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On the growth of meromorphic functions

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We obtained the estimates for upper and lower logarithmic density of the set $A(\gamma) = \left\{ r : \sum_{k=1}^{q} \mathcal{L}(r, a_k, f) < 2B(\gamma, \Delta(0, f'))T(r, f) \right\}$, where $B(\gamma, \Delta)$ is Shea's constant, $\Delta(0, f')$ is Valiron's deficiency of the derivative of the function f at zero.

We formulate the fundamental result on deficient values, obtained by Nevanlinna.

Theorem A [1]. *Let $f(z)$ be a meromorphic function in $\mathbb{C}$. Then*

$$1)\ \delta(a, f) \le 1 \text{ for each } a \in \overline{\mathbb{C}},$$

$$2)\ \sum_{a \in \overline{\mathbb{C}}} \delta(a, f) \le 2.$$

In 1969 V.P. Petrenko developed a theory of the growth of meromorphic functions. We recall its main characteristics. For each number a we set [2]

$$\mathcal{L}(r, a, f) = \max_{|z|=r} \log^+ \frac{1}{|\ f(z) - a\ |}, \quad \mathcal{L}(r, \infty, f) = \max_{|z|=r} \log^+ |\ f(z)\ | \ .$$

The magnitude of the deviation in the sence of Petrenko is defined by:

$$\beta(a, f) := \liminf_{r \to \infty} \frac{\mathcal{L}(r, a, f)}{T(r, f)}.$$

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This value characterizes the deviation of the function $f(z)$ from the value a in a stronger metric than a deficiency $\delta(a,f)$. It was shown that for meromorphic functions of a finite lower order the properties of $\beta(a,f)$ are similar to the properties of $\delta(a,f)$. Thus V.P. Petrenko has obtained a sharp estimate for $\beta(a,f)$ and also some estimate for $\sum\limits_{(a)}\beta(a,f)$.

Theorem B [2]. *For meromorphic functions of a finite lower order* λ

$$\beta(a,f) \le \begin{cases} \frac{\pi\lambda}{\sin\pi\lambda}, & \text{if} \quad \lambda < \frac{1}{2}, \\ \pi\lambda, & \text{if} \quad \lambda \ge \frac{1}{2}, \end{cases}$$

$$\sum_{(a)} \beta(a,f) \le 816\pi(\lambda+1)^2.$$

In the case $\lambda \le \frac{1}{2}$ the inequality in the theorem B was proved by A.A. Goldberg and I.V. Ostrovskii in 1961 [3].

In 1990 together with my student A.I. Shcherba we have obtained a sharp estimate for the series $\sum\limits_{(a)}\beta(a,f)$ which is an analogue of the deficiency relation for the magnitudes $\beta(a,f)$ [4].

Theorem C [4]. *If* $f(z)$ *is a meromorphic function of a finite lower order* λ*, then*

$$\sum_{(a)} \beta(a,f) \le \begin{cases} \frac{2\pi\lambda}{\sin\pi\lambda}, & \text{if} \quad \lambda < \frac{1}{2}, \\ 2\pi\lambda, & \text{if} \quad \lambda \ge \frac{1}{2}. \end{cases}$$

In 1970 D.F. Shea (see [2, 5]) has obtained the sharp estimate of the deviation of a meromorphic function of a finite lower order in terms of the Valiron deficiency. We put

$$\Lambda(\Delta) := \left\{ \lambda : 0 \le \lambda \le 0.5 \ \textit{and} \ \sin\frac{\pi\lambda}{2} \le \sqrt{\frac{\Delta}{2}} \right\},$$

$$B(\lambda,\Delta) := \begin{cases} \pi\lambda\sqrt{\Delta(2-\Delta)}, & \text{if} \quad \lambda \notin \Lambda(\Delta), \\ \frac{\pi\lambda}{\sin\pi\lambda}(1-(1-\Delta)\cos\pi\lambda), & \text{if} \quad \lambda \in \Lambda(\Delta). \end{cases} \tag{1}$$

Theorem D [5]. *Let* $f(z)$ *be a meromorphic function of a finite lower order* λ*. Then for each* $a \in \overline{\mathbb{C}}$

$$\beta(a,f) \le B(\lambda,\Delta),$$

where $\Delta = \Delta(a,f)$ *is Valiron's deficiency of* $f(z)$ *at the point* a*.*

For meromorphic functions of infinite lower order the magnitude $\beta(a,f)$ can be equal to ∞. Therefore the following result of W. Bergweiler and H. Bock [6] is interesting in this case.

Theorem F [6]. *For meromorphic functions of infinite lower order*

$$\liminf_{r\to\infty} \frac{\max\limits_{|z|=r} \log^+ |f(z)|}{rT'_-(r,f)} \le \pi. \tag{2}$$

In connection with (2) A. Eremenko has introduced the quantity

$$b(a,f) = \liminf_{r\to\infty} \frac{\mathcal{L}(r,a,f)}{A(r,f)}\,,$$

where $A(r,f)\pi$ is the area, counting multiplicity of covering, of the image on the Riemann sphere of $\{z : |z| < 1\}$ under $f(z)$ [7]. From the estimate (2) it follows that

$$b(a,f) \le \pi.$$

A. Eremenko has proved an analogue of the deficiency relation for $b(a,f)$.

Theorem G [7]. *Let $f(z)$ be a meromorphic function such that the set $\{a : b(a,f) > 0\}$ contains more than one point. Then*

$$\sum_{a\in\overline{\mathbf{C}}} b(a,f) \le 2\pi.$$

In 1998 I have obtained the estimates for $b(a,f)$ and $\sum\limits_{(a)} b(a,f)$ in the terms of Valiron's deficiency.

Theorem K [8]. *Let $f(z)$ be a meromorphic function of infinite lower order. Then*

$$(a)\quad b(a,f) \le \pi\sqrt{\Delta(a,f)(2-\Delta(a,f))},$$

$$(b)\quad \sum_{a\in\mathbf{C}} b(a,f) \le 2\pi\sqrt{\Delta(0,f')(2-\Delta(0,f'))}.$$

Corollary. *For a meromorphic function $f(z)$ we have*

$$E(f) = \{a :\ b(a,f) > 0\} \subset V(f) = \{a :\ \Delta(a,f) > 0\}.$$

We recall the definition of upper and lower logarithmic densities of a set. Let E be a measurable set on a positive half-axis. The quantities

$$\overline{\text{logdens}}\; E = \limsup_{R\to\infty} \frac{1}{\log R} \int\limits_{E\cap[1,R]} \frac{dt}{t},$$

$$\underline{\text{logdens}}\; E = \liminf_{R\to\infty} \frac{1}{\log R} \int\limits_{E\cap[1,R]} \frac{dt}{t}$$

are called an upper and a lower logarithmic density respectively. We put

$$B(\gamma) := \begin{cases} \frac{\pi\gamma}{\sin \pi\gamma}, & \text{if} \quad \gamma < \frac{1}{2}, \\ \pi\gamma, & \text{if} \quad \gamma \geq \frac{1}{2}. \end{cases}$$

Theorem L [9]. *Let $f(z)$ be a meromorphic function of a finite lower order λ and order ρ, $0 < \gamma < \infty$, $a \in \overline{\mathbb{C}}$,*

$$E_1(\gamma) = \{r : \mathcal{L}(r, a, f) < B(\gamma)T(r, f)\}.$$

Then

$$\overline{\text{logdens}}\, E_1(\gamma) \geq 1 - \frac{\lambda}{\gamma}, \quad \underline{\text{logdens}}\, E_1(\gamma) \geq 1 - \frac{\rho}{\gamma}.$$

For the case when $f(z)$ is an entire function and $a = \infty$ this theorem has been obtained by M. Essen and D.F. Shea [10].

Theorem M [9]. *Let $f(z)$ be a transcendental meromorphic function of a finite lower order λ and order ρ, $0 < \gamma < \infty$; $\{a_k\}_{k=1}^{q} \in \mathbb{C}$ is a finite set of distinct complex numbers,*

$$E_2(\gamma) = \left\{ r : \sum_{k=1}^{q} \mathcal{L}(r, a_k, f) < 2B(\gamma)T(r, f) \right\}.$$

Then

$$\overline{\text{logdens}}\; E_2(\gamma) \geq 1 - \frac{\lambda}{\gamma}, \quad \underline{\text{logdens}}\; E_2(\gamma) \geq 1 - \frac{\rho}{\gamma}.$$

We note another result in this direction.

Theorem N [11]. *Let $f(z)$ be a meromorphic function of a finite lower order λ and order ρ, $\tau > 0$, $a \in \overline{\mathbb{C}}$, $\varepsilon > 0$ is an arbitrary number,*

$$E_3(\gamma) = \{r : \mathcal{L}(r, a, f) < B(\gamma, \Delta(a, f))T(r, f)\}, \; \textit{if} \;\; \Delta(a, f) > 0,$$

$$E_3(\gamma) = \{r : \mathcal{L}(r,a,f) < \varepsilon T(r,f)\},\ if\ \Delta(a,f) = 0,$$

where $B(\gamma,\Delta)$ is the function defined in (1). Then

$$\overline{\text{logdens}}\, E_3(\gamma) \geq 1 - \frac{\lambda}{\gamma}, \quad \underline{\text{logdens}}\, E_3(\gamma) \geq 1 - \frac{\rho}{\gamma}.$$

We have the following results.

Theorem 1. *Let $f(z)$ be a meromorphic function of a finite lower order λ and order ρ, $0 < \gamma < \infty$, $\{a_k\}_{k=1}^{q} \in \mathbb{C}$ is a finite set of distinct complex numbers, $\varepsilon > 0$ is an arbitrary number,*

$$A(\gamma) = \{r : \sum_{k=1}^{q} \mathcal{L}(r,a_k,f) < B(\gamma, \Delta(0,f'))T(r,f)\},\ if\ \Delta(a,f) > 0,$$

$$A(\gamma) = \{r : \sum_{k=1}^{q} \mathcal{L}(r,a_k,f) < \varepsilon T(r,f)\},\ if\ \Delta(a,f) = 0,$$

where $B(\gamma,\Delta)$ is the function defined in (1), $\Delta(0,f')$ is Valiron's deficiency of $f'(z)$ at 0. Then

$$\overline{\text{logdens}}\ A(\gamma) \geq 1 - \frac{\lambda}{\gamma}, \quad \underline{\text{logdens}}\ A(\gamma) \geq 1 - \frac{\rho}{\gamma}.$$

Corollary 1. *For a meromorphic function $f(z)$ of a finite lower order λ it is true the inequality*

$$\sum_{a \in \mathbb{C}} \beta(a,f) \leq 2B\left(\lambda, \Delta(0,f')\right),$$

where $B(\gamma,\Delta)$ is the function defined in (1), $\Delta(0,f')$ is Valiron's deficiency of the derivative of the function f at zero.

The proof of the Theorem 1 is similar to the proof of the Theorem M [9] with use of the methods of A. Baernstein [12], R. Gariepy and J.L. Lewis [13], A. Weistman [14], P. Barry [15] and [4, 8, 9].

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