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On stabilizability of evolution partial differential equations on $\mathbb{R}^n \times [0, +\infty)$ by time-delayed feedback controls

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The problem of stabilizability by time-delayed feedback control is investigated for an evolution partial differential equation on $\mathbb{R}^n \times [0, +\infty)$. We use the Tarski–Seidenberg theorem and its corollaries to obtain some estimates of semi-algebraic functions on semi-algebraic sets and obtain estimates of the real parts of quasipolynomial zeros. These estimates make it possible to apply the Fourier transform method to investigate the stabilizability problem. We utilise some results of the theory of ordinary differential-difference equations to study a "dual" system obtained from the original system with time-delayed feedback control by applying the Fourier transform. We also give some examples of stabilizable and non-stabilizable systems.

0. Introduction

One of the most general-accepted ways to study control systems with distributed parameters is their interpretation in the form

$$\frac{d\mathrm{w}}{dt} = A\mathrm{w} + Bu, \qquad t > 0, \tag{0.1}$$

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where $\mathrm{w} : (0, +\infty) \longrightarrow \mathcal{H}$ is an unknown function; $u : (0, +\infty) \longrightarrow H$ is a control; $\mathcal{H}$, H are Banach spaces; A is an infinitesimal operator in $\mathcal{H}$; $B : H \longrightarrow \mathcal{H}$ is a linear bounded operator (see, e.g., [2, 3, 10, 12, 13, 16, 18, 19]). An important advantage of this approach is a possibility to employ the ideas and the technique of the semigroup operator theory. At the same time it should be noticed that the most substantial and important for applications results on operator semigroups deal with the case when the semigroup generator A has a discrete spectrum and may be treated in terms of its eigenvalues and its eigenelements. These assumptions correspond to differential equations in domains bounded with respect to space variables but in general they are not true for domains unbounded with respect to them.

In the present paper we consider the following equation

$$\frac{\partial^2 w(x,t)}{\partial t^2} + 2a_1(D_x)\frac{\partial w(x,t)}{\partial t} + a_0(D_x)\,w(x,t) + b(D_x)\,u(x,t) = 0, \quad x \in \mathbb{R}^n,\ t > 2h, \qquad (0.2)$$

where $D_x = (-i\partial/\partial x_1, \dots, -i\partial/\partial x_n)$, $a_0(\sigma)$, $a_1(\sigma)$ and $b(\sigma)$ are polynomials, $b(\sigma) \not\equiv 0$ on $\mathbb{R}^n$, $w : \mathbb{R}^n \times (h, +\infty) \longrightarrow \mathbb{C}$ is an unknown function, $u : \mathbb{R}^n \times (h, +\infty) \longrightarrow \mathbb{C}$ is a control, $h > 0$.

In Section 1 we investigate stabilizability of the telegraph equation, the wave equation and some model equation.

We use the following Sobolev spaces $C^q_\gamma = \{g \in C^q(\mathbb{R}^n) \mid \|g\|^q_\gamma < +\infty\}$, $\|g\|^q_\gamma = \sup\{|D^\alpha_x g(x)|(1+|x|)^{-\gamma} \mid x \in \mathbb{R}^n \wedge |\alpha| \le q\}$, $\mathcal{C}^q_\gamma = \{g \mid [\forall t \in [0,2h] g(\cdot,t) \in C^q_\gamma] \wedge [\forall \alpha \in \mathbb{N}^n_0 (|\alpha| \le q \Rightarrow D^\alpha_x g \in C(\mathbb{R}^n \times [0,2h]))] \wedge [|||g|||^q_\gamma < +\infty]\}$, $|||g|||^q_\gamma = \sup\{\|g(\cdot,t)\|^q_\gamma \mid t \in [0,2h]\}$, $\mathbf{C}^q_\gamma = \{g \mid [\forall t \in [0,+\infty) g(\cdot,t) \in C^q_\gamma] \wedge [\forall \alpha \in \mathbb{N}^n_0 (|\alpha| \le q \Rightarrow D^\alpha_x g \in C(\mathbb{R}^n \times [0,+\infty)))] \wedge [\sup\{\|g(\cdot,t)\|^q_\gamma \mid t \in [0,+\infty)\} < +\infty]\}$, where $q \in \mathbb{N}_0 = \mathbb{N} \cap \{0\}$, $\gamma \in \mathbb{R}$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0$ is a multi-index, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $|\cdot|$ is the Euclidean norm of $\mathbb{R}^k$. We also use the spaces $C^q_\infty = \bigcap_{\gamma\in\mathbb{R}} C^q_\gamma$, $C^{-\infty}_\gamma = \bigcup_{q\in\mathbb{N}_0} C^q_\gamma$, $C^q_{-\infty} = \bigcup_{\gamma\in\mathbb{R}} C^q_\gamma$, $\mathcal{M} = \bigcap_{q\in\mathbb{N}_0} C^q_{-\infty}$, $\mathcal{S} = \bigcap_{q\in\mathbb{N}_0} C^q_\infty$, and for $P \in \mathcal{M}$ denote by $P(D_x)$ the following operator $P(D_x)f = \mathcal{F}^{-1}(P\mathcal{F}f)$, $f \in \mathcal{S}'$, where $\mathcal{F}$ is the Fourier transform operator (such an operator P is called a pseudodifferential one).

Further we assume throughout the paper that $\gamma \ge 0$ and $h > 0$ are fixed.

Definition 0.1. *Equation (0.2) is said to be stabilizable in $C^{-\infty}_\gamma$ if there exist such functions $p_0, p_1, p_2 \in \mathcal{M}$ that for each $p \in \mathbb{N}_0$ there exists $q \in \mathbb{N}_0$ such that for every solution of this system with the control*

$$u(x,t) \equiv p_0(D_x)\,w(x,t-2h) + p_1(D_x)\,w(x,t-h) + p_2(D_x)\frac{\partial w(x,t-h)}{\partial t} \qquad (0.3)$$

under the initial condition $\begin{pmatrix} w \\ \partial w/\partial t \end{pmatrix} \in \mathcal{C}^q_\gamma$ *we have* $\begin{pmatrix} w \\ \partial w/\partial t \end{pmatrix} \in \mathbf{C}^p_\gamma$ *and*

$$\left\| \begin{pmatrix} w(\cdot,t) \\ \partial w(\cdot,t)/\partial t \end{pmatrix} \right\|^p_\gamma \longrightarrow 0 \qquad \text{as } t \longrightarrow +\infty. \tag{0.4}$$

Such functions p_0, p_1, p_2 are called stabilizing functions for equation (0.2), and such a control u is called a stabilizing control for this system.

In the stabilizability problem under consideration a delay appears in the control because, in fact, any control cannot be realized instantly (without a delay).

Denoting $\mathrm{w} = \begin{pmatrix} w \\ \partial w/\partial t \end{pmatrix}$, we conclude that equation (0.2) is equivalent to the following system:

$$\frac{\partial \mathrm{w}(x,t)}{\partial t} = A(D_x)\,\mathrm{w}(x,t) + b(D_x)\,\mathrm{u}(x,t), \qquad x \in \mathbb{R}^n,\ t > 2h, \tag{0.5}$$

where $A = \begin{pmatrix} 0 & 1 \\ -a_0 & -2a_1 \end{pmatrix}$. A control of the form (0.3) for equation (0.2) corresponds to the control

$$\mathrm{u}(\cdot,t) \equiv P_0(D_x)\,\mathrm{w}(\cdot,t-2h) + P_1(D_x)\,\mathrm{w}(\cdot,t-h), \tag{0.6}$$

where $P_0 = \begin{pmatrix} 0 & 0 \\ -p_0 & 0 \end{pmatrix}$, $P_1 = \begin{pmatrix} 0 & 0 \\ -p_1 & -p_2 \end{pmatrix}$.

To investigate system (0.5), we use the Fourier transform method that was proposed by I.G. Petrowsky [14] to study the well-posedness property of the Cauchy problem for evolution systems on a layer $\mathbb{R}^n \times [0,T]$. Later this method was generalized by I.M. Gelfand and G.E. Shilov [7].

It is well known [7] that in the case $b(\sigma) \equiv 0$ all solutions of the system (0.5) tend to 0 as $t \to +\infty$ iff $\forall \sigma \in \mathbb{R}^n$ $\sup\{\Re\lambda \mid \det(I\lambda - A(\sigma)) = 0\} < 0$, where I is the identity matrix. One can prove that in the case $b(\sigma) \neq 0$ the answer on the question: whether all solutions of the system (0.5) with control (0.3) tend to 0 as $t \to +\infty$ depends on the zero dispositions of the quasipolynomial $\det\left\{I\lambda - A(\sigma) - b(\sigma)\left(P_0(\sigma)e^{-2h\lambda} + P_1(\sigma)e^{-h\lambda}\right)\right\}$, where $P_0, P_1 \in \mathcal{M}$ [1]. If the answer on the question is positive then for each zero $\lambda_0(\sigma)$ of this quasipolynomial we have $\Re\lambda_0(\sigma) < 0$, $\sigma \in \mathbb{R}$ (Statement 4.2). The asymptotic behaviour of quasipolynomial roots is well known (see, e.g., [1]). Moreover, for an arbitrary quasipolynomial there are necessary and sufficient conditions for negativity of real parts of its roots [15]. Unfortunately, these conditions have so complicated form that it makes impossible to apply them to a quasipolynomial depending on a parameter. So using these conditions it is impossible to choose parameters (P_0, P_1) in order that real parts of its roots are negative. That's why we have to investigate dependence of zero dispositions of the quasipolynomial $\xi + \beta e^{-h\xi}$ on $\beta \in \mathbb{C}$ in Section 2. It makes possible to choose an appropriate P_0, P_1 and stabilize system (0.5). However A, B, P_0, P_1 also depend on a parameter $(\sigma \in \mathbb{R}^n)$. That's why in Section 3 using the Tarsky–Seidenberg theorem and its corollaries, we have to obtain some estimates for semi-algebraic functions on semi-algebraic sets and apply them to the quasipolynomial $\det\left\{I\lambda - A(\sigma) - b(\sigma)\left(P_0(\sigma)e^{-2h\lambda} + P_1(\sigma)e^{-h\lambda}\right)\right\} \equiv \lambda^2 + 2a_1(\sigma)\lambda + a_0(\sigma) + b(\sigma)\left(p_0(\sigma)e^{-2h\lambda} + p_1(\sigma)e^{-h\lambda} + p_2(\sigma)\lambda e^{-h\lambda}\right)$.

Put $\Lambda_0(s) \equiv \sup\{\Re\lambda \mid \lambda^2 + 2a_1(s)\lambda + a_0(s) = 0\}$ on $\mathbb{C}^n$, $W(\Gamma,\gamma) = \{\sigma \in \mathbb{R}^n \mid d[\sigma, N\{b\}] < \Gamma(1+|\sigma|^2)^\gamma\}$, where $\Gamma > 0$, $\gamma \in \mathbb{R}$, $N\{H\}$ is the set of real zeros of a polynomial H, $d(\sigma, M)$ is the distance between a point σ and a set $M \subset \mathbb{R}^n$ (if $N\{b\} = \emptyset$ then $W(\Gamma,\gamma) = \emptyset$ for all $\Gamma > 0$ and $\gamma \in \mathbb{R}$).

In Section 4 we prove the following

Theorem 0.1. *Assume that A and B satisfy the conditions*

$$\forall\sigma \in W(\Phi_1, \varphi_1) \qquad \Lambda_0(\sigma) < 0, \tag{0.7}$$

$$\forall\sigma \in \mathbb{R}^n \backslash W(\Phi_1, \varphi_1) \quad |\det B(\sigma)|^2 \geq \Phi_2(1+|\sigma|^2)^{\varphi_2}, \tag{0.8}$$

$$\forall\sigma \in \mathbb{R}^n \quad 1 - h\Lambda_0(\sigma) \geq \Phi_3(1+|\sigma|^2)^{\varphi_3}, \tag{0.9}$$

where $\Phi_1, \Phi_2 > 0$, $\Phi_3 \in (0,1]$, $\varphi_1, \varphi_2, \varphi_3 \in \mathbb{Q}$, $\varphi_3 \leq 0$. Assume also that $R > 0$, $\Phi_2\Phi_3 R > e$, $r \in \mathbb{Q}$, $\varphi_2 + \varphi_3 + r \geq 0$. Then equation (0.2) is stabilizable in $C_\gamma^{-\infty}$ and

$$\begin{cases} p_0(\sigma) \equiv (\psi(\sigma))^2\, b(\sigma), \\ p_1(\sigma) \equiv 2\psi(\sigma)\left[a_1(\sigma)\cosh\left(h\sqrt{D(\sigma)}\right) + \sqrt{D(\sigma)}\sinh\left(h\sqrt{D(\sigma)}\right)\right], \\ p_2(\sigma) \equiv 2\psi(\sigma)\cosh\left(h\sqrt{D(\sigma)}\right) \end{cases} \tag{0.10}$$

are stabilizing functions for this equation, where $\psi(\sigma) \equiv \frac{\overline{b(\sigma)}R\left(1+|\sigma|^2\right)^r e^{-ha_1(\sigma)}}{e|b(\sigma)|^2 Rh(1+|\sigma|^2)^r + e}$, $D(\sigma) \equiv (a_1(\sigma))^2 - a_0(\sigma)$ *(we choose the branch of $\sqrt{z}$ such that $\Re\sqrt{z} \geq 0$, $z \in \mathbb{C}$).*

In Section 2 we prove that conditions (0.7), (0.8) are necessary for (0.11) and (0.9) is necessary for (0.12). Thus we have

Theorem 0.2. *If equation (0.2) satisfies the following two conditions*

$$\forall\sigma \in \mathbb{R}^n \qquad [b(\sigma) = 0 \Longrightarrow \Lambda_0(\sigma) < 0], \tag{0.11}$$

$$\forall\sigma \in \mathbb{R}^n \qquad \Lambda_0(\sigma) < \frac{1}{2h}, \tag{0.12}$$

then this equation is stabilizable in $C_\gamma^{-\infty}$. Moreover, condition (0.11) is necessary for stabilizability of (0.2) in $C_\gamma^{-\infty}$.

Finally, in Remark 4.1 we show that for the stabilizing control u corresponding to the functions p_0, p_1, p_2 constructed in Theorem 0.1 we have $u \in \mathbf{C}_\gamma^s$ and $\|u(\cdot,t)\| \le \mu(t) \left|\left|\left| \begin{pmatrix} w \\ \partial w/\partial t \end{pmatrix} \right|\right|\right|_\gamma^q$ for some $s \in \mathbb{N}_0$, where $\mu \in C[0,+\infty)$, $\mu(t) \longrightarrow 0$ as $t \longrightarrow +\infty$.

Note that the problem of stabilizability by feedback control without delays ($h = 0$) was investigated in [5, 6] for equations and systems of the form (0.5).

1. Examples

Example 1.1. Consider **the telegraph equation**

$$\frac{\partial^2 w}{\partial t^2} - 2k\frac{\partial w}{\partial t} - \Delta w + b\left(D_x\right)u = 0, \qquad x \in \mathbb{R}^n,\ t > 2h, \tag{1.1}$$

where b is a polynomial, $k \in \mathbb{R}$. It is easy to see that

$$A(\sigma) \equiv \begin{pmatrix} 0 & 1 \\ -|\sigma|^2 & 2k \end{pmatrix}, \qquad \Lambda_0(\sigma) \equiv \left[\begin{array}{ll} k, & |\sigma| \ge |k| \\ k + \sqrt{k^2 - |\sigma|^2}, & |\sigma| \le |k| \end{array} \right. .$$

Let us consider three cases: i) $k < 0$, ii) $k = 0$, iii) $k > 0$.

i) Let $k < 0$. Then $\Lambda_0(\sigma) < 0$ if $\sigma \ne 0$ and $\Lambda_0(0) = 0$. Taking into account Theorem 0.2, we obtain

Statement 1.1. *Let $k < 0$. Then equation 0.2 is stabilizable in $C_\gamma^{-\infty}$ iff $b(0) \ne 0$.*

Let us find a stabilizing control of the form (0.10) for this equation, if $b(0) \ne 0$. Put $\Phi_2 = d(0, N\{b\})/2$, $\mathcal{N} = \{\sigma \in \mathbb{R}^n \mid |\det B(\sigma)| \le \Phi_2\}$, $\mu(r) = \sup\{\sigma \in \mathcal{N} \mid |\sigma| = r\}$, $r \ge 0$. Using the same reasoning as for obtaining estimate (0.7) in Section 2, we conclude that $\mu \le \Phi_1(1 + r^2)^{\varphi_1}$, $r \ge 0$, where $\Phi_1 > 0$, $\varphi_1 \in \mathbb{Q}$. Hence, $W(\Phi_1, \varphi_1) \supset \mathcal{N} \supset N\{b\}$. Therefore estimates (0.7), (0.8) hold with these Φ_1, Φ_2, φ_1 and $\varphi_2 = 0$. Obviously, estimate (0.9) is true for $\Phi_3 = 1$, $\varphi_3 = 0$. Put $R = 2e/\Phi_2$, $r = 0$. Due to Theorem 0.1 we conclude that p_0, p_1, p_2 defined by (0.10) are stabilizing functions for (1.1) in the case $k < 0$, where $\psi(\sigma) \equiv \dfrac{2\overline{b(\sigma)}e^{-kh}}{2eh\,|b(\sigma)|^2 + \Phi_2}$, $D(\sigma) = \sqrt{k^2 - |\sigma|^2}$.

ii) Let $k = 0$. Then (1.1) is **the wave equation**. We have $\Lambda_0(\sigma) = 0$, $\sigma \in \mathbb{R}^3$. Due to Theorem 0.2 we obtain

Statement 1.2. *Let $k = 0$. Then equation 0.2 is stabilizable in $C_\gamma^{-\infty}$ iff*

$$\forall \sigma \in \mathbb{R}^n \qquad b(\sigma) \ne 0. \tag{1.2}$$

Let us find a stabilizing control of the form (0.10) for this equation, if (1.2) holds. With regard to (3.2) we conclude that there exists $\Phi_2 > 0$, $\varphi_2 \in \mathbb{Q}$ such that $|b(\sigma)|^2 \geq \Phi_2(1 + |\sigma|^2)^{\varphi_2}$, $\sigma \in \mathbb{R}^n$. Hence, estimates (0.7), (0.8) hold with these Φ_2, φ_2 and arbitrary $\Phi_1 > 0$, $\varphi_1 \in \mathbb{Q}$. Evidently estimate (0.9) is true for $\Phi_3 = 1$, $\varphi_3 = 0$. Put $R = 2e/\Phi_2$, $r = 0$. Due to Theorem 0.1 we conclude that p_0, p_1, p_2 defined by (0.10) are stabilizing functions for (1.1) in the case $k = 0$, where $\psi(\sigma) \equiv \dfrac{2\overline{b(\sigma)}}{2eh\,|b(\sigma)|^2 + \Phi_2}$, $D(\sigma) \equiv i|\sigma|$.

iii) Let $k > 0$. Then $k \leq \Lambda_0(\sigma) \leq 2k$, $\sigma \in \mathbb{R}^n$. Applying Theorem 0.2, we obtain

Statement 1.3. *Let $k > 0$. If $h < 1/k$ and (1.2) holds then equation 0.2 is stabilizable in $C_\gamma^{-\infty}$. If (1.2) is not true then this equation is not stabilizable in $C_\gamma^{-\infty}$.*

Let us find a stabilizing control of the form (0.10) for this equation, if $h < 1/k$ and (1.2) holds. With regard to (3.2) we conclude that there exists $\Phi_2 > 0$, $\varphi_2 \in \mathbb{Q}$ such that $|b(\sigma)|^2 \geq \Phi_2(1 + |\sigma|^2)^{\varphi_2}$, $\sigma \in \mathbb{R}^n$. Hence, estimates (0.7), (0.8) hold with these Φ_2, φ_2 and arbitrary $\Phi_1 > 0$, $\varphi_1 \in \mathbb{Q}$. Obviously, estimate (0.9) is true for $\Phi_3 = 1 - 2kh$, $\varphi_3 = 0$. Put $R = 2e/(\Phi_2(1 - 2kh))$, $r = 0$. Due to Theorem 0.1 we conclude that p_0, p_1, p_2 defined by (0.10) are stabilizing functions for (1.1) in the case $k > 0$, where $\psi(\sigma) \equiv \dfrac{2\overline{b(\sigma)}e^{-kh}}{2eh\,|b(\sigma)|^2 + \Phi_2(1 - 2kh)}$, $D(\sigma) \equiv \sqrt{k^2 - |\sigma|^2}$.

E x a m p l e 1.2. Consider the equation

$$\frac{\partial^2 w}{\partial t^2} - \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)^2 \left(\frac{\partial w}{\partial t} + w\right) - w + \left(1 - \frac{\partial^2}{\partial x_1 \partial x_2}\right) u = 0,$$
$$x \in \mathbb{R}^2,\ t > 2h. \qquad (1.3)$$

Evidently $\Lambda_0(\sigma) \equiv \left[\begin{array}{ll} 1 - (\sigma_1 - \sigma_2)^2, & |\sigma_1 - \sigma_2| \leq \sqrt{2} \\ -1, & |\sigma_1 - \sigma_2| \geq \sqrt{2} \end{array}\right.$, $b(\sigma) \equiv \sigma_1\sigma_2 + 1$. Hence (0.12) holds for $h < 1/2$. We have $\Lambda_0(\sigma) \equiv -(\sigma_1^4 - \sigma_1^2 + 1)/\sigma_1^2 \leq -1 < 0$, $\sigma \in N\{b\}$, therefore (0.11) is valid. Taking into account Theorem 0.2, we get that the system (1.3) is stabilizable in $C_\gamma^{-\infty}$.

Now we construct a stabilizing matrix of the form (0.10) for this system. We have $\Lambda_0(\sigma) \leq -1/2$, $\sigma \in W(\Phi_1, \varphi_1)$, where $\Phi_1 = 1/4$, $\varphi_1 = -1/2$, $\Gamma = 1/2$, $\gamma = 0$.

Let us obtain an estimate of the form (3.2) for $b(\sigma) \equiv \sigma_1\sigma_2 + 1$. At first, assume that $|\sigma_2| \geq |\sigma_1|$ and $|\sigma| \geq 1/4$. Then $|\sigma_1 + 1/\sigma_2| \geq d\,[\sigma, N\{b\}]$. Hence,

$|\sigma_1\sigma_2+1|^2 \geq |\sigma_2|^2\,|\sigma_1+1/\sigma_2| \geq |\sigma_2|^2\,(d\,[\sigma,N\{b\}])^2$. Since $|\sigma_2| \geq (|\sigma_1|+|\sigma_2|)/2 \geq |\sigma|/2 \geq \left(1+|\sigma|^2\right)^{1/2}/10$, it follows that

$$|\sigma_1\sigma_2+1|^2 \geq \frac{1}{100}\left(1+|\sigma|^2\right)(d\,[\sigma,N\{b\}])^2. \tag{1.4}$$

Obviously, if $|\sigma_2| \leq |\sigma_1|$ and $|\sigma| \geq 1/4$ then the estimate (1.4) is also true. If $|\sigma| \leq 1/4$ then $d\,[\sigma,N\{b\}] \leq 2$ therefore $\left(1+|\sigma|^2\right)(d\,[\sigma,N\{b\}])^2 \leq 4$ and $|\sigma_1\sigma_2+1| \geq 1-|\sigma_1\sigma_2| \geq 15/16$. All this implies that (1.4) is true for all $\sigma \in \mathbb{R}^2$. Therefore $|\sigma_1\sigma_2+1|^2 \geq 1/1600$, $\sigma \in \mathbb{R}^2\backslash W(\Phi_1,\varphi_1)$. Hence $\Phi_2 = 1/1600$, $\varphi_2 = 0$. On the other hand, $|\Lambda_0(\sigma)| \leq 1$, $\sigma \in \mathbb{R}^2$. Therefore $\Phi_3 = 1-h$, $\varphi_3 = 0$. Put $r = 0$, $R = (1601e)/(1-h)$. Due to Theorem 0.1 we conclude that p_0, p_1, p_2 defined by (0.10), where $\psi(\sigma) \equiv \dfrac{1601(\sigma_1\sigma_2+1)}{1601eh(\sigma_1\sigma_2+1)^2+(1-h)}e^{-h(\sigma_1-\sigma_2)^2/2}$, $D(\sigma) \equiv (\sigma_1-\sigma_2)^2/2-1$ are stabilizing functions for equation (1.3).

2. Some forms of representations for solutions of ordinary differential-difference systems

Consider the following differential-difference system

$$v'(t) = Av(t) + B_0v(t-2h) + B_1v(t-h), \qquad t > 2h, \tag{2.1}$$

under the initial condition

$$v(t) = v^0(t), \qquad t \in [0,2h], \tag{2.2}$$

where A, B_0, B_1 are matrices (2×2) with complex coefficients.

Denote

$$K(t) = 0, \quad \text{if } t<0, \quad K(0) = I, \qquad K(t) = \int_{(c)} e^{t\lambda}H^{-1}(\lambda)\,d\lambda, \quad \text{if } t>0, \tag{2.3}$$

where c is greater then the supremum of the real parts of the roots of $\det H(\lambda)$, $H(\lambda) = \lambda I - A - B_0e^{-2h\lambda} - B_1e^{-h\lambda}$. Here and further we denote throughout the paper $\int_{(c)} f(\lambda)\,d\lambda = \text{V.P.}\int_{-\infty}^{+\infty} f(c+i\mu)\,d\mu$, V.P. means the principal value of the integral. $K(t)$ is called a resolving matrix of the system (2.1).

Due to [1, Theorems 6.2, 6.4] we obtain

Statement 2.1. *Let $v^0 \in C[0,h]$. Then there exists a unique solution $v(t)$ $(t>0)$ of the problem (2.1), (2.2) such that $v \in C[0,+\infty)$, $v \in C^1(2h,+\infty)$.*

Moreover,

$$v(t)=K(t-2h)v^0(2h)+B_0\int\limits_0^{2h}K(t-\tau-2h)v^0(\tau)\,d\tau$$

$$+B_1\int\limits_0^{h}K(t-\tau-h)v^0(\tau)\,d\tau,\qquad t>2h. \tag{2.4}$$

Set $\mathcal{H}(\beta,\xi)=\xi+\beta e^{-h\xi}$ and denote $\lambda(\beta)=\sup\{\Re\xi\mid\mathcal{H}(\beta,\xi)=0\}$.

Lemma 2.1. *If* $0\le\beta\le 1/(eh)$, *then* $\lambda(\beta)\le 0$ *and* $-1/h\le\lambda(\beta)\le(-1+\sqrt{e(1-e\beta h)})/(e\beta h^2)$.

P r o o f. Set $\xi=(x+iy)/h$, $x\in\mathbb{R}$, $y\in\mathbb{R}$. We have $\mathcal{H}(\beta,\xi)=0$ iff $x+\beta he^{-x}\cos y=0$ and $y-\beta he^{-x}\sin y=0$. It is easy to see that if the second equality is satisfied, then $y=0$ or $x\le\ln(\beta h)\le -1$. Let $y=0$ then the first equality is valid iff $x+\beta he^{-x}=0$. Since $0\le\beta\le 1/(eh)$, then this equality holds at least for one x. Let x_0 be the maximum of x such that $x+\beta he^{-x}=0$. It is easy to see that $-1\le x_0\le 0$ and $h\lambda(\beta)=x_0$. Because of $e^{-x}\ge x^2+(2-e)x+1$, $x\in[-1,0]$, we obtain that if $\beta h(x^2+(2-e)x+1)\ge -x$ and $x\ge -1$, then $x\ge x_0$. It follows from here that $h\lambda(\beta)=x_0\le x_1$ where x_1 is the greatest root of the equation $\beta h(x^2+(2-e)x+1)=-x$. It is easy to see that $x_1\le(-1+\sqrt{e(1-e\beta h)})/(e\beta h)$. The lemma is proved.

Put

$$A=\begin{pmatrix}0&1\\-a_0&-2a_1\end{pmatrix},\quad B_0=\begin{pmatrix}0&0\\-1&0\end{pmatrix}\beta^2e^{h(\lambda_1+\lambda_2)},$$

$$B_1=\begin{pmatrix}0&0\\\lambda_2&-1\end{pmatrix}\beta e^{h\lambda_1}+\begin{pmatrix}0&0\\\lambda_1&-1\end{pmatrix}\beta e^{h\lambda_2}, \tag{2.5}$$

where λ_1 λ_2 are the roots of $\lambda^2-2a_1\lambda+a_0$, $0\le\beta\le 1/(eh)$. Then

$$\det H(\lambda)\equiv\mathcal{H}(\beta,\lambda-\lambda_1)\,\mathcal{H}(\beta,\lambda-\lambda_2)\,. \tag{2.6}$$

Lemma 2.2. *Let* $0\le\beta\le 1/(eh)$, $c(\beta)\ne 0$, $\lambda(\beta)<c(\beta)<1/h$, $\Lambda_0=\Re\lambda_1\ge\Re\lambda_2$ *and* $\hat{c}(\beta)=c(\beta)+\Lambda_0$. *Then*

$$\|K(t)\|\le M\left(1+\lambda_1^2+\lambda_2^2\right)^2\frac{e^{(c(\beta)+\Lambda_0)t}}{|c(\beta)+\Lambda_0|\,|c(\beta)|\,|c(\beta)-\lambda(\beta)|^{2\omega}},\quad t>0, \tag{2.7}$$

where $M>0$, $\omega>0$.

Moreover,

$$\int\limits_{(\hat c(\beta))}\left|\frac{e^{\lambda t}}{(\det H(\lambda))^k}\right|dx\le M_k\left(1+\lambda_1^2+\lambda_2^2\right)^{1/2}\frac{e^{(c(\beta)+\Lambda_0)t}}{|c(\beta)|\,|c(\beta)-\lambda(\beta)|^{2(k+1)\omega}},\quad t>0,\quad (2.8)$$

where $M_k>0$.

P r o o f. We have

$$\begin{aligned}K(t)&\equiv I\int\limits_{(\hat c(\beta))}\frac{e^{\lambda t}}{\lambda}d\lambda+\int\limits_{(\hat c(\beta))}\frac{e^{\lambda t}}{\lambda}\left(A+B_0e^{-2h\lambda}+B_1e^{-h\lambda}\right)H^{-1}(\lambda)d\lambda\\&\equiv\int\limits_{(\hat c(\beta))}\frac{e^{\lambda t}}{\det H(\lambda)}\left(A+B_0e^{-2h\lambda}+B_1e^{-h\lambda}\right)d\lambda\\&\quad-I\int_{(\hat c(\beta))}\frac{e^{\lambda t}}{\lambda\det H(\lambda)}(\lambda_1\lambda_2+\beta^2e^{-h(\lambda-\lambda_1+\lambda-\lambda_2)}\\&\quad+\beta\lambda_2e^{-h(\lambda-\lambda_1)}+\beta\lambda_1e^{-h(\lambda-\lambda_2)})d\lambda,\qquad t>0.\end{aligned}\quad (2.9)$$

Therefore

$$\|K(t)\|\le\mathcal{K}\left(1+\lambda_1^2+\lambda_2^2\right)\frac{(1+|\hat c(\beta)|)}{|\hat c(\beta)|}e^{c(\beta)t}\left|\int\limits_{-\infty}^{\infty}\frac{d\mu}{\det H(c(\beta)+i\mu+\lambda_1)}\right|,\quad t>0,\ (2.10)$$

where $\mathcal{K}>0$.

Taking into account (2.6) and

$$2\,|\mathcal{H}(\beta,\xi)|\ge|\xi|,\qquad|\xi|\ge 2/h,\qquad 0\le\beta\le 1/(eh),\quad (2.11)$$

we conclude that

$$\left|\int\limits_{-\infty}^{\infty}\frac{d\mu}{\det H(c(\beta)+i\mu+\lambda_1)}\right|\le\frac{8\pi}{|c(\beta)|}+\int\limits_{|\mu|\le 2|\lambda_1-\lambda_2|+2/h}\frac{d\mu}{\det H(c(\beta)+i\mu+\lambda_1)}.\quad (2.12)$$

Since $|\mathcal{H}(\xi_1+i\xi_2,\beta)|^2$ is a real analytic function of (ξ_1,ξ_2,β) on $\mathbb{R}^3$ then due to [11, Section 17] we conclude that $|\mathcal{H}(\xi_1+i\xi_2,\beta)|\geq\Omega(d[(\xi_1,\xi_2,\beta),N])^\omega$, $\xi_1^2+\xi_2^2<4/h^2$, $0\leq\beta\leq 1/(eh)$, where $\Omega,\omega>0$, $N=\{(\xi_1,\xi_2,\beta)\in\mathbb{R}^3\mid\mathcal{H}(\xi_1+i\xi_2,\beta)=0\}$.

With regard to (2.11) we obtain that

$$|\mathcal{H}(\beta,c(\beta)+i\mu)|\geq\min\left\{\frac{1}{h},\Omega\,|c(\beta)-\lambda(\beta)|^\omega\right\},$$

$$|\mathcal{H}\left(\beta,c(\beta)+i\mu+\lambda_1-\lambda_2\right)|\geq\min\left\{\frac{1}{h},\Omega\,|c(\beta)-\lambda(\beta)|^\omega\right\}.$$

From (2.10) taking into account (2.12) and (2.6), we get (2.7). By analogy with obtaining the estimate of the integral in (2.10), we have (2.8). The proof is completed.

3. Auxiliary statements

Lemma 3.1. *Let for a polynomial matrix $(m\times m)$ A condition (0.11) holds. Then there exist $\Phi_1,\Gamma>0$, $\varphi_1,\gamma\in\mathbb{Q}$ such that (0.7) is true. Moreover, there exist $\Gamma>0$, $\gamma\in\mathbb{Q}$, $\Gamma_1>0$, $\gamma_2\in\mathbb{Q}$ such that*

$$\forall(\sigma,\tau)\in V(\Gamma_1,\gamma)\qquad\sigma\in W(\Phi_1,\varphi_1)\Longrightarrow\Lambda_0(\sigma)<-\Gamma(1+|\sigma|^2)^\gamma,\tag{3.1}$$

where $V(\Gamma_1,\gamma_1)=\left\{(\sigma,\tau)\in\mathbb{R}^{2n}:|\sigma+i\tau|\leq\Gamma_1\left(1+|\sigma|^2\right)^{\gamma_2}\right\}$.

P r o o f. We can represent the set $W(\Phi_1,\varphi_1)$ in the form $W(\Phi_1,\varphi_1)=\{\sigma\in\mathbb{R}^n\mid\exists\eta\in\mathbb{R}^n[\det B(\eta)=0\wedge|\eta-\sigma|<\Gamma(1+|\sigma|^1)^{\varphi_1}]\}$. Let $\nu(r)\equiv\inf\{|\sigma-\eta|\mid\sigma\in\mathbb{R}^n\wedge\eta\in\mathbb{R}^n\wedge\Lambda_0(\sigma)\geq0\wedge\det B(\eta)=0\wedge|\sigma|=r\}$. From (0.11) it follows that $\nu(r)>0$ $(r\geq0)$. It is easy to see that for every $r_0>0$ there exists $C(r_0)>0$ such that $\nu(r)\geq C(r_0)$, $r\in[0,r_0]$. Due to the Tarski–Seidenberg theorem [17] and its corollaries [8, Appendix A] we obtain that $\nu(r)=+\infty$ as $r\longrightarrow+\infty$ or $\nu(r)=Nr^{2\varphi_1}(1+o(1))$ as $r\longrightarrow+\infty$, where $N>0$, $\varphi_1\in\mathbb{Q}$. Therefore $\nu(r)\geq2\Phi_1\left(1+r^2\right)^{\varphi_1}$, $r\geq0$, where $\Phi_1>0$, $\varphi_1\in\mathbb{Q}$. Hence (0.7) holds. Applying the Tarski–Seidenberg theorem [17] and its corollaries [8, Appendix A] to $\mu(r)\equiv\sup\{\lambda_1\in\mathbb{R}\mid\det(I(\lambda_1+i\lambda_2)-A(\sigma))=0\wedge\sigma\in W(\Phi_1,\varphi_1)\wedge|\sigma|=r\}$, we conclude that $\mu(r)<\Gamma\left(1+r^2\right)^\gamma$, $r\geq0$, where $\Gamma>0$, $\gamma\in\mathbb{Q}$. For obtaining this estimate we use the same reasoning as for obtaining the analogous estimate for $\nu(r)$. Therefore $\Lambda_0(\sigma)<-\Gamma(1+|\sigma|^2)^\gamma$, $\sigma\in W(\Phi_1,\varphi_2)$. As before applying the Tarski–Seidenberg theorem [17] and its corollaries [8, Appendix A] to $\omega(r)\equiv\inf\{\eta\mid\det(I(\lambda_1+i\lambda_2)-A(\sigma+i\tau))=0\wedge\lambda_1+\Gamma(1+r^2)^\gamma\geq0\wedge|\sigma|=r\wedge\eta^2=|\sigma|^2+|\tau|^2\wedge\sigma\in\mathbb{R}^n\wedge\tau\in\mathbb{R}^n\}$, we obtain that $\omega(r)\geq\Gamma_1(1+r^2)^{\gamma_1},r\geq0$, where $\Gamma_1>0,\gamma_1\in\mathbb{Q}$. Thus (3.1) is true as was to be proved.

Lemma 3.2. *Let $\Phi_1, \Gamma > 0$, $\varphi_1, \gamma \in \mathbb{Q}$ be a constants such that (3.1) holds. Then there exist $\Phi_2 > 0$ and $\varphi_2 \in \mathbb{Q}$ such that (0.8) is true.*

P r o o f. Due to [9, Lemma 2], we get

$$|\det B(\sigma)|^2 \geq \mathcal{B}\left(1+|\sigma|^2\right)^{\alpha}\left(d[\sigma, N\{\det B\}]\right)^{\beta}, \qquad \sigma \in \mathbb{R}^n, \tag{3.2}$$

where $\mathcal{B} > 0$, $\alpha \in \mathbb{Q}$, $\beta \in \mathbb{Q}$, moreover, $\beta > 0$ if $N\{\det B\} \neq \emptyset$ and $\beta = 0$ otherwise. Hence (0.8) holds. The lemma is proved.

Lemma 3.3. *Let (0.12) holds. Then there exist $\Phi_3 \in (0,1]$ and $\varphi_3 \in \mathbb{Q}$ such that (0.9) is true. Moreover, for all $\varepsilon > 0$ there exist such $\Gamma_2 > 0, \gamma_2 \in \mathbb{Q}$ depending on ε that*

$$\forall(\sigma,\tau) \in V(\Gamma_2,\gamma_2) \quad 1 - h\Lambda_0(\sigma + i\tau) \geq (\Phi_3 - \varepsilon)(1+|\sigma|^2)^{\varphi_3}. \tag{3.3}$$

P r o o f. Taking into account (0.12) and applying the Tarski–Seidenberg theorem [17] and its corollaries [8, Appendix A] to $\mu(r) = \inf\{1 - h\lambda_1 \mid \exists \lambda_2 \in \mathbb{R}\, \exists \sigma \in \mathbb{R}^n\, [\det(A(\sigma) - (\lambda_1 + i\lambda_2)I) = 0 \wedge |\sigma| = r]\}$, we conclude that (0.9) holds. For obtaining this estimate we use the same reasoning as for obtaining the estimate (0.7). Let $\varepsilon > 0$ be fixed. As before applying the Tarski–Seidenberg theorem [17] and its corollaries [8, Appendix A] to $\nu(r) \equiv \{\eta \mid \det(I(\lambda_1 + i\lambda_2) - A(\sigma + i\tau)) \wedge (\Phi_3 - \varepsilon)(1+r^2)^{\varphi_3} + h\lambda_1 - 1 \geq 0 \wedge |\sigma| = r \wedge \eta^2 = |\sigma|^2 + |\tau|^2 \wedge \sigma \in \mathbb{R}^n \wedge \tau \in \mathbb{R}^n\}$, we get $\nu(r) \geq \Gamma_2(1+r^2)^{\gamma_2}, r \geq 0$, where $\Gamma_2 > 0, \gamma_2 \in \mathbb{Q}$. Therefore (3.3) is true. The lemma is proved.

Lemma 3.4. *Let Φ_1, $\Phi_2 > 0$, $\Phi_3 \in (0,1]$, φ_1, φ_2, $\varphi_3 \in \mathbb{Q}$, $\varphi_3 \leq 0$, be constants such that (0.7)–(0.9) hold. Let $K(\sigma,t)$ be defined by (2.3), (2.5), where $a_1(\sigma)$, $a_0(\sigma)$ are the coefficients of equation (0.2), $\beta(\sigma) \equiv \frac{1}{eh}\frac{|b(\sigma)|^2 Rh\left(1+|\sigma|^2\right)^r}{|b(\sigma)|^2 Rh(1+|\sigma|^2)^r + 1}$, $R > 0$, $\Phi_2\Phi_3 R > e$, $r \in \mathbb{Q}$, $\varphi_2 + \varphi_3 + r \geq 0$. Then for each multi-index α*

$$\|D^\alpha_\sigma K(\sigma,t)\| \leq \mathcal{K}_{|\alpha|}\left(1+|\sigma|^2\right)^{\kappa_{|\alpha|}} e^{-tL(1+|\sigma|)^l}, \quad (\sigma,t) \in \mathbb{R}^n \times [0,+\infty), \tag{3.4}$$

where $L > 0$, $l \in \mathbb{Q}$, $\mathcal{K}_{|\alpha|} > 0$, $\kappa_{|\alpha|} \in \mathbb{R}$.

P r o o f. Let $\varepsilon > 0$ such that $\Phi_3 - \varepsilon > 0, \Phi_2(\Phi_3 - \varepsilon)R > e$ be fixed. Let $\Gamma_1, \Gamma_2 > 0$ and $\gamma_1, \gamma_2 \in \mathbb{Q}$ be constants such that (3.1), (3.3) are true. Denote $\Gamma_0 = \min\{\Gamma_1, \Gamma_2\}$ $\gamma_0 = \min\{0, \gamma_1, \gamma_2\}$, $\hat{\Phi}_3 = \Phi_3 - \varepsilon$.

We assume throughout the proof that $(\sigma,\tau) \in V(\Gamma_0,\gamma_0)$.

At first, we consider $(\sigma,t) \in (\mathbb{R}^n \backslash W(\Phi_1,\varphi_1)) \times [0,+\infty)$. With regard to Lemmas 3.1–3.3 we get $\sqrt{1 - e\beta(\sigma)h} < \left(1+|\sigma|^2\right)^{\varphi_3}/(\Phi_2 Rh)$. Taking into account

Lemmas 2.1, 3.3 and setting $c_1(\beta(\sigma)) = \left(-1 + e\sqrt{1 - e\beta(\sigma)h}\right)/(e\beta(\sigma)h^2)$, we obtain that

$$\lambda(\beta(\sigma)) < c_1(\beta(\sigma)) \leq \left(e\left(1+|\sigma|^2\right)^{\varphi_3}/(\Phi_2 R) - 1\right)/h < 0 \tag{3.5}$$

and

$$\Lambda_0(\sigma + i\tau) + c_1(\beta(\sigma)) < \left(e - \Phi_2\hat{\Phi}_3 R\right)\left(1+|\sigma|^2\right)^{\varphi_3}/(\Phi_2 Rh) < 0. \tag{3.6}$$

According to Lemma 2.1, we have

$$|c_1(\beta(\sigma)) - \lambda(\beta(\sigma))| \geq (e - \sqrt{e})\sqrt{1 - e\beta(\sigma)h}/h \geq K(1+|\sigma|^2)^{-\psi + r/2}, \tag{3.7}$$

where $\psi = \deg b$, $K > 0$.

Now assume that $(\sigma, t) \in W(\Phi_1, \varphi_1) \times [0, +\infty)$. Set $c_2(\beta(\sigma)) = L'\left(1+|\sigma|^2\right)^{l'}$, where $L' = \min\{1/h, \Gamma/2\}$, $l' = \min\{0, \gamma\}$. Hence,

$$\Lambda_0(\sigma + i\tau) + c_2(\beta(\sigma)) < -\Gamma\left(1+|\sigma|^2\right)^{\gamma}/2. \tag{3.8}$$

According to Lemma 2.1, we have

$$|c_2(\beta(\sigma)) - \lambda(\beta(\sigma))| \geq \Gamma\left(1+|\sigma|^2\right)^{\gamma}/2. \tag{3.9}$$

With regard to (2.2) all this implies that (3.4) holds for $\mu = 0$.

Let us prove that it holds for $|\mu| > 0$. We have $\lambda_1(s) = -a_1(s) + \sqrt{D(s)}$, $\lambda_2(s) = -a_1(s) - \sqrt{D(s)}$ are the roots of $\lambda^2 + 2a_1(s)\lambda + a_0(s)$, $s \in \mathbb{C}$. Put $\rho_0(s) = e^{h(\lambda_1(s)+\lambda_2(s))}$, $\rho_1(s) = -\lambda_2(s)e^{h\lambda_1(s)} - \lambda_1(s)e^{-h\lambda_2(s)}$, $\rho_2(s) = e^{h\lambda_1(s)} + e^{h\lambda_2(s)}$, $h(\beta, \lambda, s) = \mathcal{H}(\beta, \lambda - \lambda_1(s))\mathcal{H}(\beta, \lambda - \lambda_2(s))$, $s \in \mathbb{C}$. Then p_0, p_1, p_2 are entire functions of s, h is a polynomial with respect to β and an entire function of s and λ.

We have

$$\begin{aligned} K(s,t) = I \int\limits_{(\hat{c}_j(\beta(\sigma)))} \frac{e^{\lambda t} d\lambda}{\lambda h(\beta(\sigma), \lambda, s)} \\ + \left[\begin{pmatrix} 2a_1(s) & 1 \\ -a_0(s) & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \beta^2(\sigma)\rho_0(s)e^{-2h\lambda}\right. \\ \left. - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \beta(\sigma)\rho_1(s)e^{-h\lambda} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \beta(\sigma)\rho_2(s)e^{-h\lambda}\right] \\ \times \int\limits_{(\hat{c}_j(\beta(\sigma)))} \frac{e^{\lambda t} d\lambda}{h(\beta(\sigma), \lambda, s)}, \quad s = \sigma + i\tau, \quad t > 0, \end{aligned} \tag{3.10}$$

where $\hat{c}_j(\beta(\sigma)) \equiv c_j(\beta(\sigma)) + \Lambda_0(\sigma + i\tau)$, $j = 1$ if $\sigma \in \mathbb{R}^n \setminus W(\Phi_1, \varphi_1)$ and $j = 2$ if $\sigma \in W(\Phi_1, \varphi_1)$.

It is clear that for each multi-index α we have

$$\left|(D_\sigma^\alpha \rho_0(\sigma))\, e^{-2h\hat{c}_j(\beta(\sigma))}\right| \le \mathcal{N}^0_{|\alpha|}\left(1+|\sigma|^2\right)^{n^0_{|\alpha|}}, \qquad \sigma \in \mathbb{R}^n. \tag{3.11}$$

To obtain analogous estimates for ρ_1 and ρ_2 we use the Cauchy formula for derivatives of a holomorphic function. Put $r(\sigma) = \Gamma_0\left(1+|\sigma|^2\right)^{\gamma_0}/2$. We integrate over the polydisk $\{|s_1 - \sigma_1| = r(\sigma)\} \times \ldots \times \{|s_n - \sigma_n| = r(\sigma)\}$. With regard to (3.6), (3.8) we conclude that for each multi-index α

$$\left|(D_\sigma^\alpha \rho_l(\sigma))\, e^{-h\hat{c}_j(\beta(\sigma))}\right| \le \mathcal{N}^l_{|\alpha|}\left(1+|\sigma|^2\right)^{n^l_{|\alpha|}}, \qquad \sigma \in \mathbb{R}^n,\ l = 1, 2. \tag{3.12}$$

Therefore for each multi-index α, $|\alpha| > 0$, we have

$$|D_\sigma^\alpha K(\sigma,t)| \le \mathcal{N}_{|\alpha|}\left(1+|\sigma|^2\right)^{n_{|\alpha|}} \frac{1}{|\hat{c}_j(\beta(\sigma))|} \max_{k\le|\alpha|} \int\limits_{\hat{c}_j(\beta)} \left|\frac{e^{\lambda t}}{(h(\beta(\sigma),\lambda,\sigma))^k}\right| d\lambda.$$

With regard to (2.8), (3.5)–(3.9) we conclude that (3.4) holds for $|\alpha| > 0$. The lemma is proved.

Lemma 3.5. *Let $\Phi_1, \Phi_2 > 0$, $\Phi_3 \in (0,1]$, $\varphi_1\varphi_2, \varphi_3 \in \mathbb{Q}$, $\varphi_3 \le 0$, be constants such that (0.7)—(0.10) hold. Then for ρ_0, ρ_1, ρ_2 defined by (0.10) we have $\rho_0, \rho_1, \rho_2 \in \mathcal{M}$.*

P r o o f. Taking into account Lemma 3.3 and using the same reasoning as for obtaining estimates (3.11), (3.12), we conclude that $\rho_0, \rho_1, \rho_2 \in \mathcal{M}$.

4. Conditions for stabilizability

Statement 4.1. *Assume that for equation (0.2) conditions (0.11) and (0.12) hold. Then there exist such functions $\rho_0, \rho_1, \rho_2 \in \mathcal{M}$ that for any $p \in \mathbb{N}_0$ there exist $q \in \mathbb{N}_0$ and a continuous function $\nu(t)$ on $[0, +\infty)$, $\nu(t) \longrightarrow 0$ as $t \longrightarrow +\infty$, such that for each solution w of equation (0.2) with control (0.3) under the initial condition $\begin{pmatrix} w \\ \partial w/\partial t \end{pmatrix} \in \mathcal{C}^q_\gamma$ we have $\begin{pmatrix} w \\ \partial w/\partial t \end{pmatrix} \in \mathbf{C}^p_\gamma$ and*

$$\forall t \ge 2h \qquad \left\|\begin{pmatrix} w(\cdot,t) \\ \partial w(\cdot,t)/\partial t \end{pmatrix}\right\|^p_\gamma \le \nu(t) \left\|\begin{pmatrix} w \\ \partial w/\partial t \end{pmatrix}\right\|^q_\gamma. \tag{4.1}$$

P r o o f. It follows from Lemmas 3.1–3.3 that conditions (0.7)–(0.9) hold. Let ρ_0, ρ_1, ρ_2 be functions of the form (0.10). Let $K(\sigma,t)$ be defined by (2.3), (2.5), where $a_1(\sigma), a_0(\sigma)$ are the coefficients of equation (0.2), β defined in Lemma 3.4. Then $B_0 = -bP_0, B_1 = -bP_1$. Due to Lemmas 3.4, 3.5 we have that $\rho_0, \rho_1, \rho_2 \in \mathcal{M}$ and estimate (3.4) is true.

Let $\mathcal{S}'$ be the dual space for $\mathcal{S}$. Let $p \in \mathbb{N}_0$ be fixed. Assume that $q \geq p + \kappa_{\gamma+n+1} + \gamma$, $w^0 \in \mathcal{C}^q_\gamma$, where $\kappa_{\gamma+n+1}$ is the constant from the estimate (3.4). For the system

$$\frac{\partial \mathrm{w}(x,t)}{\partial t} = A\left(D_x\right)\mathrm{w}(x,t) + b\left(D_x\right)\left(P_0\left(D_x\right)\mathrm{w}(x,t-2h) + P_1\left(D_x\right)\mathrm{w}(x,t-h)\right), \quad x \in \mathbb{R}^n,\ t > 2h, \tag{4.2}$$

consider a problem with the initial condition

$$\mathrm{w}(x,t) = \mathrm{w}^0(x,t), \qquad x \in \mathbb{R}^n,\ t \in [0,2h]. \tag{4.3}$$

Applying the Fourier transform (with respect to x) to problem (4.2), (4.3) in $\mathcal{S}'$, we obtain

$$\frac{dv(\sigma,t)}{dt} = A(\sigma)v(\sigma,t) + b(\sigma)\left(P_0(\sigma)v(\sigma,t-2h) + P_1(\sigma)v(\sigma,t-h)\right), \ t > 2h, \tag{4.4}$$

$$v(\sigma,t) = v^0(\sigma,t) \qquad (\mathcal{S}'), \qquad t \in [0,2h], \tag{4.5}$$

where $v(\cdot,t) = \mathcal{F}\mathrm{w}(\cdot,t)$, $v^0 = \mathcal{F}\mathrm{w}^0$. With regard to [1, Theorems 6.2, 6.4] and Lemma 3.4 we conclude that

$$v(\sigma,t) \equiv K(\sigma,t-2h)v^0(\sigma,h) + b(\sigma)P_0(\sigma)\int_0^{2h} K(\sigma,t-\tau-2h)v^0(\sigma,\tau)\,d\tau$$
$$+ b(\sigma)P_1(\sigma)\int_0^{h} K(\sigma,t-\tau-h)v^0(\sigma,\tau)d\tau, \qquad t \geq 2h, \tag{4.6}$$

is a solution of (4.4), (4.5) in $\mathcal{S}'$ and $v(\sigma,\cdot) \in C[0,+\infty)$, $v(\sigma,\cdot) \in C(2h,+\infty)$. Hence, $\mathrm{w}(\cdot,t) = \mathcal{F}^{-1}v(\cdot,t)$ is a solution of (4.2), (4.3) in $\mathcal{S}'$.

Now we prove that $\mathrm{w} \in \mathbf{C}^p_\gamma$ and (4.1) is true. Further throughout the proof we assume that $x \in \mathbb{R}^n$, $\sigma \in \mathbb{R}^n$, $t \geq 2h$. Let $e(x)$ be an infinite differentiable function on $\mathbb{R}^n$, let $\operatorname{supp} e \subset \{x \in \mathbb{R}^n \mid |x| \leq 1\}$, and let $\sum_{l \in \mathbb{Z}^n} e(x-k) \equiv 1$. Denote $\mathrm{w}^0_k(x,t) \equiv e(x)\mathrm{w}^0(x+k,t)$, $v^0_k(\cdot,t) = \mathcal{F}\mathrm{w}^0(\cdot,t)$. With regard to (4.6) we have

$$v_k(\sigma,t) \equiv K(\sigma,t-2h)v^0_k(\sigma,h) + b(\sigma)P_0(\sigma)\int_0^{2h} K(\sigma,t-\tau-2h)v^0_k(\sigma,\tau)\,d\tau$$
$$+ b(\sigma)P_1(\sigma)K(\sigma,t-\tau-h)v^0_k(\sigma,\tau)d\tau \qquad (\mathcal{S}'),\ t \geq h \tag{4.7}$$

is a solution of (4.4), (4.5) with $v^0 = v_k^0$ therefore $\mathrm{w}_k(x,t) \equiv \left(\mathcal{F}^{-1} v_k(\cdot,t)\right)(x)$ is a solution of (4.2), (4.3) in $\mathcal{S}'$ with $\mathrm{w}^0 = \mathrm{w}_k^0$, where $k \in \mathbb{Z}^n$.

Obviously, $\left\|\mathrm{w}_k^0(\cdot,\tau)\right\|_\gamma^q \le M \left\|\mathrm{w}^0(\cdot,\tau)\right\|_\gamma^q (1+|k|)^\gamma$, $\tau \in [0,2h]$, where $M > 0$ does not depend on $\tau \in [0,2h]$ and $k \in \mathbb{Z}^n$. Then we have $\left|\sigma^\lambda D_\sigma^\alpha \left(\sigma^\beta v_k^0(\sigma,t)\right)\right| \le C \left\|\left|\mathrm{w}^0\right|\right\|_\gamma^q (1+|k|)^\gamma$, where $C > 0$, $|\beta| + |\lambda| \le q$, $|\alpha| = n+\gamma+1$. With regard to (4.7) and Lemma 3.4 that gives $\left|D_\sigma^\alpha \left(\sigma^\beta v_k(\sigma,t)\right)\right| \le C' \left\|\left|\mathrm{w}^0\right|\right\|_\gamma^q (1+|\sigma|)^{-n-1} e^{-tL(1+|\sigma|)^l} (1+|k|)^\gamma$, where $C' > 0$, $|\lambda| = \deg B + \kappa_{\gamma+n+1} + n + 1$, $|\beta| \le p$. Applying the inverse Fourier transform with respect to σ, we get $\left|D_x^\beta \mathrm{w}_k(x,t)\right| \le C^* \nu(t) \left\|\left|\mathrm{w}^0\right|\right\|_\gamma^q (1+|x|)^{-(n+\gamma+1)} (1+|k|)^\gamma$, where $C^* > 0$, $\nu(t) = (1+t)^{1/l}$ if $l < 0$ and $\nu(t) = \exp\{-tL\}$ otherwise. Since $(1+|k|) \le (1+|x+k|)(1+|x|)$ that leads to $\left|D_x^\beta \mathrm{w}_k(x,t)\right| \le C^* \nu(t) \left\|\left|\mathrm{w}^0\right|\right\|_\gamma^q (1+|x+k|)^\gamma (1+|x|)^{-n-1}$. Hence, $\mathrm{w}(x,t) \equiv \sum_{k\in\mathbb{Z}^n} \mathrm{w}_k(x-k,t)$, $\mathrm{w} \in \mathbf{C}_\gamma^p$ and (4.1) is true.

It remains to show that the solution w is unique in $\mathbf{C}_\gamma^p$. It is sufficient to prove that for system (4.2) the initial problem under the condition $\mathrm{w}(x,t) = 0$, $x \in \mathbb{R}^n$, $t \in [0,2h]$, has only the trivial solution w in $\mathbf{C}_\gamma^p$. Let w be a solution of this problem and $w \in \mathbf{C}_\gamma^p$. According to the initial condition, we have $\mathrm{w}(x,t) \equiv 0$ on $\mathbb{R}^n \times [0,2h]$. Suppose $\mathrm{w}(x,t) \equiv 0$ on $\mathbb{R}^n \times [2(k-1)h, 2kh]$ and prove that $\mathrm{w}(x,t) \equiv 0$ on $\mathbb{R}^n \times [2kh, 2(k+1)h]$, $(k \in \mathbb{N})$. It is easy to see that each solution of system (4.2) on $\mathbb{R}^n \times [2kh, 2(k+1)h]$ under the initial condition $\mathrm{w}(x,t) = 0$, $x \in \mathbb{R}^n$, $t \in [2(k-1)h, 2kh]$ is a solution of the Cauchy problem

$$\frac{\partial \mathrm{w}(x,t)}{\partial t} = A(D_x)\,\mathrm{w}(x,t), \quad \mathrm{w}(x,0) = 0, \qquad x \in \mathbb{R}^n,\ t \in [2kh, 2(k+1)h]. \tag{4.8}$$

With regard to [7, §4] that gives $\mathrm{w}(x,t) \equiv 0$ on $\mathbb{R}^n \times [2kh, 2(k+1)h]$. Thus $\mathrm{w}(x,t) \equiv 0$ on $\mathbb{R}^n \times [0,+\infty)$. The statement is proved.

It follows from the proof of Statement 4.1 that conditions (0.7)–(0.9) are sufficient for stabilizability of (0.5) (moreover, in this case P defined by (0.10) is a stabilizing control). Thus Theorem 0.1 is proved.

Statement 4.2. *If for system (0.5) there exists $\sigma_0 \in \mathbb{R}^n$ such that $\Lambda_P(\sigma_0) \ge 0$ for all matrices $(m \times m)$ $P \in \mathcal{M}$, then this system is not stabilizable in $C_\gamma^{-\infty}$.*

P r o o f. Let $p_0, p_1, p_2 \in \mathcal{M}$. Let $\det\{\lambda_0 I - A(\sigma_0) + b(\sigma_0)(P_0(\sigma_0)e^{-2h\lambda_0} + P_1(\sigma_0)e^{-h\lambda_0})\} = 0$ and $\Re\lambda_0 \ge 0$ for some $\sigma_0 \in \mathbb{R}^n$, $\lambda_0 \in \mathbb{C}$, and let v_0, $|v_0| = 1$, be a vector such that $(\lambda_0 I - A(\sigma_0) + b(\sigma_0)(P_0(\sigma_0)e^{-2h\lambda_0} + P_1(\sigma_0)e^{-h\lambda_0}))v_0 = 0$. Consider system (0.5) with the control $\mathrm{u}(\cdot,t) = P_0(D_x)\,\mathrm{w}(\cdot,t-2h) + P_1(D_x)\,\mathrm{w}(\cdot,t-h)$, i.e., the system of the form (4.2), under the initial condition $\mathrm{w}(x,t) = \exp\{t\lambda_0 + i\langle x,\sigma_0\rangle\}v_0$, $x \in \mathbb{R}^n$, $t \in [0,2h]$, where $\langle\cdot,\cdot\rangle$ is the scalar product corresponding to the Euclidean norm in $\mathbb{R}^n$. It is easy to see that $\mathrm{w}(x,t) \equiv \exp\{t\lambda_0 + i\langle x, \sigma_0\rangle\}v_0$, $x \in \mathbb{R}^n$, $t \in [0,h]$, is a solution of this problem. Since $|\mathrm{w}(x,t)| \equiv \exp\{t\Re\lambda_0\}$ and $\Re\lambda_0 \ge 0$, then $\overline{\lim\limits_{t\longrightarrow+\infty}} \|\mathrm{w}(\cdot,t)\|_\gamma^0 > 0$, i.e., condition (0.4) is not satisfied. Therefore system (0.5) is not stabilizable in $C_\gamma^{-\infty}$. The statement is proved.

From this statement we obtain

Corollary 4.1. *If the system (0.5) is stabilizable in $C_\gamma^{-\infty}$, then (0.11) holds.*

With regard to Statement 4.1 and Corollary 4.1 we conclude that Theorem 0.2 is true.

R e m a r k 4.1. Let for system (0.5) conditions (0.11), (0.12) be valid. The stabilizing functions $p_0, p_1, p_2 \in \mathcal{M}$ that have been found in the proof of the Statement 4.1 have the form (0.10) and satisfy estimates (3.11), (3.12). Denote by u the control of the form (0.3). Taking into account Statement 4.1 (estimate (4.1)) and using the same reasoning as for obtaining this estimate, we conclude that for each $s \in \mathbb{N}$ there exists $q \in \mathbb{N}$ and a continuous function $\mu(t)$ on $[0,+\infty)$, $\mu(t) \longrightarrow 0$ as $t \longrightarrow +\infty$, such that for all $\begin{pmatrix} w \\ \partial w/\partial t \end{pmatrix} \in \mathcal{C}_\gamma^q$ we have $u \in \mathbf{C}_\gamma^s$ and $\forall t \ge 0$ $\|u(\cdot,t)\|_\gamma^s \le \mu(t) \left\|\!\left| \begin{pmatrix} w \\ \partial w/\partial t \end{pmatrix} \right|\!\right\|_\gamma^q$.

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