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Connection between Euler hydrodynamics and charged fluid dynamics

E.A. Kuznetsov

Landau Institute for Theoretical Physics
2 Kosygin Str., Moscow, 119334, Russia
E-mail:kuznetso@itp.ac.ru

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It is shown that the Euler hydrodynamics for vortical flows of an ideal incompressible fluid coincides with the equations of motion for a charged *compressible* fluid moving due to a self-consistent electromagnetic field. Transition to the Lagrangian description in a new hydrodynamics is equivalent for the original Euler equations to the mixed Lagrangian–Eulerian description — the vortex line representation. The correspondence with the charged hydrodynamics can be established also for the Euler equations for compressible fluids without pressure. This allows one to construct exact solutions for the charged three-dimensional hydrodynamics.

1. Introduction

In the paper [1] it was introduced a new description of vortical flows, the so-called vortex line representation, for the system of hydrodynamic type,

$$\frac{\partial \Omega}{\partial t} = \mathrm{rot}\left[\mathrm{rot}\frac{\delta \mathcal{H}}{\delta \Omega} \times \Omega\right]. \tag{1}$$

Here Ω is a generalized vorticity, $\mathbf{v} = \mathrm{rot}\, \delta\mathcal{H}/\delta\Omega$ has a meaning of the fluid velocity, and $\mathcal{H}$ is the fluid Hamiltonian. In particular, if $\mathcal{H}$ coincides with kinetic energy, $1/2\int \mathbf{v}^2 d\mathbf{r}$, vorticity is expressed through velocity by the standard formula: $\Omega = \mathrm{rot}\ \mathbf{v}$, and, respectively, the equation (1) becomes the Euler equation for vorticity:

$$\frac{\partial \Omega}{\partial t} = \mathrm{rot}\ \left[\mathbf{v} \times \Omega\right]. \tag{2}$$

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This description deals with the vortex lines and their motion. The vortex line representation is a transform to the curvilinear system coordinates moving together with vortex lines. This representation is based on the property of frozenness of vorticity into fluid according to which all fluid particles are pasted to their own vortex line and can not leave it. Locally this change of variables can be considered as a transformation to mixed Lagrangian–Eulerian description when each vortex line is labeled by two-dimensional Lagrangian marker and another coordinate defines the given vortex line.

Recently, the author of the present paper showed [2] that this transformation follows from the equivalence of the Euler hydrodynamics for incompressible fluids and the hydrodynamic type equations describing motion of charged compressible fluid moving in self-consistent electromagnetic field. Electromagnetic field in this case satisfies the Maxwell equations. The new hydrodynamics occurs compressible. Due to this fact the phenomenon of breaking becomes possible that corresponds to breaking of vortex lines for the Euler hydrodynamics. Mathematically breaking in new hydrodynamics corresponds to vanishing Jacobian of the mapping of the transition from the Eulerian description to the Lagrangian one. This results in infinite value of the vorticity field in one separate point. In this paper we develop this idea for ideal compressible hydrodynamics without pressure which can be easily integrated in terms of Lagrangian description. We show that between compressible fluids and charged fluids also there exists a familiar connection. However, this correspondence is not one-to-one. It is a homomorphism: using velocity and vorticity fields for the original fluid it is possible to construct electromagnetic fields and velocity field for charged fluid, but in the general case the velocity and vorticity fields for the original fluid can not be found through electromagnetic field and velocity of the charged fluid. In fact, by means of this approach we have opportunity to construct the whole integrable sub-manifold of solutions to the charged fluid hydrodynamics.

2. General remarks

As well known (see, for instance, [3, 4]) the Euler equations for an ideal incompressible fluid,

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p, \qquad \text{div } \mathbf{v} = 0, \tag{3}$$

in both two-dimensional and three-dimensional cases possess the infinite (continuous) number of integrals of motion. These are the so called Cauchy invariants. The most simple way to derive the Cauchy invariants is one to use the Kelvin theorem about conservation of the velocity circulation,

$$\Gamma = \oint (\mathbf{v} \cdot d\mathbf{l}), \tag{4}$$

where the integration contour $C[\mathbf{r}(t)]$ moves together with a fluid. If in this expression one makes a transform from the Eulerian coordinate $\mathbf{r}$ to the Lagrangian ones $\mathbf{a}$ then Equation (4) can be rewritten as follows:

$$\Gamma = \oint \dot{x}_i \cdot \frac{\partial x_i}{\partial a_k}\, da_k \ ,$$

where a new contour $C[\mathbf{a}]$ is already immovable. Hence, due to arbitrariness of the contour $C[\mathbf{a}]$ and using the Stokes formula one can conclude that the quantity

$$\mathbf{I} = \mathrm{rot}_a \left(\dot{x}_i \frac{\partial x_i}{\partial \mathbf{a}} \right) \tag{5}$$

conserves in time at each point $\mathbf{a}$. This is just the Cauchy invariant. If the Lagrangian coordinates $\mathbf{a}$ in (5) coincide with the initial positions of fluid particles the invariant $\mathbf{I}$ is equal to the initial vorticity $\Omega_0(\mathbf{a})$.

Conservation of these invariants, as it was shown first by Salmon [4], is consequence of the special (infinite) symmetry — the so-called relabeling symmetry. The Cauchy invariants characterize the frozenness of the vorticity into fluid. This is very important property according to which fluid (Lagrangian) particles can not leave its own vortex line where they were initially. Thus, the Lagrangian particles have one independent degree of freedom — motion along vortex line. From another side, such a motion as it follows from the equation for the vorticity (2) does not change its value. From this point of view a vortex line represents the invariant object and therefore it is natural to seek for such a transformation when this invariance is seen from the very beginning. Such type of description — the vortex line representation — was introduced in the papers [1, 7] by Ruban and the author of this paper.

3. Connection with charged fluids

According to the Equation (2) the tangent to the vector Ω velocity component $\mathbf{v}_\tau$ does not effect (directly) on the vorticity dynamics, i.e., in (2) we can put, instead of $\mathbf{v}$, its transverse component $\mathbf{v}_n$.

The equation of motion for the transverse velocity $\mathbf{v}_n$ follows directly from the Equation (3) by means of the following formula of vector analysis,

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = \nabla(\mathbf{v}^2/2) - [\mathbf{v} \times \mathrm{rot}\ \mathbf{v}]$$

and the velocity decomposition

$$\mathbf{v} = \mathbf{v}_n + \mathbf{v}_\tau \, .$$

As the result, the equation for the transverse component $\mathbf{v}_n$ can be written in the form of the equation of motion for a charged fluid moving in an electromagnetic field:

$$\frac{\partial \mathbf{v}_n}{\partial t} + (\mathbf{v}_n \cdot \nabla)\mathbf{v}_n = \mathbf{E} + [\mathbf{v}_n \times \mathbf{H}]. \tag{6}$$

Here the effective electric and magnetic fields are given by the expressions:

$$\mathbf{E} = -\nabla \left(p + \frac{v_\tau^2}{2} \right) - \frac{\partial \mathbf{v}_\tau}{\partial t}, \tag{7}$$

$$\mathbf{H} = \mathrm{rot}\ \mathbf{v}_\tau. \tag{8}$$

Interesting to note that the electric and magnetic fields introduced above are expressed through the scalar φ and vector $\mathbf{A}$ potentials by the standard way:

$$\varphi = p + \frac{\mathbf{v}_\tau^2}{2}, \qquad \mathbf{A} = \mathbf{v}_\tau, \tag{9}$$

so that two Maxwell equations

$$\mathrm{div}\ \mathbf{H} = 0, \qquad \frac{\partial \mathbf{H}}{\partial t} = -\mathrm{rot}\ \mathbf{E}$$

satisfy automatically. In this case the vector potential $\mathbf{A}$ has the gauge

$$\mathrm{div}\ \mathbf{A} = -\mathrm{div}\ \mathbf{v}_n,$$

which is equivalent to the condition div $\mathbf{v} = 0$.

Two other Maxwell equations can be written also but they can be considered as definition of the charge density ρ and the current $\mathbf{j}$ which follow from the relations (7) and (8):

$$4\pi\rho = -\mathrm{div}\ \mathbf{E}, \tag{10}$$

$$4\pi\mathbf{j} = \mathrm{rot}\ \mathbf{H} - \frac{\partial \mathbf{E}}{\partial t}, \tag{11}$$

where $\mathbf{E}$ and $\mathbf{H}$ are given by Eqs. (7) and (8).

New terms in the right hand side of Eq. (6) have also mechanical interpretation. Lorenz force $[\mathbf{v_n} \times \mathbf{H}]$ is nothing more than Cariolis force. Addition in ϕ to pressure p, equal to ${\mathbf{v}_\tau}^2/2$, has direct connection with the Bernoulli formula. The term $\partial_t \mathbf{v}_\tau$ appears due to transition to movable non-inertial system of coordinates.

The basic equation in the new hydrodynamics is the equation of motion (6) for the normal component of the velocity which represents the equation of motion for non-relativistic particle with a charge and a mass equal to unity, the light velocity in these units is equal to 1.

The obtained system of equations (6)–(11) establishes correspondence with the Euler hydrodynamics (3) of an ideal incompressible fluid. This correspondence means that any 3D solution of the Euler equations generates one solution of the system (6)–(11) but not vise versa. Only special solutions of the system give a solution to the Euler equations (3), in particular, because of the special form for the charge density and electric current. Therefore we may say that the Euler equations are embedded into the system (6)–(11) that open a new opportunity to construct solutions for the charged hydrodynamics (6)–(11).

It is necessary to emphasize that the vector potential $\mathbf{A}$ and respectively the magnetic field $\mathbf{H}$ are defined by the tangent component of the velocity. For two-dimensional flows the tangent velocity is identically equal to zero: the vorticity in this case is perpendicular to the velocity. As the result, we arrive at the potential electric field: $\mathbf{E} = -\nabla\phi$. In this case the charge density is defined from the well known in fluid dynamics relation:

$$\rho = -\frac{1}{4\pi}\Delta p = \frac{1}{4\pi}\mathrm{div}\ (\mathbf{v}\cdot\nabla)\mathbf{v}.$$

For three-dimensional vortical flows of an ideal fluid magnetic field is present already. Notice, that for the three-dimensional Bertrami flow, rot $\mathbf{v} \sim \mathbf{v}$, the magnetic field is also absent.

4. Vortex line representation in ideal incompressible hydrodynamics

The equation of motion (6) is written in the Eulerian representation. To transfer to its Lagrangian formulation one needs to consider the equations for "trajectories" given by the velocity $\mathbf{v}_n$:

$$\frac{d\mathbf{R}}{dt} = \mathbf{v}_n(\mathbf{R}, t) \tag{12}$$

with initial conditions

$$\mathbf{R}|_{t=0} = \mathbf{a}.$$

Solution of the equation (12) yields the mapping

$$\mathbf{r} = \mathbf{R}(\mathbf{a}, t), \tag{13}$$

which defines transition from the Eulerian description to a new Lagrangian one.

The equations of motion in new variables are the Hamilton equations:

$$\dot{\mathbf{P}} = -\frac{\partial h}{\partial \mathbf{R}}, \qquad \dot{\mathbf{R}} = \frac{\partial h}{\partial \mathbf{P}}, \tag{14}$$

where dot means differentiation with respect to time for fixed $\mathbf{a}$, $\mathbf{P} = \mathbf{v}_n + \mathbf{A} \equiv \mathbf{v}$ is the generalized momentum, and the Hamiltonian of a particle h being a function of momentum $\mathbf{P}$ and coordinate $\mathbf{R}$ is given by the standard expression:

$$h = \frac{1}{2}(\mathbf{P} - \mathbf{A})^2 + \varphi \equiv p + \frac{\mathbf{v}^2}{2},$$

i.e., coincides with the Bernoulli "invariant".

The first equation of the system (14) is the equation of motion (6), written in terms of $\mathbf{a}$ and t, and the second equation coincides with (12).

For new hydrodynamics (6) or for its Hamilton version (14) it is possible to formulate a "new" Kelvin theorem (it is also the Liouville theorem):

$$\Gamma = \oint (\mathbf{P} \cdot d\mathbf{R}), \tag{15}$$

where integration is taken along a loop moving together with the "fluid". Hence, analogously as it was made before while derivation of (5) we get the expression for a new Cauchy invariant:

$$\mathbf{I} = \mathrm{rot}_a \left(P_i \frac{\partial x_i}{\partial \mathbf{a}} \right). \tag{16}$$

Its difference from the original Cauchy invariant (5) consists in that in the equation of motion (12) instead of the velocity $\mathbf{v}$ stands its normal component $\mathbf{v}_n$. As consequence, the "new" hydrodynamics becomes compressible: div $\mathbf{v}_n \neq 0$. Therefore on the Jacobian J of the mapping (13) there are imposed no restrictions. The Jacobian J can take arbitrary values.

From the formula (16) it is easily to get the expression for the vorticity Ω in the given point $\mathbf{r}$ at the instant t (compare with [1, 7]):

$$\Omega(\mathbf{r}, t) = \frac{(\Omega(\mathbf{a}) \cdot \nabla_a)\mathbf{R}(a, t)}{J}, \tag{17}$$

where J is the Jacobian of the mapping (13) equal to

$$J = \frac{\partial(x_1, x_2, x_3)}{\partial(a_1, a_2, a_3)}.$$

Here we took into account that the generalized momentum $\mathbf{P}$ coincides with the velocity $\mathbf{v}$, including $t = 0$: $\mathbf{P}_0(\mathbf{a}) \equiv \mathbf{v}_0(\mathbf{a})$. $\Omega_0(\mathbf{a})$ in this relation is the "new" Cauchy invariant with zero divergence: $\mathrm{div}_a \Omega_0(a) = 0$.

Thus, in the general situation the equation of motion of vortex lines has the form (12) which is completed by the relation (17) and the equation

$$\Omega(\mathbf{r}, t) = \mathrm{rot}_r \mathbf{v}(\mathbf{r}, t) \tag{18}$$

with additional constraint $\mathrm{div}_r \mathbf{v}(\mathbf{r}, t) = 0$.

The equations of motion (12), (18) together with the relation (17) can be considered as the result of partial integration of the Euler equation (3). These new equations are resolved with respect to the Cauchy invariants — an infinite number of integrals of motion, that is a very important issue for numerical integration (see [8, 9]). For the partially integrated system the Cauchy invariants conserve automatically that, however, for direct numerical integration of the Euler equations one needs to test in which extent these invariants remain constant. Probably, this is one of the main restrictions defining accuracy of discrete algorithms for direct integration of the Euler equations.

Another very important property of the vortex line representation is absence of any restrictions on the value of the Jacobian J which do exist, for instance, for transition from the Eulerian description to the Lagrangian one in the original Euler equation (3) (when Jacobian in the simplest situation is equal to unity). The value $1/J$ for the system (12), (18), (17) has a meaning of a density n of vortex lines. This quantity as a function of $\mathbf{r}$ and t, according to (12), obeys the continuity equation:

$$\frac{\partial n}{\partial t} + \mathrm{div}_r(n\mathbf{v}_n) = 0. \tag{19}$$

In this equation $\mathrm{div}_r \mathbf{v}_n \neq 0$ because only the total velocity has zero divergence.

The vortex line representation as a local change of variables $\mathbf{r} = \mathbf{r}(\mathbf{a}, t)$ does not work in singular points, when the vorticity is equal to zero and respectively the normal velocity occurs uncertain. Due to the frozenness of vorticity such points remain in time, advected by fluid. Really, let us consider the point $\mathbf{r} = \mathbf{r}(t)$ which defines from the equation

$$\Omega(\mathbf{r}(t), t) = 0. \tag{20}$$

Differentiate this equation with respect to time we arrive at the equation,

$$\frac{\partial \Omega}{\partial t} + (\dot{\mathbf{r}}(t) \cdot \nabla)\Omega = 0,$$

coinciding with the Euler equation for vorticity in this partial case, $\Omega(\mathbf{r}(t), t) = 0$. Here $\dot{\mathbf{r}}(t) = \mathbf{v}(\mathbf{r}(t), t)$. This proves that these points are advected by flows and can not dissipate or, for instance, transform into cuts.

The velocity $\mathbf{v}$ in these points is defined by inverting the curl operator:

$$\mathbf{v} = \mathrm{curl}^{-1}\Omega. \tag{21}$$

The normal velocity $\mathbf{v}_n$ is not defined in these points. By this reason for the vector field $\tau(\mathbf{r}) \equiv \Omega/|\Omega|$, i.e., for the unit tangent vector to vortex lines, the null points represent topological singularities which can be classified by means of topological methods. This classification is defined by the topological charge as a degree of mapping $\mathcal{S}^2 \to \mathcal{S}^2$ given by the integral,

$$\int_{\partial V} \epsilon_{\alpha\beta\gamma}\ (\tau\cdot[\partial_\beta\tau\times\partial_\gamma\tau])\ dS_\gamma = 4\pi m, \tag{22}$$

where integration is performed over the boundary ∂V of the region V containing the points and the topological charge m takes integer numbers.

The vortex line representation, as a change of variables (17), can be used not only to the Euler equations (2), but in the general case (1) also. In the partial case $\mathcal{H} = \int |\Omega| d\mathbf{r}$ the system (2) becomes integrable [1, 6]: each vortex line represents as a free, but nonlinear object and the system itself as continuous distributed free vortices. This is just a reason of breaking of vortex lines in the integrable three-dimensional hydrodynamics.

5. Connection of charged fluid dynamics with the Hopf equations

Consider the Euler equations for a compressible fluid without pressure (the hydrodynamics of dust) which are sometimes called the Hopf equations:

$$\frac{\partial\rho}{\partial t} + \mathrm{div}\ \rho\mathbf{v} = 0, \tag{23}$$

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = 0 \tag{24}$$

with a sufficient decrease of the velocity $\mathbf{v}$ at infinity.

We would like to note that the model (23), (24) has a lot of astrophysical applications. According to the pioneering idea of Ya.B. Zeldovich [10], the formation of proto-galaxies due to breaking of stellar dust can be described by the system (23), (24) (see also the review [11]).

The equations (23, 24) can be successfully solved by means of the Lagrangian description. In terms of Lagrangian variables the Equation (24) describes the motion of free fluid particles:

$$\mathbf{r} = \mathbf{a} + \mathbf{v}_0(\mathbf{a})t,\ \ \mathbf{v} = \mathbf{v}_0(\mathbf{a}). \tag{25}$$

Here $\mathbf{v}_0(\mathbf{a})$ is initial velocity and $\mathbf{a}$ are initial coordinates of a fluid particle.

For this reason breaking is possible in this model: it happens when trajectories of fluid particles intersect. In terms of the mapping (25), describing the change of variables from the Eulerian description to the Lagrangian one, the process of breaking corresponds to a vanishing Jacobian, J, of the mapping. This process can be considered as a collapse in this system: the density

$$\rho(\mathbf{r},t) = \frac{\rho_0(\mathbf{a})}{J}$$

as well as the velocity gradient become infinite at a finite time in the points where the Jacobian vanishes. The latter follows from the equation for the matrix U with matrix elements

$$U_{ij} = \frac{\partial v_j}{\partial x_i}.$$

This equation can be easily got from (24):

$$\frac{dU}{dt} = -U^2, \tag{26}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla).$$

This equation has the exact solution of the form

$$U = U_0(\mathbf{a})(1 + U_0(\mathbf{a})t)^{-1}. \tag{27}$$

Here $\mathbf{a}$ is the initial coordinates of a fluid particle and $U_0(\mathbf{a})$ is the initial value of the matrix U.

The Jacobian J in this case is expressed as follows:

$$J = \det(1 + U_0(\mathbf{a})t).$$

Thus, at the points where the Jacobian vanishes the Jacoby matrix $\hat{J} = 1 + U_0(\mathbf{a})t$ becomes degenerated.

Assume that the matrix U_0 has three real eigenvalues $\lambda_{0k}(\mathbf{a})$. This assumption means that symmetric part of U, $S = 1/2(U + U^T)$, is greater than its antisymmetric part, $\Omega = 1/2(U - U^T)$ — the vorticity tensor. It takes place if the following inequality takes place:

$$2(\mathrm{tr}\ S)^2 - \frac{2}{3}\mathrm{tr}(S^2) \geq \Omega^2.$$

Then, introducing the projectors $P^{(k)}$ of the matrix $U_0(\mathbf{a})$ ($P^{(k)2} = P^{(k)}$ corresponding to each of the eigenvalues $\lambda_{0k}(\mathbf{a})$), the expression (27) can be rewritten in the form of a spectral expansion:

$$U = \sum_{k=1}^{D} \frac{\lambda_{0k}}{1 + \lambda_{0k}t} P^{(k)}. \tag{28}$$

The projector $P^{(k)}$, being a matrix function of $\mathbf{a}$, is expressed through the eigenvectors for the direct ($U_0(\mathbf{a})\psi = \lambda_0\psi$) and conjugated ($\phi U_0(\mathbf{a}) = \phi\lambda_0$) spectral problems for the matrix $U_0(\mathbf{a})$:

$$P_{ij}^{(k)} = \psi_i^{(k)}\phi_j^{(k)},$$

where the vectors $\psi^{(n)}$ and $\phi^{(m)}$ with different n and m are mutually orthogonal:

$$\psi_i^{(m)}\phi_i^{(n)} = \delta_{mn}.$$

Hence, the determinant of the matrix U is defined by the product,

$$\det\ U = \prod_{k=1}^{D} \frac{\lambda_{0k}}{1+\lambda_{0k}t},$$

and, respectively, the Jacobian J of the mapping $\mathbf{r} = \mathbf{a} + \mathbf{v_0}(\mathbf{a})t$ is given by the expression:

$$J = \prod_{k=1}^{D}(1+\lambda_{0k}t). \tag{29}$$

From (28) it follows also that singularity in U first time appears at $t = t_0$, defined from the condition [11, 13]:

$$t_0 = \min_{k,a}[-1/\lambda_{0k}(\mathbf{a})]. \tag{30}$$

From (28), one can see that near the singular point only one term in the sum (28) survives,

$$U \approx -\frac{P^{(n)}}{\tau + \gamma_{\alpha\beta}\Delta a_\alpha \Delta a_\beta}, \tag{31}$$

where the projector $P^{(n)}$ is evaluated at the point $\mathbf{a} = \mathbf{a_0}$ and $k = n$, corresponding to the minimum (30), $\tau = t_0 - t$, $\Delta\mathbf{a} = \mathbf{a} - \mathbf{a_0}$, and

$$2\gamma_{\alpha\beta} = -\left.\frac{\partial^2\lambda_{0n}^{-1}}{\partial a_\alpha \partial a_\beta}\right|_{a=a_0}$$

is a positive definite matrix.

The remarkable formula (31) demonstrates that i) the matrix U tends to the degenerate one as $t \to t_0$ and ii) both parts of the matrix U in this limit, i.e., the stress tensor S and the vorticity tensor Ω, become simultaneously infinite (compare with [12]). It is interesting to note that in the vicinity of the singular time the ratio between both parts is fixed and governs by two relations following from the definition of the projector P:

$$P_S = P_S^2 + P_A^2, \;\; P_A = P_S P_A + P_A P_S \,, \tag{32}$$

where P_S and P_A are respectively symmetric ("potential") and antisymmetric (vortical) parts of the projector P. Because the antisymmetric part of the Equation (26) coincides with the equation for vorticity, i.e., Equation (2) the second relation of (32), in particular, provides existence of the collapsing solution for this equation. Important, that the equation for vorticity (2) has the same form for both compressible and incompressible cases.

The formulas (27), (25) together with (28) yield the complete (implicit) solution of the equations (23),(24). The singularity formation described by (31) should be related to the gradient catastrophe [5].

Let us now apply to the Hopf equation (24) the same procedure as to the Euler equation (3) by introducing an effective electromagnetic field. In the given case the formulas, analogous to (6)–(11), read as follows:

$$\frac{\partial \mathbf{v}_n}{\partial t} + (\mathbf{v}_n \cdot \nabla)\mathbf{v}_n = \mathbf{E} + [\mathbf{v}_n \times \mathbf{H}] \tag{33}$$

with the electric and magnetic fields,

$$\mathbf{E} = -\nabla \left(\frac{v_\tau^2}{2} \right) - \frac{\partial \mathbf{v}_\tau}{\partial t} \,, \tag{34}$$

$$\mathbf{H} = \mathrm{rot}\ \mathbf{v}_\tau \,. \tag{35}$$

Here, as before, $\mathbf{v}_n$ and $\mathbf{v}_\tau$ are normal and tangent (relative to the vortex lines) components of the velocity $\mathbf{v}$.

Due to (34) and (35), two Maxwell equations will be satisfied also and two others can be considered as definition of the current and the charge density. Because of the model (24) is integrable, solutions to the charged hydrodynamics (33) together with the Maxwell equations, corresponding to arbitrary solution of (24), can be written. In opposite case any solution of the charged hydrodynamics and Maxwell equations in the generally situation is not a solution of the Euler equation (24). Thus, we can say the equation of motion (33) plus the Maxwell equations have the whole integrable sub-manifold of solutions which can be constructed by means of solution given by (25) and (27). It is necessary to mention that the corresponding solution will be represented in the implicit form also.

6. Conclusion

Thus, we demonstrate a new possibility of constructing the whole integrable sub-manifold of solutions to the charged hydrodynamics. If for the Euler equations for incompressible fluids (3) the transformation to the charged *compressible* hydrodynamics can be considered as a technical detail to derive the vortex line representation, for the Hopf equations this approach yields new solutions to the charged hydrodynamics. Important also that all these solutions, written in the implicit form, describe breaking. It seems that this is a typical phenomenon not only for compressible hydrodynamics but for incompressible Euler equations also. At least, all numerical experiments (besides [8, 9] see also [14–17] demonstrate formation of singularities with blow-up behavior for the vorticity like $\sim (t_0 - t)^{-1}$. The papers [8, 9] based on the vortex line representation show that blow-up tendency can be interpreted as breaking of vortex lines. For the Hopf equations appearance of singularities takes place simultaneously for symmetric ("potential") and antisymmetric (vortical) parts of the matrix U. In this case the vorticity tensor Ω defines the tangent component of the velocity in the Hopf equations that allows one to construct the effective electric and magnetic fields. This means that any solution of the Hopf equations yields some solution to the charged hydrodynamics, but not vise versa. The same statement is valid also for correspondence between the Euler equations for incompressible fluids and the charged hydrodynamics. The similar correspondence can be established also between the Euler equations for barotropic flows and the charged hydrodynamics. The latter question, however, lies beyond the scope of this paper and will be published in another place.

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