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Some stability theorems on narrow operators acting in L_1 and $C(K)$

V.M. Kadets

Department of Mechanics and Mathematics, V.N. Karazin Kharkov National University
4 Svobody Sq., Kharkov, 61077, Ukraine
E-mail:Anna.M.Vishnyakova@univer.kharkov.ua

M.M. Popov

Department of Mechanics and Mathematics, Chernivtsi National University
2 Kotsiubyns'kogo Str., Chernivtsi, 58012, Ukraine
E-mail:popov@chv.ukrpack.net

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A new proof of two stability theorems concerning narrow operators acting from L_1 to L_1 or from $C(K)$ to an arbitrary Banach space is given. Namely a sum of two such operators and moreover a sum of a point-wise unconditionally convergent series of such operators is a narrow operator again. The relations between several possible definitions of narrow operators on L_1 are also discussed.

1. Introduction

We use standard notation such as B_X and S_X for the unit ball and the unit sphere of a Banach space X and $\mathcal{L}(E, X)$ for the space of all linear bounded operators acting from E to X. Allover the text (Ω, Σ, μ) is a fixed non-atomic measure space, K is a fixed compact without isolated points, $L_1 = L_1(\Omega, \Sigma, \mu)$ and $C = C(K)$. By Σ^+ we denote the collection of all measurable subsets of Ω having non-zero measure. In this paper we deal with real Banach spaces.

Let X be a subspace of a Banach space Y and let J: $X \to Y$ denote the inclusion operator. We say that the pair (X, Y) has the *Daugavet property* for a class $\mathcal{M}$ of operators, where $\mathcal{M} \subset \mathcal{L}(X, Y)$, if

$$\|J + T\| = 1 + \|T\| \tag{1}$$

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for all $T \in \mathcal{M}$. If $X = Y$, we simply say that X has the Daugavet property with respect to $\mathcal{M}$, and if $\mathcal{M}$ is the class of rank-1 operators, we just say that X or the pair (X, Y) has the Daugavet property.

Classical results due to I.K. Daugavet [2], G.Ya. Lozanovskii [10], and C. Foiaş, I. Singer and A. Pełczyński [3] state that $C(K)$, $L_1(\Omega, \Sigma, \mu)$ and $L_\infty(\Omega, \Sigma, \mu)$ have the Daugavet property provided that K is perfect and μ is non-atomic. Recently, corresponding results in the non-commutative setting were obtained by T. Oikhberg [11]. The papers [7] and [15] study Banach spaces with the Daugavet property from a structural point of view; for example, it is shown that such a space never embeds into a space having an unconditional basis, and it contains (many) subspaces isomorphic to ℓ_1. For a detailed survey of the recent progress on the Daugavet propertysee [18].

In order to perform a unified approach to the study of the Daugavet property the following concept was introduced [8]:

Definition 1.1. An operator $T \in \mathcal{L}(X, Y)$ is said to be a *narrow operator* if for every two elements $x, y \in S_X$, for every $x^* \in X^*$ and for every $\varepsilon > 0$ there is an element $z \in S_X$ such that $\|x + z\| > 2 - \varepsilon$ and $\|T(y - z)\| + |x^*(y - z)| < \varepsilon$. We denote the subset of $\mathcal{L}(X, Y)$ consisting of all narrow operators by $\mathcal{NAR}(X, Y)$.

It was shown in [8] that every weakly compact operator and every operator which does not fix a copy of ℓ_1 on a Banach space with the Daugavet propertyis narrow, and that every Banach space X with the Daugavet propertyhas the Daugavet propertywith respect to $\mathcal{NAR}(X, X)$. Although $\mathcal{NAR}(X, Y)$ has some stability properties (for example, a sum of a narrow operator and a weakly compact one is narrow), a sum of two narrow operators is not necessarily a narrow operator again [1]. The class of narrow operators forms a left ideal in the following sense: if $T \in \mathcal{NAR}(X, Y)$, $V \in \mathcal{L}(Y, Z)$, then $VT \in \mathcal{NAR}(X, Z)$.

In the case of $X = L_1$ Definition 1.1 can be equivalently reformulated in terms of so-called balanced ε-peaks.

Definition 1.2. A function $f \in L_1$ is said to be a *balanced ε-peak* on $A \in \Sigma^+$ if $f \geq -1$, $\operatorname{supp} f \subset A$, $\int_\Omega f \, d\mu = 0$ and $\mu\{t\colon f(t) = -1\} > \mu(A) - \varepsilon$.

According to [8], Theorem 6.1, an operator $T \in \mathcal{L}(L_1, X)$ is narrow if and only if for every $\varepsilon > 0$ and every $A \in \Sigma^+$ there exists a balanced ε-peak g on A with $\|Tg\| \leq \varepsilon$.

Definition 1.3. Let $A \in \Sigma^+$. A function $x \in L_1$ is said to be a *sign* supported on A if $x = \chi_{B_1} - \chi_{B_2}$, where B_1 and B_2 form a partition of A into two subsets of equal measure. An operator $T \in \mathcal{L}(L_1, X)$ is said to be L_1-*narrow* if for every set $A \in \Sigma^+$ and every $\varepsilon > 0$ there is a sign x, supported on A, with $\|Tx\| < \varepsilon$.

The concept of L_1-narrow operator was introduced in [12] under the name "narrow operator", but we prefer to use the name "narrow operator" for Definition 1.1.

Definition 1.4. An operator $T \in \mathcal{L}(L_1, X)$ is called $L_1(A)$-*singular* if for every $A \in \Sigma^+$ the restriction of T to $L_1(A)$ is unbounded from below.

Let us note that every L_1-narrow operator is narrow, and every narrow operator on L_1 is $L_1(A)$-singular. For operators acting from L_1 to L_1 (or even between two different L_1 spaces) the inverse inclusions are true too, which follows from Rosenthal's papers [13, 14]. As shows the quotient map from Talagrand's example [17] in general an $L_1(A)$-singular operator is not necessarily L_1-narrow. We don't know the answer to the following questions:

Problem 1.1. *Is it true that every narrow operator acting from L_1 to a Banach space X is L_1-narrow? In other words, is it sufficient to consider balanced ε-peaks instead of signs in the definition of L_1-narrow operator?*

Problem 1.2. *Is it true that a sum of two narrow operators acting from L_1 to a Banach space X is narrow?*

For an arbitrary open subset $U \subset K$ denote by $C_0(U)$ a subspace of $C(K)$ consisting of functions, vanishing on the complement of U.

Theorem 1.1. [8] *For an operator $T \in \mathcal{L}(C, X)$ the following conditions are equivalent:*

1. *$T \in \mathcal{NAR}(C, X)$,*
2. *for every non-empty open subset $U \subset K$ the restriction of T to $C_0(U)$ is unbounded from below,*
3. *for every non-empty open subset $U \subset K$ the restriction of T to $C_0(U)$ is narrow.*

In the second section of this paper we give a new proof of two stability theorems concerning narrow operators acting from L_1 to L_1 or from $C(K)$ to arbitrary Banach space. Namely we prove that a sum of two such operators and moreover a sum of a point-wise unconditionally convergent series of such operators is a narrow operator again. In L_1 case the original proof [12] of the first of this statements contained a gap. The corrected proof of this statement and the proof of the second one as well was done recently by Shvydkoy* [16] in his Thesis. The $C(K)$

* By the way, Roman Shvydkoy and Roman Shvidkoy in the references below is the same person: these are just Ukrainian and Russian spellings of the same name.

case was studied for two operators in [5] and for a series of operators in [1]. The advantage of our approach is its applicability for L_1 and $C(K)$ cases simultaneously.

In the last section we prove a bit technical reformulation of the notion of L_1-narrow operator, which looks as a first step toward the solution of the Problem 1.1.

2. Stability theorems

First, remind some definitions and results from the paper [6]. Let X be a Banach space. Denote by B^* the closed unit ball of X^* equipped with weak* topology. Recall that $A \subset B^*$ is said to be a first category (f.c.) set if $A = \cup_{i=1}^{\infty} A_i$, where A_i are nowhere-dense sets.

Let us introduce the following Banach spaces:

$$l_\infty(B^*) = \{f : B^* \mapsto R, \quad \sup\{|f(s)|, s \in B^*\} = \|f\|_\infty < \infty\},$$

$$\mathrm{fc}(B^*) = \{f \in l_\infty(B^*) : \mathrm{supp}(f) \text{ is a f.c.set}\}.$$

$\mathrm{fc}(B^*)$ is a closed linear subspace of $l_\infty(B^*)$, so we can consider a Banach space $m_0(B^*) = l_\infty(B^*)/\,\mathrm{fc}(B^*)$ with the norm $\|[f]\| = \inf\{\sup\{|f(s)|, s \in B^* \backslash F\}, F$ is a f.c.set$\}$. Since every $x \in X$ may be considered as a continuous function on B^*, and the sup-norm of this function coincides with its norm in $m_0(B^*)$ and coincides with $\|x\|$, we will consider below the inclusions $X \subset m_0(B^*)$ and $X \subset l_\infty(B^*)$ in the sense described. According to [15] for every space with the Daugavet property, the pair $(X, m_0(B^*))$ has the Daugavet property, i.e., it has the Daugavet propertyfor the classes of compact and weakly compact operators (see [7]).

Let us introduce a new definition.

Definition 2.1. A Banach space X is said to be *D-acceptable,* if the pair $(X, m_0(B^*))$ has the Daugavet propertyfor the class of narrow operators.

Due to [6] and [16] the classical spaces L_1 and C are D-acceptable. We don't know whether every Banach space X with the Daugavet propertyis D-acceptable.

Lemma 2.1. *Let E be a subspace of a Banach space F, the pair $(X, m_0(B^*))$ be as above, $V \in \mathcal{L}(E, X)$. Then there exists an extension of V to a bounded operator $\tilde{V}$: $F \to m_0(B^*)$.*

P r o o f. By injectivity of $l_\infty(B^*)$, or, in other words, by Hahn–Banach theorem for $l_\infty(B^*)$-valued operators instead of functionals (see [9, Ch. 2.f]), V can be extended to an operator W: $F \to l_\infty(B^*)$. To obtain the needed operator $\tilde{V}$ it suffices to compose this extension with the natural quotient map q: $l_\infty(B^*) \to m_0(B^*)$. ∎

The following observation is extracted from [12].

Lemma 2.2. *Let X be a subspace of a Banach space Y, $T_1, T_2 \in \mathcal{L}(X,Y)$. If both $-T_1$ and $-T_2$ satisfy the Daugavet equation* (1)*, then $T_1 + T_2 \neq J$.*

P r o o f. Assume to the contrary that $T_1 + T_2 = J$. Then

$$\|T_1\| = \|J - T_2\| = 1 + \|T_2\| = 1 + \|J - T_1\| = 2 + \|T_1\|.$$

A contradiction. ∎

Theorem 2.1. *Let X, F be Banach spaces, X be D-acceptable, and $T_1, T_2 \in \mathcal{NAR}(X,F)$. Then $T_1 + T_2$ is unbounded from below.*

P r o o f. Assume to the contrary that $T_1 + T_2$ is an into isomorphism, and denote its image by E. Applying Lemma 2.1 to $V = (T_1 + T_2)^{-1}$, we obtain an operator $\tilde{V}\colon F \to m_0(B^*)$ for which $\tilde{V}(T_1 + T_2) = J$, where J is the canonical embedding of X into $m_0(B^*)$. But both the operators $-\tilde{V}T_1$ and $-\tilde{V}T_2$ are narrow, which means for a D-acceptable space that both of them satisfy the Daugavet equation. By the previous Lemma 2.2 it is impossible. ∎

Corollary 2.1. *Let $T_1, T_2 \in \mathcal{NAR}(L_1, L_1)$. Then $T_1 + T_2 \in \mathcal{NAR}(L_1, L_1)$*

P r o o f. According to properties of narrow operators acting from L_1 to L_1 listed in the introduction, the restrictions of both operators to all the subspaces of the form $L_1(A)$ are narrow. According to the previous theorem, the operator $T_1 + T_2$ is $L_1(A)$-singular, which means in turn, that it is narrow. ∎

Corollary 2.2. *Let F be a Banach space, and $T_1, T_2 \in \mathcal{NAR}(C,F)$. Then $T_1 + T_2 \in \mathcal{NAR}(C,F)$.*

P r o o f. By the same argument as before this follows from Theorem 2.1 and the characterization of narrow operators on C, given in the introduction (Theorem 1.1). ∎

We turn now to the case of point-wise unconditionally convergent series of narrow operators. Let X be a subspace of a Banach space Y. Let us remind the following Lemma ([7, Lemma 2.6], or [4] for the case $X = Y$).

Lemma 2.3. *If a pair (X,Y) has the Daugavet property for a class $\mathcal{M} \subset \mathcal{L}(X,Y)$ of operators, and $\mathcal{M}$ is a linear space, then the natural embedding operator J cannot be represented as a sum of point-wise unconditionally convergent series of operators from $\mathcal{M}$.*

We say that a pair (X, F) of Banach spaces is *completely acceptable*, if X is D-acceptable and $\mathcal{NAR}(X, F)$ is a linear space. For an operator $V \in \mathcal{L}(F, m_0(B^*))$ denote by $\mathcal{M}_V(X, F)$ the set of all operators of the form VT, where $T \in \mathcal{NAR}(X, F)$.

Lemma 2.4. *If a pair (X, F) of Banach spaces is completely acceptable, $V \in \mathcal{L}(F, m_0(B^*))$, then the canonical embedding J: $X \to m_0(B^*)$ cannot be represented as a sum of point-wise unconditionally convergent series of operators from $\mathcal{M}_V(X, F)$.*

P r o o f. All the operators from the class $\mathcal{M}_V(X, F)$ are narrow, which means for a D-acceptable space that all of them satisfy the Daugavet equation. Since $\mathcal{M}_V(X, m_0(B^*))$ is a linear space for a completely acceptable pair (X, F), we may apply Lemma 2.3. ■

Theorem 2.2. *Let X, F be Banach spaces, the pair (X, F) be completely acceptable, $T_n \in \mathcal{NAR}(X, F), n = 1, 2, \ldots$, and the series $\sum_{n=1}^{\infty} T_n$ be point-wise unconditionally convergent. Then the operator $\sum_{n=1}^{\infty} T_n$ is unbounded from below.*

P r o o f. The idea of the proof comes from Theorem 2.1. Assume contrary that $\sum_{n=1}^{\infty} T_n$ is an into isomorphism, and denote its image by E. Applying Lemma 2.1 to $V = (\sum_{n=1}^{\infty} T_n)^{-1}$, we obtain an operator $\tilde{V}$: $F \to m_0(B^*)$ for which $\sum_{n=1}^{\infty} \tilde{V} T_n = J$, where J is the canonical embedding of X into $m_0(B^*)$ and the series converges point-wise unconditionally. By the previous Lemma 2.4 it is impossible. ■

According to Corollaries 2.1 and 2.2, the pairs of the form (L_1, L_1) or (C, F) are completely acceptable. By the same argument as in Corollaries 2.1 and 2.2, this implies two corollaries more:

Corollary 2.3. *Let $T_n \in \mathcal{NAR}(L_1, L_1)$ and the series $\sum_{n=1}^{\infty} T_n$ be point-wise unconditionally convergent. Then $\sum_{n=1}^{\infty} T_n \in \mathcal{NAR}(L_1, L_1)$.*

Corollary 2.4. *Let F be a Banach space, $T_n \in \mathcal{NAR}(C, F)$ and $\sum_{n=1}^{\infty} T_n$ be point-wise unconditionally convergent. Then $\sum_{n=1}^{\infty} T_n \in \mathcal{NAR}(C, F)$.*

3. Signs and balanced ε-peaks

In the sequel, by a *biased sign* (a special kind of balanced ε-peaks) we mean any $h \in L_1$ of the form:

$$h = \frac{\mu(A \setminus B)}{\mu(B)} \chi_B - \chi_{A \setminus B}, \quad \text{where} \quad A, B \in \Sigma^+, \ \ B \subset A.$$

We answer below in affirmative to the following weaker version of Problem 1.1: *may one consider biased signs in the definition of an L_1-narrow operator instead of signs?*

Note that a converse in some sense statement was proved in [12, Lemma 1, p. 55].

Theorem 3.1. *Let X be a Banach space and $T \in \mathcal{L}(L_1, X)$ have the following property: for each $\varepsilon > 0$ and $A \in \Sigma^+$ there exists a measurable $B \subset A$ such that:*

$$0 < \mu(B) < \mu(A); \qquad (a)$$

$$\|Th\| < \varepsilon\|h\| \ \ \textit{where} \ \ h = \frac{\mu(A \setminus B)}{\mu(B)}\chi_B - \chi_{A\setminus B}. \qquad (b)$$

(In other words, T is unbounded from below at biased signs having the common support A for every $A \in \Sigma^+$.) Then for each $\delta \in (0,1)$, each $\varepsilon > 0$ and each $A \in \Sigma^+$ there exists a measurable $B \subset A$ such that (b) *holds together with*

$$\mu(B) = \delta\mu(A). \qquad (a')$$

In particular ($\delta = \frac{1}{2}$), T is L_1-narrow.

The proof contains several auxiliary statements. Note first, that h could be written as

$$h = \frac{\mu(A)}{\mu(B)}\chi_B - \chi_A.$$

Indeed,

$$\frac{\mu(A)}{\mu(B)}\chi_B - \chi_A = \frac{\mu(A)}{\mu(B)}\chi_B - \chi_B - \chi_{A\setminus B} = \frac{\mu(A) - \mu(B)}{\mu(B)}\chi_B - \chi_{A\setminus B}.$$

Lemma 3.1. *T possesses the following property: for each $\varepsilon > 0$, $\varepsilon_1 > 0$ and each $A \in \Sigma^+$ there exists a measurable $B \subset A$ such that (b) holds together with:*

$$0 < \mu(B) < \varepsilon_1\mu(A). \qquad (a'')$$

P r o o f of Lemma 3.1. Pick an integer n so that $2^{-n} < \varepsilon_1$ and choose a measurable $B \subset A$ so that (a) holds together with

$$\|Th_0\| < \frac{\varepsilon}{2n}\|h_0\| \le \frac{\varepsilon}{n}\mu(A), \quad \text{where } h_0 = \frac{\mu(A)}{\mu(B)}\chi_B - \chi_A.$$

Put $A_1 = B$ if $\mu(B) \le \frac{1}{2}\mu(A)$ and $A_1 = A \setminus B$ if $\mu(B) > \frac{1}{2}\mu(A)$. Anyway, $A_1 \subset A$, $0 < \mu(A_1) \le \frac{1}{2}\mu(A)$ and

$$\|Th_1\| < \frac{\varepsilon}{n}\mu(A), \quad \text{where } h_1 = \frac{\mu(A)}{\mu(A_1)}\chi_{A_1} - \chi_A.$$

Indeed, if $A_1 = A \setminus B$ then

$$h_1 = \frac{\mu(A)}{\mu(A \setminus B)} \chi_{A\setminus B} - \chi_A = \frac{\mu(A)}{\mu(A \setminus B)} \Big(\chi_A - \chi_B \Big) - \chi_A$$

$$= \Big(\frac{\mu(A)}{\mu(A \setminus B)} - 1 \Big) \chi_A - \frac{\mu(A)}{\mu(A \setminus B)} \chi_B$$

$$= \frac{\mu(B)}{\mu(A \setminus B)} \chi_A - \frac{\mu(A)}{\mu(A \setminus B)} \chi_B = - \frac{\mu(B)}{\mu(A \setminus B)} \Big(\frac{\mu(A)}{\mu(B)} \chi_B - \chi_A \Big) = - \frac{\mu(B)}{\mu(A \setminus B)} h_0$$

and therefore

$$\|Th_1\| = \frac{\mu(B)}{\mu(A \setminus B)} \ \|Th_0\| < \frac{\mu(B)}{\mu(A \setminus B)} \ \frac{\varepsilon}{2n} \ \|h_0\| = \frac{\varepsilon}{2n} \ \|h_1\|.$$

Then choose as above a measurable $A_2 \subset A_1$ with $0 < \mu(A_2) \le \frac{1}{2}\mu(A_1) \le \frac{1}{4}\mu(A)$ such that

$$\|Th_2\| < \frac{\mu(A_1)}{\mu(A)} \ \frac{\varepsilon}{2n} \ \|h_2\| \le \mu(A_1) \ \frac{\varepsilon}{n}, \quad \text{where } \ h_2 = \frac{\mu(A_1)}{\mu(A_2)} \chi_{A_2} - \chi_{A_1}.$$

Going like that, choose at the last step $A_n \subset A_{n-1}$ with $0 < \mu(A_n) \le \frac{1}{2}\mu(A_{n-1}) \le 2^{-n}\mu(A) \le \varepsilon_1 \mu(A)$ such that

$$\|Th_n\| < \frac{\mu(A_{n-1})}{\mu(A)} \ \frac{\varepsilon}{2n} \ \|h_n\| \le \mu(A_{n-1}) \ \frac{\varepsilon}{n}, \quad \text{where } \ h_n = \frac{\mu(A_{n-1})}{\mu(A_n)} \chi_{A_n} - \chi_{A_{n-1}}.$$

Put

$$h = h_1 + \sum_{k=2}^{n} \frac{\mu(A)}{\mu(A_{k-1})} h_k.$$

Evidently

$$h = \frac{\mu(A)}{\mu(A_n)} \chi_{A_n} - \chi_A, \|h\| \ge \mu(A).$$

So

$$\|Th\| < \frac{\varepsilon}{n} \mu(A) + \sum_{k=2}^{n} \frac{\mu(A)}{\mu(A_{k-1})} \|Th_k\| < \frac{1}{2} \varepsilon \mu(A) \le \varepsilon \|h\|.$$

Thus, Lemma 3.1 is proved.

Now fix $\delta \in (0,1)$, $\varepsilon > 0$ and $A \in \Sigma^+$. Put $\varepsilon_2 = 2\delta\varepsilon\mu(A)$. For measurable $C \subset B \subset A$ $(\mu(C) > 0)$ denote:

$$\phi_B(C) = \|T \frac{\mu(B)}{\mu(C)} \chi_C - T \chi_B\|.$$

It is an easy exercise to show that $\phi_B(C)$ is separately continuous, i.e., for fixed measurable sets $C \subset B$ and each $\varepsilon_1 > 0$ there is a $\delta_1 > 0$ such that for every $C_1 \subset B$ ($\mu(C_1) > 0$) and every $B_1 \supset C$ if $\mu(C_1 \triangle C) < \delta_1$ then $|\phi_B(C_1) - \phi_B(C)| < \varepsilon_1$ and if $\mu(B_1 \triangle B) < \delta_1$ then $|\phi_{B_1}(C) - \phi_B(C)| < \varepsilon_1$.

For a measurable $B \subset A$ with $\mu(B) \geq \delta\mu(A)$ denote by $F(B)$ (respectively, $\overline{F}(B)$) the collection of all measurable subsets $C \subset B$ with $\mu(C) \leq \mu(B) - \delta\mu(A)$ and either $\mu(C) = 0$ or $\phi_B(C) < \varepsilon_2$ (respectively, $\phi_B(C) \leq \varepsilon_2$) if $\mu(C) > 0$.

Lemma 3.2. *Let $B \in F(A)$ and $C \in \overline{F}(A \setminus B)$. Then $B \bigcup C \in F(A)$.*

P r o o f o f L e m m a 3.2. Since $\mu(C) \leq \mu(A \setminus B) - \delta\mu(A)$, we have

$$\mu(B \bigcup C) = \mu(B) + \mu(C) \leq \mu(B) + \mu(A \setminus B) - \delta\mu(A) = \mu(A) - \delta\mu(A).$$

Estimate

$$\begin{aligned}
\phi_{B\cup C}(A) &= \|T\frac{\mu(A)}{\mu(B\bigcup C)}\chi_{B\cup C} - T\chi_A\| \\
&= \|T\frac{\mu(A)}{\mu(B\bigcup C)}\chi_C + \frac{\mu(C)}{\mu(A\setminus B)}T\frac{\mu(A)}{\mu(B\bigcup C)}\chi_B \\
&+\Big(1 - \frac{\mu(C)}{\mu(A\setminus B)}\Big)T\frac{\mu(A)}{\mu(B\bigcup C)}\chi_B - \Big(1 - \frac{\mu(C)\mu(A)}{\mu(A\setminus B)\mu(B\bigcup C)}\Big)T\chi_A \\
&-\frac{\mu(C)\mu(A)}{\mu(A\setminus B)\mu(B\bigcup C)}T\chi_B - \frac{\mu(C)\mu(A)}{\mu(A\setminus B)\mu(B\bigcup C)}T\chi_{A\setminus B}\| \\
&\leq \|\frac{\mu(C)\mu(A)}{\mu(A\setminus B)\mu(B\bigcup C)}T\frac{\mu(A\setminus B)}{\mu(C)}\chi_C - \frac{\mu(C)\mu(A)}{\mu(A\setminus B)\mu(B\bigcup C)}T\chi_{A\setminus B}\| \\
&+\|\Big(1 - \frac{\mu(C)}{\mu(A\setminus B)}\Big)\frac{\mu(B)}{\mu(B\bigcup C)}T\frac{\mu(A)}{\mu(B)}\chi_B - \Big(1 - \frac{\mu(C)\mu(A)}{\mu(A\setminus B)\mu(B\bigcup C)}\Big)T\chi_A\| \\
&\leq \frac{\mu(C)\mu(A)}{\mu(A\setminus B)\mu(B\bigcup C)}\ \varepsilon_2 + \frac{\mu(A)\mu(B) - \mu(B)\mu(B\bigcup C)}{\mu(A\setminus B)\mu(B\bigcup C)}\ \varepsilon_2 = \varepsilon_2.
\end{aligned}$$

Thus, Lemma 3.2 is proved.

Lemma 3.3. *Let $C_1 \in F(A)$, $C_n \in F(A \setminus \bigcup\limits_{k=1}^{n-1} C_k)$ for $n \geq 2$ and $D = \bigcup\limits_{n=1}^{\infty} C_n$. Then $D \in F(A)$.*

P r o o f o f L e m m a 3.3. Note that $C_2 \in F(A \setminus C_1)$; $C_n \in F((A \setminus C_1) - \bigcup_{k=2}^{n-1} C_k)$ for $n \geq 3$. Put $D_n = \bigcup_{k=2}^{n} C_k$ for $n \geq 3$. Lemma 3.2 implies that $D_n \in F(A \setminus C_1)$ for each $n \geq 3$. Then

$$\mu(\bigcup_{n=2}^{\infty} C_n) = \lim_n \mu(D_n) \leq \mu(A \setminus C_1) - \delta\mu(A)$$

and

$$\phi_{A \setminus C_1}(\bigcup_{n=2}^{\infty} C_n) = \lim_n \phi_{A \setminus C_1}(D_n) \leq \varepsilon_2.$$

Thus, $\bigcup_{n=2}^{\infty} C_n \in \overline{F}(A \setminus C_1)$. By Lemma 3.2, $D = C_1 \bigcup \bigcup_{n=2}^{\infty} C_n \in F(A)$. Lemma 3.3 is proved.

For every measurable $B \subset A$ with $\mu(B) \geq \delta\mu(A)$ consider

$$\nu(B) = \sup\{\mu(C) : \ C \in F(B)\}.$$

Lemma 3.4. *1) $\nu(B) = 0$ if and only if $\mu(B) = \delta\mu(A)$.*

2) ν is semicontinuous in the following sense: if $B_1 \supset B_2 \supset \ldots$; $B = \bigcap_{n=1}^{\infty} B_n$, then $\nu(B) \leq \liminf_n \nu(B_n)$.

P r o o f o f L e m m a 3.4. 1) Let $\nu(B) = 0$. It means that there is no $C \in F(B)$ with $\mu(C) > 0$. But if $\mu(B) - \delta\mu(A) > 0$ then by Lemma 3.1 there exists $C \in F(B)$ with $\mu(C) > 0$, — a contradiction. The converse is trivial.

2) Let $C \in F(B)$. It means that $\mu(C) \leq \mu(B) - \delta\mu(A)$ and $\phi_B(C) < \varepsilon_2$. Then for each n we have $\mu(C) \leq \mu(B) - \delta\mu(A) \leq \mu(B_n) - \delta\mu(A)$. Since $\liminf_n \phi_{B_n}(C) = \phi_B(C) < \varepsilon_2$, there is an n_0 such that $\phi_{B_n}(C) < \varepsilon_2$ for $n \geq n_0$, and therefore $C \in F(B_n)$. Thus, $\mu(C) \leq \liminf_n \nu(B_n)$. By arbitrariness of C, Lemma 3.4 is proved.

Continue the proof of Theorem 3.1. Put $B_1 = A$ and construct two sequences of subsets (B_n) and (C_n) so that:

(i) $C_n \in F(B_n)$,
(ii) $\mu(C_n) \geq \frac{1}{2}\nu(B_n)$,
(iii) $B_{n+1} = B_n \setminus C_n$.

Then put $D = \bigcup_{n=1}^{\infty} C_n$. By Lemma 3.3, $D \in F(A)$. Since C_n are disjoint, we have $\mu(C_n) \longrightarrow 0$ and by (ii), $\nu(B_n) \longrightarrow 0$ as well. By Lemma 3.4 for $B = \bigcap_{n=1}^{\infty} B_n$

we have $\nu(B) = 0$ and $\mu(B) = \delta\mu(A)$. It is not hard to see that $D = A \setminus B$ and therefore (like in the proof of Lemma 3.1):

$$\frac{\mu(A)}{\mu(B)}\chi_B - \chi_A = -\frac{\mu(D)}{\mu(A \setminus D)}\Big(\frac{\mu(A)}{\mu(D)}\chi_D - \chi_A\Big) = -\frac{1-\delta}{\delta}\Big(\frac{\mu(A)}{\mu(D)}\chi_D - \chi_A\Big).$$

Thus, for $h = \frac{\mu(A)}{\mu(B)}\chi_B - \chi_A$ we obtain $\|h\| = 2(1-\delta)\mu(A)$ and, since $D \in F(A)$,

$$\|Th\| = \frac{1-\delta}{\delta}\ \phi_A(D) < \frac{1-\delta}{\delta}\ \varepsilon_2 < \varepsilon\|h\|.$$

The theorem is proved.

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