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On variation preserving operators

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For a piecewise-continuous function f on $[0,1]$ we denote by $\nu(f)$ the number of its sign changes. By $K_n[0,1]$ we denote the set of piecewise-continuous functions f on $[0,1]$ such that $\nu(f) \leq n$. We prove that for any $n \geq 2$ there are no integral transforms $\tilde{K}f(x) = \int\limits_0^1 K(x,y)f(y)dy$ with a continuous kernel $K(x,y)$ such that $\nu(\tilde{K}f) = \nu(f)$, for every $f \in K_n[0,1]$. We give an example of a continuous kernel $K(x,y)$ such that $\nu(\tilde{K}f) = \nu(f)$, for every $f \in K_1[0,1]$.

Introduction and statement of results

The variation-diminishing property was studied by G. Pólya, I.J. Schoenberg, T.S. Motzkin, A. Whitney and many others (see [1, 2]). To formulate some of their results let us give a few definitions.

For a vector $x \in \mathbf{R}^n$ we will denote by x_j the j-th coordinate of x. By $(x_1, x_2, \dots, x_n)^t$ we will designate the corresponding column-vector. We will denote by $\nu(x_1, \dots, x_n)$ the number of sign changes of the real sequence $x_1, \dots, x_n$, zero terms being discarded. For a column-vector $x \in \mathbf{R^n}$ we will denote by $\nu(x)$ the number of sign changes in the sequence of its components.

Definition 1. *The real $m \times n$ matrix A is said to have a variation-diminishing property if*

$$\nu(Ax) \leq \nu(x), \quad \forall x \in \mathbf{R^n}. \tag{1}$$

The following theorem gives the full description of real matrixes having the variation-diminishing property.

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Theorem A (Pólya, Schoenberg, Motzkin, see [2, Ch. 4, p. 118]). *The real $m \times n$ matrix A has a variation-diminishing property if and only if two conditions hold ($r = \mathrm{rank A}$):*

(i) *for any k $1 \le k < r$, all nonzero minors of A of order k have the same sign (depending on k);*

(ii) *for arbitrary r columns of A all minors of order r formed by these columns have the same sign (depending on the set of columns).*

Denote by $PWC[a,b]$ (*piecewise-continuous on* $[a,b]$) the set of functions $f : [a,b] \to \mathbf{R}$ satisfying the following conditions:

1) $f(a) = f(a+0),\ f(b) = f(b-0)$;

2) $\exists m \in \mathbf{N}\ \exists a = x_1 < x_2 < \ldots < x_m = b$,
$\forall i = 1, 2, \ldots, m-1 : \quad f \in C(x_i, x_{i+1})$;

3) $\forall i = 1, 2, \ldots, m \quad \exists f(x_i - 0) \neq \infty,\ \exists f(x_i + 0) \neq \infty$
and $f(x_i) = f(x_i + 0)$ or $f(x_i) = f(x_i - 0)$.

Definition 2. *For a function $f \in PWC[a,b]$ let us denote by*

$$\nu(f) = \sup \nu(f(t_1), \ldots, f(t_m)),$$

where the supremum is extended over all $m \in \mathbf{N}$ and all ordered sets $a \le t_1 < t_2 < \ldots < t_m \le b$. We will denote by $K_n[a,b]$ the set $\{f \in PWC[a,b] : \nu(f) \le n\}$.

Let $K(x,y)$ be a real continuous function defined on $[a,b] \times [c,d]$. Then for any function $f \in PWC[c,d]$ the integral $\int\limits_{[c,d]} |K(x,y) f(y)| dy$ is finite. Let us introduce the integral transform

$$\tilde{K} f = \int\limits_{[c,d]} K(x,y) f(y)\, dy. \tag{2}$$

Definition 3. *The kernel $K(x,y)$ (the corresponding integral transform $\tilde{K}$) is said to have a variation-diminishing property on $D \subset PWC[c,d]$ if*

$$\nu(\tilde{K} f) \le \nu(f),\ \forall f \in D.$$

Theorem B (see [1, Ch. 1, p. 21]). *The integral transform (2) has a variation-diminishing property on $K_n[a,b]$ if there exists a sequence $\varepsilon_1, \dots, \varepsilon_{n+1}$, all $\varepsilon_i = \pm 1$, such that for any $1 \le p \le n+1$ and for any $a \le x_1 < \ \dots < \ x_p \le \ b$, $c \le y_1 < \dots < y_p \le d$*

$$\varepsilon_p \det(K(x_i, y_j))_{i,j=1}^{p} \ge 0.$$

There are many interesting publications devoted to the class of linear operators which diminish variation. In this paper we study a narrower class: operators, which preserve variation.

Definition 4. *We will say that a real $n \times n$ matrix A possesses a variation-preserving property if*

$$\nu(Ax) = \nu(x), \quad \forall x \in \mathbf{R}^{\mathbf{n}}.$$

Definition 5. *We will say that the kernel $K(x,y)$ (the corresponding integral transform $\tilde{K}$), defined by (2), possesses a variation-preserving property on $D \subset PWC[c,d]$ if*

$$\nu(\tilde{K}f) = \nu(f), \quad \forall f \in D.$$

Theorem 1. *The real $n \times n$ matrix A preserves variation if and only if*

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ & & \dots & & \\ 0 & 0 & 0 & \dots & \lambda_n \end{pmatrix} \ or \ A = \begin{pmatrix} 0 & \dots & 0 & 0 & \lambda_1 \\ 0 & \dots & 0 & \lambda_2 & 0 \\ 0 & \dots & \lambda_3 & 0 & 0 \\ & & \dots & & \\ \lambda_n & \dots & 0 & 0 & 0 \end{pmatrix}, \qquad (3)$$

where $\lambda_j \neq 0,\ j = 1, \dots, n,$ and $\mathrm{sign}(\lambda_1) = \mathrm{sign}(\lambda_2) = \dots = \mathrm{sign}(\lambda_\mathrm{n})$.

The main result of this paper is the following theorem.

Theorem 2. *Let $n \in \mathbf{N}$, $n \ge 2$, be a fixed number. There is no kernel $K \in C([0,1]^2)$ such that the corresponding integral transform $\tilde{K}$ preserves variation on $K_n[0,1]$.*

We will also construct an example of a ke rnel $K \in C([0,1]^2)$ such that the corresponding integral transform $\tilde{K}$ preserves variation on $K_1[0,1]$.

1. Proof of Theorem 1

The sufficiency in Theorem 1 is obvious. We will prove the necessity.

Let A be a real $n \times n$ matrix which preserves variation. Let us fix any j, $1 \le j \le n$, and consider a vector $x = (0, \dots, 0, 1, 0, \dots, 0)^t$, $x_j = 1$. Since $\nu(Ax) = \nu(x) = 0$, we obtain that $\nu(a_{1j}, a_{2j}, \dots, a_{nj}) = 0$, $j = 1, 2, \dots, n$.

Fix any $l \in \{1, 2, \dots, n\}$. Since the matrix A preserves variation if and only if the matrix $-A$ preserves variation, without loss of generality we can assume that $a_{kl} \ge 0$, $k = 1, 2, \dots, n$.

Now we will prove that $a_{il} \times a_{(i+1)l} = 0$ for any $1 \le i \le n-1$. Assume that $\exists i \in \{1, \dots, n-1\} :\ a_{il} > 0,\ a_{i+1,l} > 0$. Let us consider $x = (\dots, \varepsilon, -\varepsilon, 1, -\varepsilon, \varepsilon, \dots)^t$, where $x_l = 1$, $\varepsilon > 0$, $\forall k \ne l$ $x_k = (-1)^{(l-k)}\varepsilon$, $\nu(x) = n - 1$. For sufficiently small ε we have: $\mathrm{sign}((\mathrm{Ax})_\mathrm{i}) = \mathrm{sign}(\mathrm{a_{il}}) = 1$ and $\mathrm{sign}((\mathrm{Ax})_{\mathrm{i+1}}) = \mathrm{sign}(\mathrm{a_{i+1,l}}) = 1$, therefore $\nu(Ax) \le n - 2$, and it is a contradiction.

Assume that there exist i, j, $j > i + 1$, such that $a_{il} > 0$, $a_{i+1,l} = 0$ and $a_{jl} > 0$. Obviously, if the matrix A has a vanished row, then A has no preserving sign property, therefore there exists $k \ne l$ such that $a_{i+1,k} \ne 0$.

Assume that $a_{i+1,k} < 0$. Let us consider a column-vector x such that $x_l = 1$, $x_k = \varepsilon$, $\varepsilon > 0$, $x_m = 0$, $\forall m \notin \{l, k\}$, $\nu(x) = 0$. We have

$$Ax = (a_{1l}+\varepsilon a_{1k}, \dots, a_{il}+\varepsilon a_{ik}, \varepsilon a_{(i+1)k}, a_{(i+2)l}+\varepsilon a_{(i+2)k}, \dots, a_{jl}+\varepsilon a_{jk}, \dots)^t,$$

and for ε being sufficiently small $\nu(Ax) \ge 2$ holds, and it is a contradiction. Analogously assume that $a_{(i+1)k} > 0$. Then let us consider a column-vector x such that $x_l = 1$, $x_k = -\varepsilon$, $\varepsilon > 0$, $x_m = 0$, $\forall m \notin \{l, k\}$, $\nu(x) = 1$. We have

$$Ax = (a_{1l}\varepsilon a_{1k}, \dots, a_{il}-\varepsilon a_{ik}, -\varepsilon a_{(i+1)k}, a_{(i+2)l}-\varepsilon a_{(i+2)k}, \dots, a_{jl}-\varepsilon a_{jk}, \dots)^t,$$

and for ε being sufficiently small $\nu(Ax) \ge 2$ holds, and it is also a contradiction.

So we have proved that there exists not more than one nonzero element in any column of A. Since a matrix with a vanished column has no variation preserving property, there exists one and only one nonzero element in any column of A. Since the matrix A preserves variation, A has no vanishing row, so by the reasons mentioned above every row of A has one and only one nonzero element. Since $\nu(Ax) = \nu(x) = 0$ for a vector $x = (1, \dots, 1)^t$, all these nonzero elements of matrix A have the same sign.

So we have shown that there exist a set of nonzero numbers $\lambda_1, \dots, \lambda_n$ such that $\mathrm{sign}(\lambda_1) = \ldots = \mathrm{sign}(\lambda_\mathrm{n})$ and for any $x = (x_1, \dots, x_n)^t$

$$Ax = (\lambda_{i_1} x_{i_1}, \dots, \lambda_{i_n} x_{i_n})^t,$$

where $(i_1, i_2, \dots, i_n)$ is a perturbation of $(1, 2, \dots, n)$.

It is easy to verify that the sign preserving is possible if and only if $i_k = k,\ k = 1,2,\dots,n$ or $i_k = n-k,\ k = 1,2,\dots,n$, holds, and this concludes the proof. ■

2. Proof of Theorem 2

Since $K_n \subset K_{n+1},\ n \in \mathbf{N}$, it is enough to prove Theorem 2 for $n = 2$. Assume that there exists a kernel $K(x,y)$ which preserves variation on $K_2[0,1]$.

It is obvious that $K(x,y) \not\equiv 0$ on $[0,1]^2$. Let us prove that $K(x,y) \geq 0$, $\forall (x,y) \in [0,1]^2$ or $K(x,y) \leq 0, \quad \forall (x,y) \in [0,1]^2$.

Let us introduce several notations, which will be used only in this section. For $x \in [0,1]$ we denote

$$k(x,J) := \int\limits_J K(x,y)dy,$$

where $J \subset [0,1]$ is a measurable set, and for $A \subset [0,1]$ we denote

$$I_A(y) := \begin{cases} 1, & \text{if } y \in A, \\ 0, & \text{if } y \in [0,1] \setminus A. \end{cases}$$

At first, we will prove that for any $y_0 \in [0,1]$

$$K(x,y_0) \geq 0, \forall x \in [0,1] \quad \text{or} \quad K(x,y_0) \leq 0, \forall x \in [0,1]. \tag{4}$$

Let us fix any $y_0 \in [0,1]$. Assume that $\exists x_1, x_2 \in [0,1],\ x_1 \neq x_2$ such that $K(x_1,y_0) > 0$ and $K(x_2,y_0) < 0$. Then $\exists \varepsilon > 0$ such that $K(x_1,y) > 0$ and $K(x_2,y) < 0$ for any $y \in U_\varepsilon(y_0) = \{y \in [0,1] : |y - y_0| < \varepsilon\}$. Let us consider a function

$$f(y) = I_{U_\varepsilon(y_0)}(y),\ y \in [0,1].$$

We have $\nu(f) = 0$ and

$$\tilde{K}f(x) = k(x, U_\varepsilon(y_0)), \quad \forall x \in [0,1],$$

so $\tilde{K}f(x_1) > 0$, $\tilde{K}f(x_2) < 0$ and $\nu(\tilde{K}f) \geq 1$. This contradicts our assumption that $K(x,y)$ preserves variation, so (4) holds.

Let us assume now that $\exists\ x_1, x_2, y_1, y_2 \in [0,1],\ y_1 \neq y_2$ such that $K(x_1,y_1) > 0$ and $K(x_2,y_2) < 0$. Then for some $\varepsilon > 0\ \ U_\varepsilon(y_1) \cap U_\varepsilon(y_2) = \emptyset$ and $\forall y' \in U_\varepsilon(y_1)$, $\forall y'' \in U_\varepsilon(y_2)$ we have $K(x_1,y') > 0$ and $K(x_2,y'') < 0$. Then from (4) $K(x,y) \geq 0$ for $(x,y) \in [0,1] \times U_\varepsilon(y_1)$ and $K(x,y) \leq 0$ for $(x,y) \in [0,1] \times U_\varepsilon(y_2)$. Let us consider a function

$$f(y) = I_{U_\varepsilon(y_1)}(y) - I_{U_\varepsilon(y_2)}(y),\ y \in [0,1]. \tag{5}$$

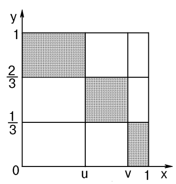
We have $\nu(f) = 1$ and

$$\tilde{K}f(x) = k(x, U_\varepsilon(y_1)) - k(x, U_\varepsilon(y_2)) \geq 0, \forall x \in [0,1],$$

i.e., $\nu(\tilde{K}f) \neq \nu(f)$. So $K(x,y) \geq 0,\ \forall x, y \in [0,1]$ or $K(x,y) \leq 0,\ \forall x, y \in\ [0,1]$.
Without loss of generality we can assume that

$$K(x,y) \geq 0, \quad \forall x, y \in [0,1].$$

Let us prove that there exist numbers $0 < u < v < 1$ such that $K(x,y)$ vanishes outside the set, shaded at the Fig. 1.



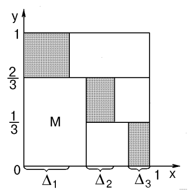


Fig. 1 Fig. 2

Let us consider

$$\varphi_1(y) = I_{[2/3,\ 1]}(y) - I_{[0,\ 2/3)}(y),\ \ y \in [0;1].$$

Since $\nu(\tilde{K}\varphi_1) = \nu(\varphi_1) = 1$, there exists $x_1 \in (0,1)$ such that $\tilde{K}\varphi_1(x_1) > 0$ and, moreover, either $\tilde{K}\varphi_1(x) \geq 0$ for $x > x_1$ or $\tilde{K}\varphi_1(x) \geq 0$ for $x < x_1$.

Notice that for any function $g(x) \in PWC[0,1]$ we have $\nu(g(x)) = \nu(g(1-x))$. Therefore the kernel $K(x,y)$ preserves variation if and only if the kernel $K(1-x,y)$ preserves variation, and we can suppose that

$$\tilde{K}\varphi_1(x) = k(x, [\tfrac{2}{3}, 1]) - k(x, [0, \tfrac{2}{3}]) \geq 0, \quad \forall x < x_1. \tag{6}$$

Denote by $\Delta_1 := [0, x^1]$, where $x^1 = \sup\{x : \tilde{K}\varphi_1(x) > 0\}$. Since $x_1 \in \Delta_1$, we obtain $\text{inter}\Delta_1 \neq \emptyset$ (henceforth by interA we denote the interior of the set A) and $[0, x_1] \subset \Delta_1$. We have $\forall x \in \Delta_1\ \tilde{K}\varphi_1(x) \geq 0$, i.e.

$$k(x, [\tfrac{2}{3}, 1]) \geq k(x, [0, \tfrac{2}{3}]), \quad \forall x \in \Delta_1. \tag{7}$$

Let us consider

$$\varphi_3(y) = I_{[0,\ 1/3)}(y) - I_{[1/3,\ 1]}(y),\ \ y \in [0,1].$$

Since $\nu(\tilde{K}\varphi_3) = \nu(\varphi_3) = 1$, there exists $x_3 \in (0,1)$ such that $\tilde{K}\varphi_3(x_3) > 0$ and, moreover, either $\tilde{K}\varphi_3(x) \geq 0$ for $x > x_3$ or $\tilde{K}\varphi_3(x) \geq 0$ for $x < x_3$ holds.

Now we will show that $x_3 \notin \Delta_1$ and $\tilde{K}\varphi_3(x) \geq 0,\ \forall x \geq x_3$. Since $\forall x \in \Delta_1$,

$$k(x,[0,\tfrac{1}{3}]) \leq k(x,[0,\tfrac{2}{3}]) \leq [\text{by } (7)]$$
$$k \leq (x,[\tfrac{2}{3},1]) \leq k(x,[\tfrac{1}{3},1]),$$

we have $\forall x \in \Delta_1$ $\tilde{K}\varphi_3(x) \leq 0$ and $\tilde{K}\varphi_3(x_1) < 0$. Therefore, $x_3 \notin \Delta_1$ and $\forall x > x_3$ $\tilde{K}\varphi_3(x) \geq 0$.

Denote by $\Delta_3 := [x^3, 1]$, where $x^3 = \inf\{x : \tilde{K}\varphi_3(x) > 0\}$. Since $x_3 \in \Delta_3$, we obtain that $\text{inter}\Delta_3 \neq \emptyset$ and $[x_3, 1] \subset \Delta_3$. We have $\tilde{K}\varphi_3(x) \geq 0\ \ \forall x \in \Delta_3$, i.e.,

$$k(x,[0,\tfrac{1}{3}]) \geq k(x,[\tfrac{1}{3},1]), \quad \forall x \in \Delta_3.$$

Let us consider

$$\varphi_2(y) = I_{[1/3,\ 2/3]}(y) - I_{[0,\ 1/3)\cup(2/3,\ 1]}(y),\ \ y \in [0,1]. \tag{8}$$

Denote by $\Delta_2 := [x_1^2, x_2^2]$ an intersection of all closed intervals, which contain the set $\{x : \tilde{K}\varphi_2(x) > 0\}$. Notice that $\text{inter}\Delta_2 \neq \emptyset$, since $\nu(\tilde{K}\varphi_2) = \nu(\varphi_2) = 2$ and $\tilde{K}\varphi_2 \in C[0,1]$.

Since $\forall x \in \Delta_1$,

$$k(x,[\tfrac{1}{3},\tfrac{2}{3}]) \leq k(x,[0,\tfrac{2}{3}]) \leq [\text{by } (7)]$$
$$\leq k(x,[\tfrac{2}{3},1]) \leq k(x,[0,\tfrac{1}{3}]) + k(x,[\tfrac{2}{3},1]),$$

we have $(\forall x \in \Delta_1)$ $\tilde{K}\varphi_2(x) \leq 0$. Therefore $\text{inter}\Delta_2 \cap \Delta_1 = \emptyset$. Analogously, using (8), we can show that $\text{inter}\Delta_2 \cap \Delta_3 = \emptyset$.

Let us prove that we can take the left end of the interval Δ_2 as u and the right end of Δ_2 as v.

At first, we will prove that $K(x,y) = 0$ for all $(x,y) \in M$, where $M = [0, x_1^2] \times [0, \frac{2}{3}]$, where x_1^2 is the left end of interval Δ_2 (the set M is shown at the Fig. 2).

Let us consider for every $k \geq 4$

$$f_k(y) = I_{[2/3-1/k,\ 2/3]}(y) - I_{[0,\ 2/3-1/k)\cup(2/3,\ 1]}(y),\ \ y \in [0,1].$$

Since $\nu(\tilde{K}f_k) = \nu(f_k) = 2$, there exists $\xi_k \in (0,1)$ such that $\tilde{K}f_k(\xi_k) > 0$.

We will show that $\tilde{K}\varphi_2(\xi_k) > 0$. We have

$$\begin{aligned} k(\xi_k, [\tfrac{1}{3}, \tfrac{2}{3}]) &\geq k(\xi_k, [\tfrac{2}{3} - \tfrac{1}{k}, \tfrac{2}{3}]) \\ > k(\xi_k, [0,1] \setminus [\tfrac{2}{3} - \tfrac{1}{k}, \tfrac{2}{3}]) &\geq k(\xi_k, [0,1] \setminus [\tfrac{1}{3}, \tfrac{2}{3}]), \end{aligned}$$

and therefore $\xi_k \in \Delta_2$. Then $\forall k \geq 4$ we have $\xi_k \geq x_1^2$, where x_1^2 is the left end of Δ_2.

Let us consider

$$f_k^1(y) = I_{[2/3-1/k,\ 2/3)}(y) - I_{[0,\ 2/3-1/k)}(y),\ y \in [0,1].$$

We will show that $\tilde{K}f_k^1(x) \geq 0,\ \forall x < x_1^2$. Notice that $\nu(\tilde{K}f_k^1) = \nu(f_k^1) = 1$ and $\tilde{K}f_k^1(x) = k(x, [\frac{2}{3} - \frac{1}{k}, \frac{2}{3}]) - k(x, [0, \frac{2}{3} - \frac{1}{k}]) \geq\ \tilde{K}f_k(x)$, for $x \in [0,1]$.

Since

$$k(x_3, [\tfrac{2}{3} - \tfrac{1}{k}, \tfrac{2}{3}]) \leq k(x_3, [\tfrac{1}{3}, 1]) < k(x_3, [0, \tfrac{1}{3}]) \leq k(x_3, [\tfrac{2}{3} - \tfrac{1}{k}]),$$

it follows that $\tilde{K}f_k^1(x_3) < 0$. Moreover, $\tilde{K}f_k^1(\xi_k) \geq \tilde{K}f_k(\xi_k) > 0$ and $\xi_k < x_3$, since $\xi_k \in \Delta_2,\ x_3 \in \Delta_3$ and Δ_2 lies to the left of Δ_3.

As $\nu(\tilde{K}f_k^1) = 1$, then $\tilde{K}f_k^1(x) \geq 0\ \ \forall x \leq \xi_k$. And since $\xi_k > x_1^2$, we have $\tilde{K}f_k^1(x) \geq 0\ \ \forall x \leq x_1^2$. So for $\forall k \geq 4$ and $\forall x \leq x_1^2$ we get

$$k(x, [\tfrac{2}{3} - \tfrac{1}{k}, \tfrac{2}{3}]) \geq k(x, [0, \tfrac{2}{3} - \tfrac{1}{k}]).$$

As $k \to \infty$ we obtain $0 \geq k(x, [0, \frac{2}{3}])$ provided $x < x_1^2$, i.e., $K(x,y) = 0$ for all $(x,y) \in M = [0, x_1^2] \times [0, \frac{2}{3}]$, where x_1^2 is the left end of interval Δ_2.

Repeating this reasoning for the kernel $K(1-x, 1-y)$, which also preserves variation, and using property (6) , we obtain that $K(x,y) = 0$ for all $(x,y) \in\ [x_2^2, 1] \times [\frac{1}{3}, 1]$, where x_2^2 is the right end of the interval Δ_2.

We will show that $K(x,y) = 0$ for all $(x,y) \in S$, where $S = \Delta_2 \times [0, \frac{1}{3}]$ (the set S is shown at the Fig. 3).

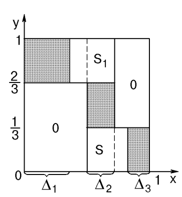


Fig. 3

Let us consider the kernel $K_1(x,y) : [0,1] \times [0, \frac{2}{3}] \to \mathbf{R}$, which is a restriction of the kernel $K(x,y)$. Since for any $f_1(y) \in PWC[0, \frac{2}{3}]$

$$\nu(f_1) = \nu(f) = \nu(\tilde{K}f) = \nu\left(\int\limits_0^{2/3} K(x,y)f(y)dy\right) = \nu(\tilde{K}_1 f_1),$$

where

$$f(y) = \begin{cases} f_1(y), & \text{if } y \in [0, \frac{2}{3}], \\ 0, & \text{if } y \in (\frac{2}{3}, 1], \end{cases}$$

the kernel $K_1(x,y)$ also preserves variation.

Let us take a partition $0 < \frac{1}{6} < \frac{1}{3} < \frac{2}{3}$ of the interval $[0, \frac{2}{3}]$. Our further construction will be analogous to the previous one.

Let us consider

$$\tilde{\varphi_1}(y) = I_{[1/3,\ 2/3]}(y) - I_{[0,\ 1/3)}(y), \quad y \in [0, \tfrac{2}{3}].$$

Notice that for $x \in \Delta_2$ in view of definition Δ_2 we have

$$\tilde{K_1}\tilde{\varphi_1}(x) = k(x, [\tfrac{1}{3}, \tfrac{2}{3}]) - k(x, [0, \tfrac{1}{3}]) \ge k(x, [\tfrac{1}{3}, \tfrac{2}{3}])$$
$$-k(x, [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1]) = \tilde{K}\varphi_2(x) \ge 0$$

and $\tilde{K_1}\tilde{\varphi_1}(x) > 0$, for $x \in \text{inter}\Delta_2$. Moreover since $K_1(x,y) = 0$ on M, $\tilde{K_1}\tilde{\varphi_1}(x) = 0 \ \ \forall x < x_1^2$, where x_1^2 is the left end of Δ_2. So we obtain

$$\tilde{K_1}\tilde{\varphi_1}(x) \ge 0, \ \forall x \le x_2^2,$$

where x_2^2 is the right end of Δ_2, and on $\{x \in \Delta_2 : \tilde{K}\varphi_2(x) > 0\}$ this inequality is strict (analogously to the inequality (6)), therefore $\tilde{\Delta}_1 \supset [0, x_2^2]$, where $\tilde{\Delta}_1 = [0, \tilde{x}^1]$ and $\tilde{x}^1 = \sup\{x : \tilde{K_1}\tilde{\varphi_1}(x) > 0\}$.

Let us consider also

$$\tilde{\varphi}_3(y) = I_{[0,\ 1/6)}(y) - I_{[1/6,\ 2/3]}(y), \ y \in [0, \tfrac{2}{3}],$$
$$\tilde{\varphi}_2(y) = I_{[1/6,\ 1/3]}(y) - I_{[0,\ 1/6) \cup (1/3,\ 2/3]}(y), \ y \in [0, \tfrac{2}{3}],$$

$\tilde{\Delta}_3 := [\tilde{x}^3, 1]$, where $\tilde{x}^3 = \inf\{x : \tilde{K_1}\tilde{\varphi}_3(x) > 0\}$ and $\tilde{\Delta}_2 := [\tilde{x}_1^2, \tilde{x}_2^2]$ — an intersection of all closed intervals, which contain the set $\{x : \tilde{K_1}\tilde{\varphi}_2(x) > 0\}$.

By the same arguments as in the proof that $K(x,y)$ vanishes on M, we get $K_1(x,y) = 0$ on $[0, \tilde{x}_1^2] \times [0, \frac{1}{3}]$, therefore $K(x,y) = K_1(x,y) = 0$ on $(x,y) \in S = \Delta_2 \times [0, \frac{1}{3}]$ (we take into account that $\Delta_2 \subset \tilde{\Delta}_1 \subset [0, \tilde{x}_1^2]$).

Let us take a partition $\frac{1}{3} < \frac{2}{3} < \frac{5}{6} < 1$ of $[\frac{1}{3}, 1]$ and consider a restriction of the kernel $K(x,y)$ on $[0,1] \times [\frac{1}{3}, 1]$. Analogously to the proof that $K(x,y)$ vanishes on $[x_2^2, 1] \times [\frac{1}{3}, 1]$ and our previous reasoning we obtain that $K(x,y) = 0$ for any $(x,y) \in S_1$, where $S_1 = \Delta_2 \times [\frac{2}{3}, 1]$ (the set S_1 is shown at the Fig. 3).

So we have proved that, taking left and right ends of interval Δ_2 as u and v correspondingly, $K(x,y) = 0$ outside the set, shown at the Fig. 1.

Since $K(x,y) \not\equiv 0$ on $[0,1]^2$ and $K(x,y) \in C([0,1]^2)$, $K(x,y) > 0$ on some $[\alpha, \beta] \times [\gamma, \delta] \subset [0,1]^2$, where $\alpha < \beta,\ \gamma < \delta$. We have $\delta - \gamma \le \frac{1}{3}$.

Let us consider kernels

$$K_{[0,u]}(x,y), K_{[u,v]}(x,y) \text{ and } K_{[v,1]}(x,y),$$

which are the restriction of $K(x,y)$ to the sets $\Delta_1 \times [\frac{2}{3}, 1]$, $\Delta_2 \times [\frac{1}{3}, \frac{2}{3}]$ and $\Delta_3 \times [0, \frac{1}{3}]$ correspondingly.

These kernels, obviously, preserve variation, therefore we can repeat our reasoning, hence $\delta - \gamma \le \frac{1}{9}$.

Repeating our argument, we obtain that $\delta - \gamma \le \frac{1}{3^n}$ for any $n \in \mathbf{N}$, which is impossible. This contradiction concludes the proof. ∎

3. Example

Now we present an example of a kernel $K \in C([0,1]^2)$ such that the corresponding integral transform preserves variation on $K_1[0,1]$.

Let us consider a kernel

$$K(x,y) = \begin{cases} 0, & \text{if } x \in \{0,1\}, y \in [0,1], \\ x(\frac{1}{2} - x) y^{\frac{1}{x}}, & \text{if } (x,y) \in (0, \frac{1}{2}] \times [0,1], \\ (1-x)(x - \frac{1}{2})(1-y)^{\frac{1}{1-x}}, & \text{if } (x,y) \in [\frac{1}{2}, 1) \times [0,1]. \end{cases}$$

This kernel is continuous on $[0,1]^2$. Moreover, $\nu(\tilde{K}f) = \nu(f)$, provided $\nu(f) = 0$ (since $K(x,y) \ge 0$ on $[0,1]^2$). We will show that $\nu(\tilde{K}f) = \nu(f)$, provided $\nu(f) = 1$.

Let $f(y) \in PWC[0,1]$ and $\nu(f) = 1$. Then one of the following conditions holds:

(i) $\exists y_0 \in (0,1)$ such that $f(y) \ge 0$, if $y > y_0$ and $f(y) \le 0$, if $y < y_0$;

(ii) $\exists y_0 \in (0,1)$ such that $f(y) \le 0$, if $y > y_0$ and $f(y) \ge 0$, if $y < y_0$.

Without loss of generality we can assume that (i) holds. We will show that $\nu(\tilde{K}f) \ge 1$. Since $f(y) \in PWC[0,1]$ and $\nu(f) = 1$, we have $\inf_{y \in [0,1]} f(y) < 0$ and $\exists [y_1, y_2] \subset [0,1]$ such that $\inf_{y \in [y_1, y_2]} f(y) > 0$. Denote by

$$m := -\inf_{y \in [0,1]} f(y) > 0, \quad M := \inf_{y \in [y_1, y_2]} f(y) > 0.$$

Since for any $x \in (0, \frac{1}{2})$ and for any y_1, y_2 $(y_0 \le y_1 < y_2)$:

$$\frac{\int\limits_{y_1}^{y_2} K(x,y)dy}{\int\limits_{0}^{y_1} K(x,y)dy} = \left(\frac{y_2}{y_1}\right)^{\frac{1}{x}+1} - 1 \to +\infty, \quad x \to 0,$$

there exists $x_1 \in (0, \frac{1}{2})$ such that

$$M \int\limits_{y_1}^{y_2} K(x_1,y)dy > m \int\limits_{0}^{y_1} K(x_1,y)dy, \tag{9}$$

therefore $\tilde{K}f(x_1) > 0$.

In the same way, using the fact that for any $x \in (\frac{1}{2}, 1)$ and for any y_3, y_4 $(y_3 < y_4 \le y_0)$, we have

$$\frac{\int\limits_{y_3}^{y_4} K(x,y)dy}{\int\limits_{y_4}^{1} K(x,y)dy} = \left(\frac{1-y_3}{1-y_4}\right)^{\frac{1}{1-x}+1} - 1 \to +\infty, \quad x \to 1,$$

we can prove that there exists $x_2 \in (\frac{1}{2}, 1)$ such that $\tilde{K}f(x_2) < 0$. So $\nu(\tilde{K}f) \ge 1$.

Now we will show that $\nu(\tilde{K}f) \le 1$. Let us consider a continuous on $(0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ function:

$$g(x) = \frac{1}{K(x, y_0)} \ : \ (0, \tfrac{1}{2}) \cup (\tfrac{1}{2}, 1) \to (0, +\infty).$$

We will prove that $g(x)\tilde{K}f(x)$ is monotonically nonincreasing on $(0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Let $0 < x' < x'' < \frac{1}{2}$. Then

$$\begin{aligned} g(x')\tilde{K}f(x') &= \int\limits_0^{y_0} \left(\frac{y}{y_0}\right)^{\frac{1}{x'}} f(y)dy + \int\limits_{y_0}^{1} \left(\frac{y}{y_0}\right)^{\frac{1}{x'}} f(y)dy \ge [\text{by } (i)] \\ &\ge \int\limits_0^{y_0} \left(\frac{y}{y_0}\right)^{\frac{1}{x''}} f(y)dy + \int\limits_{y_0}^{1} \left(\frac{y}{y_0}\right)^{\frac{1}{x''}} f(y)dy = g(x'')\tilde{K}f(x''), \end{aligned}$$

so the function $g(x)\tilde{K}f(x)$ is monotonically nonincreasing on $(0, \frac{1}{2})$. In the same way we can prove that $g(x)\tilde{K}f(x)$ is monotonically nonincreasing on $(\frac{1}{2}, 1)$. Moreover, we have

$$\lim_{x'\to\frac{1}{2}-0} g(x')\tilde{K}f(x') = \int_0^{y_0}\left(\frac{y}{y_0}\right)^2 f(y)dy + \int_{y_0}^{1}\left(\frac{y}{y_0}\right)^2 f(y)dy \geq [\text{by } (i)]$$

$$\geq \int_0^{y_0} f(y)dy + \int_{y_0}^{1} f(y)dy \geq \int_0^{y_0}\left(\frac{1-y}{1-y_0}\right)^2 f(y)dy + \int_{y_0}^{1}\left(\frac{1-y}{1-y_0}\right)^2 f(y)dy$$

$$= \lim_{x'\to\ frac12+0} g(x'')\tilde{K}f(x''),$$

So for any $x_1, x_2 \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, $x_1 < x_2$, we have $g(x_1)\tilde{K}(x_1) < g(x_2)\tilde{K}(x_2)$. Therefore $\nu(g\tilde{K}f) \leq 1$. And since $g(x) > 0$ on $(0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, we have

$$\nu(\tilde{K}f) = \nu(g\tilde{K}f) \leq 1.$$

We have proved, that $\nu(\tilde{K}f) = 1$, so $\tilde{K}$ really preserves variation on $K_1[0,1]$.

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