, x)$ be an arbitrary continuous function that coincides with the function $p$ on the set $[-2T, 2T] \times [-2T, 2T] \times [0, 2\pi]$ and it is equal to zero outside the set $[-3T, 3T] \times [-3T, 3T] \times [0, 2\pi]$. Then $\{(\lambda, u)\}_{p_T}^n = \{(\lambda, u)\}_p^n$.* ■

Denote by $Pr_k$ the operator of the orthogonal projection

$$Pr_k(u) = \sum_{i=0}^{k} \left( \int_0^{2\pi} u u_i dx \right) u_i,$$

which generated by the orthogonal system of fucntions:

$$u_0 = 1/\sqrt{2\pi},\ u_1 = 1/\sqrt{\pi}\sin x,\ u_2 = 1/\sqrt{\pi}\cos x,\ u_3 = 1/\sqrt{\pi}\sin 2x,\ \dots\ .$$

At the same time with problem (10) let us consider the sequence of auxiliary boundary problems:

$$-u''(x) + p_T(Pr_k(u), (Pr_k(u))', x)u(x) = \lambda u(x), \quad V(u) = 0, \quad \int_0^{2\pi} u^2 dx = 1, \tag{10, (k)}$$

where $k = 0, 1, \dots$ . Reduce each of problems $(10, (k))$ to the problem of the form (1).

As usually, denote by $W_2^i(0, 2\pi)$ the Hilbert separable space of functions with distributional derivatives up through order $i$, which are 2-integrable (i.e., the Sobolev space). We recall that $W_2^0(0, 2\pi) = L_2(0, 2\pi) = H$ is the space of functions, which are 2-integrable. Consider the next mappings.

1) Denote by $C_+^0[0, 2\pi] \subset C^0[0, 2\pi]$ the open set of positive continuous functions. Consider the completely continuous mapping

$$\widehat{p}_{T,k} : H \to C_+^0(0, 2\pi), \quad \widehat{p}_{T,k}(u) = p_T(Pr_k(u), (Pr_k(u))', x),$$

which generated by the $k$-dimensional projection operator and the function $p_T$.

2) Denote by $L(W_2^2(0, 2\pi), H)$ the Banah space of continuous linear operator from $W_2^2(0, 2\pi)$ to $H$ and by $L_{is}(W_2^2(0, 2\pi), H) \subset L(W_2^2(0, 2\pi), H)$ the open subset of linear isomorphisms. Consider the smooth mapping

$$D : C_+^0(0, 2\pi) \to L_{is}(W_2^2(0, 2\pi), H), \quad D(q) = -d^2/dx^2 + q,$$

which takes a function $q$ to the continuous positive linear differential operator. (The positivity means that for any function $u \in H$ the inequality $\int_0^{2\pi} [D(q)u(x)]u(x)dx > 0$ is true.)

3) Consider the smooth mapping

$$inv : L_{is}(W_2^2(0, 2\pi), H) \to L_{is}(H, W_2^2(0, 2\pi)), \quad inv(F) = F^{-1},$$

which takes a continuous linear isomorphism $F$ to the inverse isomorphism.

4) Denote by $j : W_2^2(0, 2\pi) \to H$ the imbedding operator. The operator $j$ is Hilbert–Schmidt one [11]. Consider the continuous linear operator $h$ that takes each linear isomorphism $\mathbf{C} \in L_{is}(H, W_2^2(0, 2\pi))$ to the linear Hilbert–Schmidt operator by $h(\mathbf{C}) = j\mathbf{C}$.

Now, consider the mapping product $A_k = h \cdot inv \cdot D \cdot \widehat{p}_{T,k} : H \to L$. By the complete continuity of $\widehat{p}_{T,k}$, the mapping $A_k$ is completely continuous. By the positivity of $D(q)$, for any $u \in H$ the operator $A_k(u)$ is positive; in particularly, one is self-adjoint. Thus, $A_k \in C$, and furthermore, $A_k : H \to L_n$ for any natural $n$. From the given in this item definitions of the space $H$ and the mapping $A_k$ it follows that problem (10,(k)) is identified to problem (1), where $A = A_k$ and $\gamma = 1/\lambda$. Moreover, by the definition of the mapping $inv$ and by Lemma 10, each of n.e.v. $u \in H$ of problem (1) with $A = A_k$ is the classic eigenfunction of problem (1,(k)), i.e., $u \in C^2[0, 2\pi]$.

Now, we must prove that the mappings $A_k$ are typical and calculate their intersection vectors. In general, this problem is very difficult. We shall prove a simple theorem. Denote by $O = \{u \in H : ||u||_H \leqslant 1\}$ the unit ball. Denote by $L^{tip}_{(\leqslant n)} \subset L$ the open subset of Hilbert–Schmidt self-adjoint operators that have positive and *simple* eigenvalues with numbers $i = 1, \dots , n$.

**Theorem 6.** *Let the mapping $\widehat{A} : O \to L^{tip}_{(\leqslant n)}$ be completely continuous. Then the rectriction $A$ of the mapping $\widehat{A}$ to the sphere $S$ is $(\leqslant n)$-typical and its intersection $n$-vector is equal to $(2, -2, \dots)$.*

P r o o f. The first assertion is obvious. To prove the second assertion, let us join the given mapping $A$ with the constant mapping $\mathbf{A} = \widehat{A}(0)$ by following $(\leqslant n)$-homotopy: $G(u, t) = \widehat{A}((1 - t)u)$. Now, the assertion of Theorem 6 is a consequence of Item 2 of Theorem 5. ■

**Corollary 3.** *For any $k = 0, 1, \dots$ problem (10,(k)) with a separating boundary condition has at least one simple n.s. with any number $i$.*

P r o o f. It is known [15] that the linear boundary problem

$$-u''(x) + q(x)u(x) = \lambda u(x), \quad V(u) = 0$$

with a separating boundary condition has simple eigenvalues only. Therefore for the mapping $A_k$ under any $n$ all conditions of Theorem 6 are satisfied. By Item 4 of Theorem 5, problem (10,(k)) has at least one simple n.s. with any number $i$. ■

Return to investigation of problem (10).

**Theorem 7.** *Problem (10) with a separating boundary condition has at least one simple n.s. with any number i.*

P r o o f. Denote by $\{(\lambda,u)\}^n_{T,k}$ the set of normalised solutions of problem (10,(k)), which have the number $n$. By the compactness of the support of the function $p_T$, the set $\cup_{k=1}^{\infty}\{(\lambda,u)\}^n_{T,k}$ is compact in $\mathbf{R}\times C^2[0,2\pi]$. Therefore, there exists a sequence $(\lambda_k,u_k)\subset\{(\lambda,u)\}^n_{T,k}$ $(k=1,2,\dots)$, which has a limit point $(\lambda^0,u^0)\in\mathbf{R}\times C^2[0,2\pi]$. It is not difficult to show that the pair $(\lambda^0,u^0)$ is the simple solution with the number $n$ of problem (10), where $p=p_T$. From Corollary 2 it follows that the pair $(\lambda^0,u^0)$ is at the same time the solution of input problem (10). ■

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