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Sturm–Liouville problem with a distributed condition

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A special problem for the standard liner differential equation of 2-nd order on $[0, 1]$ is investigated when one of boundary conditions must be orthogonal to a given measure on $[0, 1]$. The measure and the potential are complex-valued. The main theorem yields some conditions for the alternative: the codimension or the linear span of the root functions in $C[0, 1]$ is either 1 or ∞. The transformation operators are applied to reduce the problem to the theory of entire functions.

Dedicated to the 80-th birthday of Vladimir Alexandrovich Marchenko

We consider the spectral problem

$$-y'' + q(x)y = \lambda^2 y,\ 0 \le x \le 1, \tag{0.0.1}$$

under the conditions

$$y'(0) - hy(0) = 0,\ \int_0^1 y\,\mathrm{d}\omega = 0, \tag{0.0.2}$$

where ω is a complex valued Borel measure on $[0, 1]$ *the support of which contains the endpoint* $x = 1$; $h \in \mathbb{C}$ and $q : [0, 1] \to \mathbb{C}$ is a continuous function. Even if h and q are real, the problem is not selfadjoint as long as $\mathrm{supp}(\omega)$ consists of at least two points. Such "distributed boundary conditions" appeared in the classicals works of Tamarkin [6] devoted to the n-th order problems where the so-called regular boundary conditions can be disturbed by some integral terms (see also [1] and [2] for a further development). Tamarkin's method is based on the asymptotics for the finite Laplace transform of ω under some natural assumptions. In application to our problem (1)–(2) this yields the following result.

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Let us say that ω is of class $\mathcal{D}_2$ if it is absolutely continuous on $[0,1]$ together with its first derivative ω' and, moreover, the second derivative ω'' has bounded variation. If ω is of class $\mathcal{D}_2$ and, in addition, $\omega'(1) \neq 0$ then the system of root functions (the eigen- and adjoint functions) arising in the problem (1)–(2) is complete in the domain of the corresponding differential operator in the space $C[0,1]$. This means that the codimension of the closed linear span of root functions in $C[0,1]$ is equal to 1. Indeed, the closure of the domain of the operator consists of those continuous functions which satisfy the second of boundary conditions (2).

In the present paper we prove the following

Main Theorem. *If ω does not belong to the class $\mathcal{D}_2$ or $\omega'(1) \neq 0$ then the closure in $C[0,1]$ of the linear span of root functions has codimension 1 or ∞.*

In other words, if $\omega \notin \mathcal{D}_2$ or $\omega'(1) \neq 0$ then either any function $f \in C[0,1]$ with

$$\int_0^1 f \, \mathrm{d}\omega = 0 \qquad (0.0.3)$$

is the uniform limit of linear combinations of the root functions or there is an infinite-dimensional subspace of $C[0,1]$ consisting of functions which do not admit the above mentioned approximation. Note that the condition (0.0.3) is necessary for this approximation.

A similar result for the 1st order problem was obtained in [4] by an entire function theory method. Now we combine this method with the transformation operator technique to investigate the problem (0.0.1)–(0.0.2). Let us stress that the transformation operators were introduced by J. Delsart and applied by V. Marchenko and others to a wide range of very important direct and inverse problems of spectral theory of differential operators, see e.g. [5] and the references there.

We include the Main Theorem into a wider context arising in some natural applications of entire functions to the problem (0.0.1)–(0.0.2). All information from the entire function theory we need can be found in [3, Ch. 1, 3, 5].

For any complex λ we denote by $\varphi(x,\lambda)$ the solution of the equation (0.0.1) under the Cauchy conditions $y'(0) = h$, $y(0) = 1$, so that $\varphi(x,\lambda)$ satisfies the first boundary conditions (0.0.2). All solutions of the problem (0.0.1)–(0.0.2) are proportional to $\varphi(x,\lambda)$. Hence, the problem has a nontrivial solution if and only if λ is a root of the function

$$\Delta(\lambda) = \int_0^1 \varphi(x,\lambda) \, \mathrm{d}\omega(x) \qquad (0.0.4)$$

and the solution is $\varphi(x,\lambda)$ up to proportionality.

Obviously, $\varphi(x,-\lambda)=\varphi(x,\lambda)$ hence, $\varphi(x,\sqrt{\mu})$ is uniquely determined. This is an entire function of μ as it follows from the general theory of ordinary differential equations. The set

$$S=\{\mu\in\mathbb{C}:\ \mu=\lambda^2,\ \Delta(\lambda)=0\} \tag{0.0.5}$$

is the discrete spectrum (the set of eigenvalues) of the operator $-y''+q(x)y$ under the conditions (0.0.2), the eigenfunctions are $\varphi(x,\sqrt{\mu})$, $\mu\in S$. Now let m_μ be the multiplicity of the root $\lambda=\sqrt{\mu}$ in (0.0.4). If $m_\mu\geq 2$ then

$$\Delta^{(j)}(\sqrt{\mu})=0,\ 1\leq j\leq m_\mu-1 \tag{0.0.6}$$

and the corresponding adjoint functions are

$$\frac{\mathrm{d}^j\varphi(x,\sqrt{\mu})}{\mathrm{d}\mu^j},\ 1\leq j\leq m_\mu-1. \tag{0.0.7}$$

All of them satisfy the second condition (0.0.2).

We denote by $\mathcal{R}$ the system of the above mentioned root functions corresponding to all $\mu\in S$. Consider its annihilator $\mathcal{R}^\perp$ in the dual space $C[0,1]^*$. (The latter is the space of complex valued Borel measures on $[0,1]$.) The quantity $\dim\mathcal{R}^\perp$ is called the *deficiency* of the system $\mathcal{R}$ and denoted by $\mathrm{def}\mathcal{R}$. Since $\omega\in\mathcal{R}^\perp$, we have

$$1\leq \mathrm{def}\mathcal{R}\leq\infty. \tag{0.0.8}$$

The Main Theorem states that the only values of $\mathrm{def}\mathcal{R}$ are 1 and ∞ if $\omega\notin\mathcal{D}_2$ or $\omega'(1)\neq 0$.

Now recall that the transformation operator is a Volterra integral operator

$$(Tz)(x)=z(x)+\int_0^x K(x,t)z(t)\,\mathrm{d}t,\ 0\leq x\leq 1, \tag{0.0.9}$$

which makes $\varphi(x,\lambda)$ from $\cos\lambda x$, so that

$$\varphi(x,\lambda)=\cos\lambda x+\int_0^x K(x,t)\cos\lambda t\,\mathrm{d}t,\ \lambda\in\mathbb{C}. \tag{0.0.10}$$

This operator with a smooth kernel $K(x,t)$ does exist, see [5]. The dual operator is

$$T^*:\mathrm{d}\sigma(t)\mapsto\mathrm{d}\sigma(t)+\left[\int_t^1 K(x,t)\,\mathrm{d}\sigma(x)\right]\mathrm{d}t,\ 0\leq t\leq 1. \tag{0.0.11}$$

Obviously, T^* preserves the absolute continuity and after the derivation T^* turns into the operator

$$\tilde{T}^* : \sigma'(t) \mapsto \sigma'(t) + \int_t^1 K(x,t)\sigma'(x)\,\mathrm{d}x \tag{0.0.12}$$

acting in $L_1(0,1)$. The operators $T, T^*, \tilde{T}^*$ are invertible and their inverse have the same Volterra form with smooth kernels. Further we set $\sigma^* = T^*\sigma$.

Lemma 1. $1 \in \operatorname{supp}(\omega^*)$.

P r o o f. Suppose the contrary. According to (0.0.11) $\omega\{1\} = 0$ and

$$\mathrm{d}\omega(t) + \left[\int_t^1 K(x,t)\,\mathrm{d}\omega(x)\right]\mathrm{d}t = 0,\ 1-\varepsilon < t < 1,$$

for some $\varepsilon > 0$. We see that ω is absolutely continuous on $(1-\varepsilon, 1)$ and

$$\omega'(t) + \int_t^1 K(x,t)\omega'(x)\,\mathrm{d}x = 0,\ 1-\varepsilon < t < 1.$$

This Volterra's integral equation has the only solution $\omega' = 0$. Ultimately, $1 \notin \operatorname{supp}(\omega)$ that contradicts a preliminary assumption on ω. ■

By substitution from (0.0.10) into (0.0.4) we obtain

$$\Delta(\lambda) = \int_0^1 \cos\lambda t\,\mathrm{d}\omega^*(t). \tag{0.0.13}$$

We see that $\Delta(\lambda)$ is an entire even function of exponential type $\tau \leq 1$.

Lemma 2. *The exponential type τ of $\Delta(\lambda)$ is equal to* 1.

P r o o f. It follows from (0.0.13) that

$$\int_0^\infty \Delta(\lambda)e^{-\zeta\lambda}\,\mathrm{d}\lambda = \int_0^1 \frac{\zeta\mathrm{d}\omega^*(t)}{\zeta^2 + t^2} \tag{0.0.14}$$

for $\mathrm{Re}\zeta > \tau$. This Laplace transform coincides with the Borel transform of $\Delta(\lambda)$ in the half-plane $\mathrm{Re}\zeta > \tau$. Formula (0.0.14) shows that the singularities of the

Borel transform lie on the segment $[-i,i]\subset i\mathbb{R}$. On the other hand, the minimal segment containing the singularities is the mirror reflection in $\mathbb{R}$ of the indicator diagram of $\Delta(\lambda)$, according to the well known Polya Theorem. By (0.0.13) the indicator diagram of $\Delta(\lambda)$ is the segment $[-i\tau, i\tau]$. If $\tau<1$ then the point $\zeta=i$ is regular for the integral on the right hand side of (0.0.14). This contradicts Lemma 1. ■

Lemma 3. *The spectrum $\mathcal{S}$ is an infinite sequence going to infinity.*

P r o o f. By the Hadamard Factorization Theorem

$$\Delta(\lambda)=c\lambda^{2m}\prod_n\left(1-\frac{\lambda^2}{\lambda_n^2}\right), \tag{0.0.15}$$

where $c=\text{const}\neq 0$, $m\geq 0$ and $(\pm\lambda_n)$ is the sequence of nonzero roots of $\Delta(\lambda)$ repeated according to their multiplicities. A priori, this sequence may be finite (even empty). This is just the possibility we have to disprove.

Suppose to the contrary. Then $\Delta(\lambda)=P(\lambda^2)$ where $P(\mu)$ is a polynomial. If $\deg P=p$ then $(2p+1)$-th derivative of $\Delta(\lambda)$ is identically zero. By (0.0.13) we obtain

$$\int_0^1 t^{2p+1}\sin\lambda t\,\mathrm{d}\omega^*(t)\equiv 0.$$

However, the system $\left\{t^{2p+1}\right\}_{p=0}^{\infty}$ is complete in the space of $f\in C[0,1]$ such that $f(0)=0$. Hence, $\mathrm{supp}(\omega^*)\subset\{0\}$ that contradicts Lemma 1. ■

It is interesting to note that

$$\sum_n\frac{1}{|\lambda_n|^{1+\alpha}}<\infty$$

for any $\alpha>0$ since $\Delta(\lambda)$ is an entire function of order 1. Hence,

$$\sum_{\mu\in\mathcal{S}}\frac{m_\mu}{|\mu|^{\frac{1}{2}+\beta}}<\infty,\ \beta>0. \tag{0.0.16}$$

A more subtle information about the distribution of the spectrum in the complex plane can be extracted from the theory of functions of completely regular growth. The function $\Delta(\lambda)$ belongs to this class because of its boundedness on the real axis. Moreover, its indicator diagram is the segment $[-i,i]$. According to the above mentioned theory, the part of spectrum $\mathcal{S}$ lying out any angle $|\arg\mu|<\eta$ is a set of zero density at the order 1/2, i.e.

$$\lim_{r\to\infty}\frac{\#\{\mu\in\mathcal{S}:|\,\mu|<r,|\arg\mu|\geq\eta\}}{\sqrt{r}}=0, \tag{0.0.17}$$

meanwhile,

$$\lim_{r\to\infty}\frac{\#\{\mu\in\mathcal{S}:|\mu|<r,|\arg\mu|<\eta\}}{\sqrt{r}}=\frac{1}{\pi}. \tag{0.0.18}$$

Let us come back to the system $\mathcal{R}$ of root functions. A measure σ belongs to $\mathcal{R}^\perp$ if and only if the entire function

$$\int_0^1\varphi(x,\lambda)\,\mathrm{d}\sigma(x)$$

vanishes at all roots of $\Delta(\lambda)$ with at least the same multiplicities. This is equivalent to that the quotient

$$\Phi_\sigma(\lambda)=\frac{\int_0^1\varphi(x,\lambda)\,\mathrm{d}\sigma(x)}{\Delta(\lambda)}=\frac{\int_0^1\cos\lambda t\,\mathrm{d}\sigma^*(t)}{\Delta(\lambda)} \tag{0.0.19}$$

is an entire function. The indicator diagram of $\Phi_\sigma(\lambda)$ is the singleton $\{0\}$. Indeed, the indicator diagram of the numerator in (0.0.19) is contained in $[-i,i]$ while the latter is the indicator diagram of $\Delta(\lambda)$ by (0.0.13) and Lemma 2. It remains to refer to the addition theorem for the indicator diagrams of functions of completely regular growth. Thus, we have

Lemma 4. *The exponential type of the entire function $\Phi_\sigma(\lambda)$, $\sigma\in\mathcal{R}^\perp$, is equal to zero.*

Formula (0.0.19) defines the linear mapping $\sigma\mapsto\Phi_\sigma$ from $\mathcal{R}^\perp$ onto a subspace $\mathcal{F}$ of the space of entire even functions of zero exponential type. This mapping is an isomorphism because of invertibility of the dual transformation operator T^*. Therefore,

$$\dim\mathcal{F}=\dim\mathcal{R}^\perp=\mathrm{def}\mathcal{R}. \tag{0.0.20}$$

The space $\mathcal{F}$ contains all constants since $\omega\in\mathcal{R}^\perp$.

If $\dim\mathcal{F}>1$ then there is $\sigma\in\mathcal{R}^\perp$ with $\Phi_\sigma\neq\mathrm{const}$. Every such Φ_σ has a root by the Hadamard Factorization Theorem applying to the functions of zero exponential type.

Lemma 5. *If $\Phi_\sigma\in\mathcal{F}$ and $\Phi_\sigma(\alpha)=0$ then*

$$\frac{\Phi_\sigma(\lambda)}{\lambda^2-\alpha^2}\in\mathcal{F}. \tag{0.0.21}$$

P r o o f. We have to show that there exists $\tau \in \mathcal{R}^{\perp}$ such that

$$\left(\lambda^2-\alpha^2\right)\int_0^1 \varphi(x,\lambda)\,\mathrm{d}\tau(x)=\int_0^1 \varphi(x,\lambda)\,\mathrm{d}\sigma(x)$$

or, equivalently,

$$\left(\lambda^2-\alpha^2\right)\int_0^1 \cos\lambda t\,\mathrm{d}\tau^*(t)=\int_0^1 \cos\lambda t\,\mathrm{d}\sigma^*(t). \tag{0.0.22}$$

However, the latter equation can be explicitly resolved. The answer

$$\mathrm{d}\tau^*(t)=\left[\int_t^1 \frac{\sin\alpha(t-x)}{\alpha}\,\mathrm{d}\sigma^*(x)\right]\mathrm{d}t \tag{0.0.23}$$

can be directly verified by substitution into (0.0.22). (For $\alpha=0$ the kernel in (0.0.23) must be replaced by $t-x$.) ■

Lemma 6. *If* $\dim\mathcal{F}=\delta$, $1<\delta<\infty$, *then* $\mathcal{F}$ *is the space of all even polynomials of degrees* $<2\delta$.

P r o o f. Let Φ_σ be not a polynomial. Then the set of roots of Φ_σ is infinite. Take some pairwise distinct roots $\pm\alpha_1,\pm\alpha_2,\dots,\pm\alpha_\delta$ and get $\delta+1$ linearly independent functions

$$\Phi_\sigma(\lambda),\frac{\Phi_\sigma(\lambda)}{\lambda^2-\alpha_1^2},\frac{\Phi_\sigma(\lambda)}{(\lambda^2-\alpha_1^2)(\lambda^2-\alpha_2^2)},\dots,\frac{\Phi_\sigma(\lambda)}{(\lambda^2-\alpha_1^2)(\lambda^2-\alpha_2^2)\dots(\lambda^2-\lambda_\delta^2)}. \tag{0.0.24}$$

All of them belong to $\mathcal{F}$ by Lemma 5. This contradicts $\dim\mathcal{F}=\delta$. Thus, all functions from $\mathcal{F}$ are polynomials.

Now let $\Phi_\sigma\in\mathcal{F}$ be a polynomial of the maximal possible degree which we denote by $2\delta'-2$. Using the roots of Φ_σ for the construction of type (0.0.24) we obtain δ' polynomials of degrees $2\delta'-2, 2\delta'-4,\dots,0$ respectively. All these polynomials belong to $\mathcal{F}$. Therefore, $\mathcal{F}$ contains all even polynomials of degrees $<2\delta'$. As a result, $\mathcal{F}$ coincides with the space of such polynomials. Hence, $\dim\mathcal{F}=\delta'$ and then $\delta'=\delta$. ■

Note that the Main Theorem conditions were still not used. Now we pass to the

P r o o f o f t h e M a i n T h e o r e m. Suppose to the contrary. Then Lemma 6 is applicable. In particular, $\mathcal{F}$ contains the polynomial λ^2, i.e., there is $\sigma \in \mathcal{R}^{\perp}$ such that

$$\int_0^1 \cos\lambda t \,\mathrm{d}\sigma^*(t) = \lambda^2 \int_0^1 \cos\lambda t \,\mathrm{d}\omega^*(t).$$

Hence,

$$\mathrm{d}\omega^*(t) = \left[\int_t^1 (t-x)\,\mathrm{d}\sigma^*(x)\right]\mathrm{d}t,\ \ 0 \le t \le 1. \tag{0.0.25}$$

Applying the inverse transformation operator we obtain

$$\mathrm{d}\omega(t) = \mathrm{d}\omega^*(t) + \left[\int_t^1 Q(x,t)\,\mathrm{d}\omega^*(x)\right]\mathrm{d}t, \tag{0.0.26}$$

where $Q(x,t)$ is a smooth kernel. By (0.0.25) and (0.0.26) ω turns out to be absolutely continuous on $[0,1]$ and

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \int_t^1 \left[(t-y) + \int_t^y Q(x,t)(x-y)\,\mathrm{d}x\right]\mathrm{d}\sigma^*(y). \tag{0.0.27}$$

In turn, the derivative $\omega'(t)$ is absolutely continuous on $[0,1]$. Finally,

$$\frac{\mathrm{d}^2\omega}{\mathrm{d}t^2} = \sigma^*[1,t) + \int_t^1 \left[\int_t^y \frac{\partial Q(x,t)}{\partial t}(x-y)\,\mathrm{d}x - Q(t,t)(t-y)\right]\mathrm{d}\sigma^*(y).$$

So, $\omega''(t)$ has bounded variation. As a result, $\omega \in \mathcal{D}_2$ and $\omega'(1) = 0$ as (0.0.27) shows. This situation contradicts our assumptions. ■

A difficult problem is to completely distinguish between two cases in the Main Theorem. At present do not know any example of infinite deficiency in the frameworks of the Main Theorem. (Such an example exists in a first order problem, cf. [4].)

Regardless of the conditions of the Main Theorem we can suggest a simple construction which allows us to pass from any given finite deficiency δ to $\delta + \nu$ with any ν, $1 \le \nu < \infty$. For simplicity we restrict ourself to the case $q(x) \equiv 0$, $h = 0$.

For a measure ω_0 we consider its integral of order 2ν

$$\mathrm{d}\omega_1 = \left[\frac{1}{\Gamma(2\nu)}\int\limits_t^1 (x-t)^{2\nu-1}\,\mathrm{d}\omega_0(x)\right]\mathrm{d}t. \qquad (0.0.28)$$

Obviously,

$$\int\limits_0^1 \cos\lambda t\,\mathrm{d}\omega_1(t) = \int_0^1 \mathrm{d}\omega_0(x)\left[\frac{1}{\Gamma(2\nu)}\int\limits_0^x (x-t)^{2\nu-1}\cos\lambda t\,\mathrm{d}t\right].$$

The interior integral equals

$$\frac{\cos\lambda x}{\lambda^{2\nu}} - \sum_{k=0}^{\nu-1}(-1)^k\frac{\lambda^{2(k-\nu)}x^{2k}}{(2k)!}.$$

Suppose that

$$\int\limits_0^1 x^{2k}\,\mathrm{d}\omega_)(x) = 0, \quad 0 \le k \le \nu-1. \qquad (0.0.29)$$

Then

$$\int\limits_0^1 \cos\lambda t\,\mathrm{d}\omega_0(t) = \lambda^{2\nu}\int\limits_0^1 \cos\lambda t\,\mathrm{d}\omega_1(t).$$

Hence for any measure σ we have

$$\frac{\lambda^{2\nu}\int_0^1\cos\lambda t\,\mathrm{d}\sigma(t)}{\int_0^1\cos\lambda t\,\mathrm{d}\omega_0(t)} = \frac{\int_0^1\cos\lambda t\,\mathrm{d}\sigma(t)}{\int_0^1\cos\lambda t\,\mathrm{d}\omega_1(t)}. \qquad (0.0.30)$$

Denote by $\mathcal{F}_0$ and $\mathcal{F}_1$ the spaces in role of $\mathcal{F}$ for ω_0 and ω_1 respectively. Let $\dim\mathcal{F}_0 = \delta$, $1 < \delta < \infty$. We prove that

$$\delta_1 = \dim\mathcal{F}_1 = \delta + \nu. \qquad (0.0.31)$$

First of all, $\delta_1 < \infty$, otherwise, there is a measure σ which yields on the right hand side of (0.0.29) a function $\Phi_\sigma \in \mathcal{F}_1$ such that $\Phi_\sigma \neq 0$, $\Phi_\sigma^{(2k)}(0) = 0$, $0 \le k \le \delta+\nu$. Then $\Psi(\lambda) =\equiv \lambda^{-2\nu}\Phi_\sigma(\lambda) \in \mathcal{F}_0$, hence, $\Psi(\lambda)$ is an even polynomial of degree $< 2\delta_0$. However, it must be zero having the root $\lambda = 0$ with multiplicity $\ge 2\delta$.

By Lemma 6 the space $\mathcal{F}_1$ consists of all even polynomials of degree $< 2\delta_1$. On the other hand, $\mathcal{F}_1$ contains $\lambda^{2\delta+2\nu-2}$ since $\lambda^{2\delta-2} \in \mathcal{F}_0$. Hence, $\delta+\nu \le \delta_1$. Finally, $\lambda^{2\delta_1-2\nu-2} \in \mathcal{F}_0$ which contradicts Lemma 6. Thus, we have (0.0.31).

As a more or less concrete example we consider $\omega_0 = \varepsilon + \rho$ where ε is Dirac's measure concentrated at the point 1 and ρ is absolutely continuous, i.e. $\mathrm{d}\rho = p\mathrm{d}t$ where $p \in \mathrm{L}_1(0,1)$. Hence,

$$\int_0^1 \cos\lambda t\,\mathrm{d}\omega(t) = \cos\lambda + \int_0^1 p(t)\cos\lambda t\,\mathrm{d}t.$$

On the imaginary axis we have

$$\int_0^1 \cos i\mu t\,\mathrm{d}\omega(t) = \frac{1}{2}e^{|\mu|} + o(1), \quad |\mu| \to \infty.$$

Then (0.0.19) shows that $\Phi_\sigma(i\mu)$ is bounded on $i\mathbb{R}$. With Lemma 4 we conclude that $\Phi_\sigma(\lambda)$ is constant. Thus, $\dim\mathcal{F} = 1$ so, $\operatorname{def}\mathcal{R} = 1$ by (0.0.20).

The construction (0.0.28) provide us with an example of deficiency $\nu + 1$, $1 \le \nu < \infty$, as soon as (0.0.29) is fulfilled for the above considered measure ω_0. We have faced the power moment problem

$$\int_0^1 x^{2k}p(x)\,\mathrm{d}x = -1, \quad 0 \le k \le \nu - 1$$

for $p \in L_1(0,1)$. A solution p does exist in the space of all odd polynomials of degree $< 2\nu$.

Let us conclude a simple example of infinite deficiency. Take the measure $\mathrm{d}\omega(t) = p(t)\mathrm{d}t$ where $p \in C^\infty[0,1]$ is such that $p^{(2k+1)}(0) = 0$, $p^{(k)}(1) = 0$ for all $k \ge 0$ while $p(t)$ is not identically zero in any neighborhood of the endpoint $t = 1$. Then

$$\int_0^1 \cos\lambda t\,\mathrm{d}\omega(t) = \frac{1}{\lambda^{2k}}\int_0^1 p^{(2k)}(t)\cos\lambda t\,\mathrm{d}t \quad (k \ge 0)$$

and we see that $\mathcal{F}$ contains all even polynomials.

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