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On power series whose tails have multiply positive coefficients

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Entire functions whose tails have 4-times positive coefficients are studied. The growth estimate for such functions is given. It is shown that the same growth estimate holds for entire functions of order < 1 whose tails do not have zeros in the angle $\{z : |\arg z| \le 4\pi/5\}$.

1. Introduction and statement of results

Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad a_0 > 0, \tag{1}$$

be a formal power series. Denote by $R(f)$ its radius of convergence. For $R(f) > 0$, denote by

$$t_n(z; f) = \sum_{k=n}^{\infty} a_k z^k, \qquad n = 0, 1, 2, \dots, \tag{2}$$

the tails of the series (1), and set

$$M(r, f) = \max_{|z|=r} |f(z)|, \qquad 0 < r < R(f).$$

In 1997, I.V. Ostrovskii proved the following theorem.

Theorem A ([2]). *Assume $R(f) = \infty$. If all zeros of tails (2), for all sufficiently large n, are real nonpositive, then*

$$\log M(r, f) = O((\log r)^2). \tag{3}$$

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Theorem A is an analogue of Pólya's result [5] of 1913 on power series (1) whose sections $\sum_{k=0}^{n} a_k z^k$ have only real negative zeros for all sufficiently large n. In the joint work of I.V. Ostrovskii and the author [4], a new analogue of Theorem A has been obtained (see Theorem B later in the text). It is based on concept of multiple positivity introduced by M. Fekete [1] in 1912.

Definition. A sequence $\{a_k\}_{k=0}^{\infty}$ of real numbers is said to be *m-times positive* for $m \in \mathbb{N} \cup \{\infty\}$ if all minors of orders less than $m+1$ of the infinite matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \dots \\ 0 & a_0 & a_1 & a_2 & \dots \\ 0 & 0 & a_0 & a_1 & \dots \\ . & . & . & . & \dots \end{pmatrix}$$

are nonnegative.

To formulate precisely the analogue of Theorem A mentioned earlier, let us introduce the following classes.

Denote by R_m, $m \in \mathbb{N} \cup \{\infty\}$, the class of all power series (1) such that the sequences $\{a_n, a_{n+1}, \dots\}$ are m–times positive for all n large enough. Evidently,

$$R_1 \supset R_2 \supset R_3 \supset \dots \supset R_\infty.$$

Series (1) belonging to R_2 may have singularities of rather arbitrary kind and location:

E x a m p l e. ([4]). Let (1) be the power series expansion of

$$f(z) = 1 + \frac{z}{(1-z)^2} + h(z),$$

where $h(z)$ is an arbitrary power series with real coefficients whose radius of convergence is strictly greater than 1. Evidently, for some $\varepsilon > 0$,

$$a_k = k + O((1-\varepsilon)^k), \quad k \to \infty.$$

Hence, $a_k^2 \geq a_{k-1}a_{k+1}$ for k large enough and therefore $\{a_n, a_{n+1}, \dots\}$ is 2-times positive for all sufficiently large n.

The next theorem from [4] shows that for $m \geq 3$ the situation is quite different.

Theorem B ([4]). *If $f \in R_m$ for some $m \geq 3$, then: either*
(i) $R(f) = \infty$ and

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{(\log r)^2} \leq \frac{1}{2\log c}, \qquad c = \frac{1+\sqrt{5}}{2} = 1.613\dots\,; \tag{4}$$

or
(ii) $0 < R(f) < \infty$ *and*

$$f(z) = \frac{A}{1 - z/R(f)} - g(z),$$

where A *is a positive constant and* g *is an entire function with nonnegative coefficients (except at most a finite number) satisfying the condition*

$$\log M(r,g) \le \frac{\log r \cdot \log\log r}{\log 2} + O(\log r), \quad r \to \infty. \tag{5}$$

The bound (4) cannot be improved for entire functions from R_3*. The bound (5) cannot be improved for non entire functions from* R_m *for any* $m \ge 3$*.*

The last theorem means that a series (1) from R_m, $m \ge 3$, either represents an entire function satisfying (4) or has exactly one singularity (a simple pole) in the whole complex plane and satisfies the very restrictive condition (5).

The question arises whether the bound (4) can be improved for entire functions from R_m for $m \ge 4$. In the present work we show that, for *entire* functions belonging to R_m, $m \ge 4$, an estimate better than (4) holds.

Theorem 1. *If* $m \ge 4$*, then an entire function* $f \in R_m$ *satisfies the condition*

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{(\log r)^2} \le \frac{1}{2\log d(R_4)}, \tag{6}$$

where the constant $d(R_4) > 2$ *is independent of* f *and*

$$2.0833 \le d(R_4) \le 2.087.$$

Note that for $m < \infty$ there is a relation between m-times positivity of the sequence $\{a_k\}_{k=0}^n$ and zeros of the corresponding section $\sum_{k=0}^n a_k z^k$ but this relation is a less strict one than for $m = \infty$. I.J. Schoenberg ([6, p. 397, 415]) proved: (i) the necessary condition for $\{a_k\}_{k=0}^n$ to be m-times positive is the nonvanishing of $\sum_{k=0}^n a_k z^k$ in the angle $\{z : |\arg z| \le (\pi m)/(m+n-1)\}$, (ii) the sufficient condition is its nonvanishing in the larger angle $\{z : |\arg z| \le (\pi m)/(m+1)\}$. Both conditions are best possible in terms of the sizes of angles.

As a consequence of Theorem 1 and I.J. Shoenberg's result (ii) we obtain the following

Corollary 1. *Let* $f(z)$ *be an entire function (1) of order strictly less than 1 with real coefficients* a_k*. Assume that for all sufficiently large* n*, the zeros of tails (2) are located in the angle* $\left\{z : |\arg z - \pi| < \frac{\pi}{5}\right\}$*. Then the estimate (6) holds.*

2. Preliminaries

Let

$$\{a_n\}_{n=0}^{\infty}, \quad a_0 > 0, \tag{7}$$

be a sequence of real numbers. Recall two lemmas from [3].

Lemma 1. *Let (7) be a 2–times positive sequence. Set $n = \min\{k : a_k = 0\}$. If n is finite, then $a_k = 0$ for any $k \geq n$.*

Lemma 2. *Let (7) be a 2–times positive sequence without zero terms. Then the sequence $\left\{\frac{a_{k-1}}{a_k}\right\}_{k=1}^{\infty}$ is nondecreasing.*

By Lemma 1, a 2-times positive sequence (7) is either finite (has only finitely many nonzero terms), or is composed entirely of positive terms.

With (7), we associate the generating function (1). It follows from Lemma 2 that the generating function (1) of 2-times positive sequence (7) has a nonzero radius of convergence.

Assume an entire transcendental function f belongs to R_m, $m \geq 2$. Choose n such that $\{a_n, a_{n+1}, \dots\}$ is m-times positive and moreover $a_n > 0$. By Lemma 1, $a_k > 0$ for all $k \geq n$. This permits us to introduce the positive numbers

$$\rho_k = \frac{a_{k-1}}{a_k}, \quad k = n+1, n+2, \dots \tag{8}$$

and

$$\delta_k = \frac{\rho_k}{\rho_{k-1}} = \frac{a_{k-1}^2}{a_k a_{k-2}}, \quad k = n+2, n+3, \dots . \tag{9}$$

It is easy to see that

$$a_k = \frac{a_n}{\rho_{n+1}^{k-n} \prod_{j=n+2}^{k} \delta_j^{k-j+1}} . \tag{10}$$

Since the sequence $\{a_n, a_{n+1}, \dots\}$ is 2-times positive, we have

$$\begin{vmatrix} a_{k-1} & a_k \\ a_{k-2} & a_{k-1} \end{vmatrix} \geq 0, \qquad k \geq n+2,$$

i.e., $a_{k-1}^2 \geq a_k a_{k-2}$, $k \geq n+2$. Therefore,

$$\delta_k \geq 1, \quad k = n+2, n+3, \dots . \tag{11}$$

It turns out that if sequence (7) of positive numbers satisfies a stronger condition than (11), namely $\delta_n \geq A > 1$, then the corresponding generating function (1) is an entire function of order 0. More precisely, the following lemma holds.

Lemma 3 ([3]). *Let (1) be a formal power series with all positive coefficients $a_k > 0$. Assume that there exists a constant $A > 1$ such that*

$$\delta_k \geq A, \quad k \geq k_0, \tag{12}$$

where numbers δ_k are defined by (9). Then the series (1) converges in the whole complex plane $\mathbb{C}$, and

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{(\log r)^2} \leq \frac{1}{2\log A}. \tag{13}$$

The next lemma from [4] gives information about series (1) from R_m, $m \geq 3$.

Lemma 4. *Let sequence (7) belong to R_3. Let n_0 be an integer such that the sequences $\{a_n, a_{n+1}, \dots\}$ are 3-times positive for any $n \geq n_0$. There is the following alternative:*

(I) for any $n \geq n_0 + 2$, $\quad \delta_n \geq c = \frac{1+\sqrt{5}}{2}$,

(II) there exists $q \geq n_0 + 2$, $\quad \delta_q < c$.

In case (I) the assertion (i) of Theorem B is valid while in the case (II) the assertion (ii) is.

We will need the following test of m-times positivity that is contained in [4] and which is due to I.J. Schoenberg (see [6]).

Lemma 5 (Schoenberg's Theorem) ([6]). *Let $\{b_k\}_{k=0}^{\infty}$ be a sequence of positive numbers. Consider the m matrices*

$$B_\nu = \begin{pmatrix} b_0 & b_1 & b_2 & \dots & b_{\nu-1} & \dots \\ 0 & b_0 & b_1 & \dots & b_{\nu-2} & \dots \\ 0 & 0 & b_0 & \dots & b_{\nu-3} & \dots \\ . & . & . & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & b_0 & \dots \end{pmatrix} \qquad \nu = 1, 2, \dots, m,$$

consisting of ν rows and infinitely many columns. Assume that for $\nu = 1, 2, \dots, m$, the matrix B_ν satisfies the condition: all its $\nu \times \nu$–block-minors (i.e., minors consisting of ν consecutive rows and ν consecutive columns) are positive. Then the sequence $\{b_k\}_{k=0}^{\infty}$ is m–times positive.

3. Proof of Theorem 1

3.1. The lower bound for $d(R_4)$

Assuming $f \in R_m$, $m \geq 4$, we choose n_0 such that $\{a_n, a_{n+1}, \dots\}$ are m-times positive for all $n \geq n_0$. The next lemma plays a basic role in the proof of Theorem 1. Without loss of generality we may assume that $a_k > 0$ for all $k \geq n_0$. Thus, the numbers ρ_k, $k \geq n_0 + 1$, and δ_k, $k \geq n_0 + 2$, defined by (8) and (9) respectively, are well defined.

Lemma 6. *Let $\{a_k\}_{k=n}^{\infty}$, $a_n > 0$, $n \geq 1$, be a 4–times positive sequence. Then*

$$\left(\delta_{n+2} - \frac{3}{2}\right)^2 \geq \frac{5}{4} - \frac{2}{\delta_{n+3}} + \frac{1}{\delta_{n+2}\delta_{n+3}^2\delta_{n+4}}. \tag{14}$$

P r o o f. Using (9), we have

$$I_{n+1} = \begin{vmatrix} a_{n+1} & a_{n+2} & a_{n+3} & a_{n+4} \\ a_n & a_{n+1} & a_{n+2} & a_{n+3} \\ 0 & a_n & a_{n+1} & a_{n+2} \\ 0 & 0 & a_n & a_{n+1} \end{vmatrix}$$

$$= a_{n+1}^2 a_n a_{n+2} \left(\frac{a_{n+1}^2}{a_n a_{n+2}} + 2\frac{a_n a_{n+3}}{a_{n+1}a_{n+2}} - 3 - \frac{a_n^2 a_{n+4}}{a_{n+2}a_{n+1}^2} + \frac{a_n a_{n+2}}{a_{n+1}^2}\right)$$

$$= a_{n+1}^2 a_n a_{n+2} \left(\delta_{n+2} + \frac{2}{\delta_{n+2}\delta_{n+3}} - 3 - \frac{1}{\delta_{n+2}^2\delta_{n+3}^2\delta_{n+4}} + \frac{1}{\delta_{n+2}}\right).$$

4–times positivity of $\{a_k\}_{k=n}^{\infty}$ implies $I_{n+1} \geq 0$, from which the inequality (14) follows. ■

Lemma 7. *Let an entire function (1) belong to R_4. Then there exists an integer n_0 such that $\delta_k > 2$ for all $k \geq n_0 + 2$.*

P r o o f. Let an entire function f belong to $R_4 \subset R_3$. By Lemma 4, there exists an integer n_0 such that $\delta_n \geq c$, $n \geq n_0 + 2$. Assume that there exists $n' \geq n_0 + 2$ such that $\delta_{n'} < 2$. Then, by (14),

$$\frac{1}{4} > \left(\delta_{n'} - \frac{3}{2}\right)^2 > \frac{5}{4} - \frac{2}{\delta_{n'+1}},$$

whence $\delta_{n'+1} < 2$. Applying the same procedure successively, we get $\delta_n < 2$ for all $n \geq n' + 1$. Since

$$\frac{5}{4} - \frac{2}{x} \geq \left(x - \frac{3}{2}\right)^2,$$

for $c \le x \le 2$, then for $n \ge n' + 1$ we have

$$\frac{5}{4} - \frac{2}{\delta_n} \ge \left(\delta_n - \frac{3}{2}\right)^2 > \frac{5}{4} - \frac{2}{\delta_{n+1}},$$

whence $\delta_{n+1} < \delta_n$. Therefore, there exists

$$\lim_{n\to\infty} \delta_n =: b \in [c, 2]. \tag{15}$$

On the other hand, this limit, by (14), satisfies the condition

$$\left(b - \frac{3}{2}\right)^2 \ge \frac{5}{4} - \frac{2}{b} + \frac{1}{b^4} > \frac{5}{4} - \frac{2}{b}$$

and, hence, $b \notin [c; 2]$ in contradiction with (15). ■

Hence, by Lemma 7, for any function $f \in R_4$,

$$\delta(f) := \liminf_{n\to\infty} \delta_n \ge 2. \tag{16}$$

Define

$$\delta(R_4) = \inf\{\delta(f) : f \in R_4\}. \tag{17}$$

Lemma 8. *We have*

$$\delta(R_4) \ge d,$$

where $d(= 2.0833\dots)$ *is the biggest positive root of*

$$2x^5 - 6x^4 + 2x^3 + 5x^2 - 3x + 1 = 0.$$

P r o o f. Let f belong to R_4 and n_0 be such an integer that sequences $\{a_n, a_{n+1}, \dots\}$ are 4–times positive for all $n \ge n_0$. By (14),

$$\delta_{n+1} + \frac{2}{\delta_{n+1}\delta_{n+2}} - 3 - \frac{1}{\delta_{n+1}^2\delta_{n+2}^2\delta_{n+3}} + \frac{1}{\delta_{n+1}} \ge 0, \qquad n \ge n_0 + 1, \tag{18}$$

whence

$$\delta_{n+1} + \frac{2}{\delta_{n+1}\delta_{n+2}} - 3 + \frac{1}{\delta_{n+1}} \ge 0, \qquad n \ge n_0 + 1. \tag{19}$$

It follows from (19) that

$$\delta_{n+2} + \frac{2}{\delta_{n+2}\delta_{n+3}} - 3 + \frac{1}{\delta_{n+2}} \ge 0, \qquad n \ge n_0,$$

or,

$$\frac{1}{2\delta_{n+1}^2} + \frac{1}{\delta_{n+1}^2\delta_{n+2}^2\delta_{n+3}} - \frac{3}{2\delta_{n+1}^2\delta_{n+2}} + \frac{1}{2\delta_{n+2}^2\delta_{n+1}^2} \geq 0, \qquad n \geq n_0. \tag{20}$$

By adding (18) to (20), we obtain

$$f(x,y) := x + \frac{2}{xy} - 3 + \frac{1}{x} + \frac{1}{2x^2} - \frac{3}{2x^2y} + \frac{1}{2x^2y^2} \geq 0, \tag{21}$$

where $x = \delta_{n+1}$ and $y = \delta_{n+2}$.

Evidently,

$$\frac{\partial f(x,y)}{\partial y} = -\frac{2}{xy^2} + \frac{3}{2x^2y^2} - \frac{1}{x^2y^3} \leq -\frac{2}{xy^2} + \frac{3}{2x^2y^2} \leq 0$$

for $x \geq 3/4$.

Note that if $x \leq y$ then $f(x,x) \geq f(x,y) \geq 0$, or the same

$$2x^4 f(x,x) = 2x^5 - 6x^4 + 2x^3 + 5x^2 - 3x + 1 \geq 0. \tag{22}$$

Two the biggest positive zeros of $2x^4 f(x,x)$ are approximately equal to $d := 2.0833\ldots$ and $1.335 < 2$. It follows that if $x \geq y$ and $x \geq 2$ then $x \geq d$.

Assume that there exists δ_s such that $\delta_s < d$. Then $\delta_{s+1} < \delta_s < d$, $\delta_{s+2} < \delta_{s+1} < d$, $\ldots$, i.e., $\{\delta_n\}_{n=s}^{\infty}$ is a nonincreasing sequence. Therefore there exists $\lim_{s_n \to \infty} \delta_n =: \delta_0$. It follows from (21) that $\delta_0 \geq d$. This contradiction implies that $\delta_n \geq d$ for all $n \geq n_0 + 2$. ■

Define $d(f)$ and $d(R_4)$ by formulas

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{(\log r)^2} =: \frac{1}{2\log d(f)} \tag{23}$$

and

$$d(R_4) := \inf\{d(f) : f \in R_4\}. \tag{24}$$

It follows from Lemma that $d(f) \geq \delta(f)$ and hence $d(R_4) \geq \delta(R_4)$. Then, by Lemma 8, $d(R_4) \geq 2.0833\ldots$.

3.2. The upper bound for $d(R_4)$

Lemma 9. *The entire function*

$$f_q(z) = \sum_{k=0}^{\infty} q^{-\frac{(k-1)k}{2}} z^k$$

belongs to R_4 for all $q > a$, where $a (= 2.08679\ldots)$ is the biggest positive root of

$$x^6 - 3x^5 + x^4 + 2x^3 - 1 = 0. \tag{25}$$

P r o o f. Consider the functions

$$f_q(z) = \sum_{k=0}^{\infty} q^{-\frac{k(k-1)}{2}} z^k.$$

We have

$$t_n(z, f_q) = z^n \sum_{k=0}^{\infty} q^{-\frac{(k+n)(k+n-1)}{2}} z^k$$

$$= q^{-\frac{n^2}{2}+\frac{n}{2}} z^n \sum_{k=0}^{\infty} q^{-\frac{k(k-1)}{2}} (zq^{-n})^k = q^{-\frac{n^2}{2}+\frac{n}{2}} z^n f_q(q^{-n}z).$$

Thus, if for some q, the sequence $\left\{q^{-\frac{k(k-1)}{2}}\right\}_{k=0}^{\infty}$ is 4-times positive, then all the sequences $\left\{q^{-\frac{k(k-1)}{2}}\right\}_{k=n}^{\infty}$, $n \geq 1$, are also 4-times positive. Therefore, to show that $f_q(z) \in R_4$ for all $q > a$, it is enough to show that the sequences $\left\{q^{-\frac{k(k-1)}{2}}\right\}_{k=0}^{\infty}$ are 4-times positive for all $q > a$.

To do this we will use Lemma 5. Construct

$$B_1 := (b_0, b_1, b_2, b_3, \dots), \qquad B_2 = \begin{pmatrix} b_0 & b_1 & b_2 & b_3 & \dots \\ 0 & b_0 & b_1 & b_2 & \dots \end{pmatrix},$$

$$B_3 = \begin{pmatrix} b_0 & b_1 & b_2 & b_3 & \dots \\ 0 & b_0 & b_1 & b_2 & \dots \\ 0 & 0 & b_0 & b_1 & \end{pmatrix}, \qquad B_4 = \begin{pmatrix} b_0 & b_1 & b_2 & b_3 & b_4 & \dots \\ 0 & b_0 & b_1 & b_2 & b_3 & \dots \\ 0 & 0 & b_0 & b_1 & b_2 & \dots \\ 0 & 0 & 0 & b_0 & b_1 & \dots \end{pmatrix}.$$

Positivity of all 1×1-block-minors of B_1 is trivial. All 2×2-block-minors of B_2 are positive for all $q > 1$ and hence they are positive for $q > a > 2$. All 3×3-block-minors of B_3 are one of the kind:

$$A_1 = \begin{vmatrix} 1 & 1 & q^{-1} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}, \quad A_2 = \begin{vmatrix} 1 & q^{-1} & q^{-3} \\ 1 & 1 & q^{-1} \\ 0 & 1 & 1 \end{vmatrix} = q^{-3}(q-1)(q^2-q-1),$$

$$\begin{aligned} A_n &= \begin{vmatrix} q^{-\frac{n(n-1)}{2}} & q^{-\frac{(n+1)n}{2}} & q^{-\frac{(n+2)(n+1)}{2}} \\ q^{-\frac{(n-1)(n-2)}{2}} & q^{-\frac{n(n-1)}{2}} & q^{-\frac{(n+1)n}{2}} \\ q^{-\frac{(n-2)(n-3)}{2}} & q^{-\frac{(n-1)(n-2)}{2}} & q^{-\frac{n(n-1)}{2}} \end{vmatrix} \\ &= q^{(-3n^2+3n-10)/2} \begin{vmatrix} 1 & 1 & 1 \\ 1 & q & q^2 \\ 1 & q^2 & q^4 \end{vmatrix}, \, n \geq 3. \end{aligned}$$

Thus, for all $n \geq 3$, $A_n > 0$, if $q > 1$ and $A_2 > 0$, if $q > \frac{1+\sqrt{5}}{2}$. Hence, if $q > \frac{1+\sqrt{5}}{2}$, then all 3×3-block-minors of B_3 are positive.

All 4×4-block -minors of B_4 are one of the kind

$$C_1 = \begin{vmatrix} 1 & 1 & q^{-1} & q^{-3} \\ 0 & 1 & 1 & q^{-1} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix},$$

$$C_2 = \begin{vmatrix} 1 & q^{-1} & q^{-3} & q^{-6} \\ 1 & 1 & q^{-1} & q^{-3} \\ 0 & 1 & 1 & q^{-1} \\ 0 & 0 & 1 & 1 \end{vmatrix} = q^{-6}(q^6 - 3q^5 + q^4 + 2q^3 - 1),$$

$$C_n = \begin{vmatrix} q^{-\frac{n(n-1)}{2}} & q^{-\frac{(n+1)n}{2}} & q^{-\frac{(n+2)(n+1)}{2}} & q^{-\frac{(n+3)(n+2)}{2}} \\ q^{-\frac{(n-1)(n-2)}{2}} & q^{-\frac{n(n-1)}{2}} & q^{-\frac{(n+1)n}{2}} & q^{-\frac{(n+2)(n+1)}{2}} \\ q^{-\frac{(n-2)(n-3)}{2}} & q^{-\frac{(n-1)(n-2)}{2}} & q^{-\frac{n(n-1)}{2}} & q^{-\frac{(n+1)n}{2}} \\ q^{-\frac{(n-3)(n-4)}{2}} & q^{-\frac{(n-2)(n-3)}{2}} & q^{-\frac{(n-1)(n-2)}{2}} & q^{-\frac{n(n-1)}{2}} \end{vmatrix}$$

$$= q^{-4n^2+10n-24} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & q & q^2 & q^3 \\ 1 & q^2 & q^4 & q^6 \\ 1 & q^3 & q^6 & q^9 \end{vmatrix}, \quad n \geq 4,$$

$$C_3 = \begin{vmatrix} q^{-1} & q^{-3} & q^{-6} & q^{-10} \\ 1 & q^{-1} & q^{-3} & q^{-6} \\ 1 & 1 & q^{-1} & q^{-3} \\ 0 & 1 & 1 & q^{-1} \end{vmatrix} = q^{-2} \begin{vmatrix} 1 & q^{-2} & q^{-5} & q^{-9} \\ 1 & q^{-1} & q^{-3} & q^{-6} \\ 1 & 1 & q^{-1} & q^{-3} \\ 0 & q & q & 1 \end{vmatrix}$$

$$= q^{-18} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & q & q^2 & q^3 \\ 1 & q^2 & q^4 & q^6 \\ 0 & q^3 & q^6 & q^9 \end{vmatrix} \geq q^{-18} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & q & q^2 & q^3 \\ 1 & q^2 & q^4 & q^6 \\ 1 & q^3 & q^6 & q^9 \end{vmatrix}.$$

We have, $C_2 > 0$, if $q > a$. For all $n \geq 3$, $C_n > 0$, if $q > 1$. Thus all 4×4-block-minors of B_4 are positive if $q > a$.

By Lemma 5, the sequences $\left\{q^{-\frac{k(k-1)}{2}}\right\}_{k=0}^{\infty}$ are 4-times positive for all $q > a$.

■

Using Cauchy's inequality $M(r, f_q) \geq q^{-\frac{k(k-1)}{2}} r^k$ with k equal to the integer part of $\log r / \log q$, we get

$$\log M(r, f_q) \geq \frac{\log^2 r}{2 \log q} + O(\log r), \quad r \to \infty.$$

Thus, $d(f_q) \leq q$ for all $q > a$. On the other hand, by Lemma 3, $d(f_q) \geq q$ for all $q > a$. It implies $d(f_q) = q$ and hence, by Lemma 9, $d(R_4) \leq a$.

4. Proof of Corollary 1

Let $t_n(z, f)$ be a tail of $f(z)$ such that all its zeros $z_1^{(n)}, z_2^{(n)}, \dots$, are situated in the angle $\{z : |argz - \pi| < \pi/5\}$. Then

$$t_n(z,f) = a_n z^n \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k^{(n)}}\right).$$

By Schoenberg's result (ii) the sequence of coefficients of the polynomial

$$P_m^{(n)}(z) = a_n z^n \prod_{|z_k^{(n)}|<m} \left(1 - \frac{z}{z_k^{(n)}}\right)$$

is 4-times positive. Since coefficients of $t_n(z, f)$ can be approximated by those of $P_m^{(n)}$ then $t_n(z, f)$ has also 4-times positive coefficients. Therefore, $f \in R_4$. ■

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