Matematicheskaya fizika, analiz, geometriya
2003, v. 10, No. 3, p. 301–306

About the string with beads

A.D. Myshkis and A.M. Filimonov

Moscow State University of Communications
15 Obraztsova Str., Moscow, 127994, Russia
E-mail:afilimonov@mtu-net.ru

Received April 8, 2003

We consider a classical problem of free oscillations of the elastic weightless string with N beads which has been originally studied by Lagrange. We prove that for N being prime or a power of 2 the maximal displacement of the bead from its equilibrium position increases to infinity as $N \to \infty$ while the total energy of system remains bounded by independent on N constant.

We consider a classical problem of free oscillations of an elastic weightless string with beads. This problem is given by the following system:

$$m\ddot{z}_j = c(z_{j+1} - 2z_j + z_{j-1}),\ j = 1, \dots, N-1,\ m > 0,\ c > 0; \tag{1}$$

with the following boundary and initial conditions

$$z_0(t) = 0; \tag{2}$$

$$z_N(t) = 0; \tag{3}$$

$$z_j(0) = z_j^0,\ \dot{z}_j(0) = z_j^1,\ j = 1, \dots, N-1. \tag{4}$$

Here $(j, t) \mapsto z_j(t)$ $(0 \le t < \infty)$ are unknown functions and the dots over z denote the derivatives with respect to the time variable t. The initial conditions $j \mapsto z_j^0$, $j \mapsto z_j^1$ $(j = 1, \dots, N-1)$ in (4) are assumed to be known.

For the first time the solution of the problem above was given by Lagrange [1]:

$$z_j(t) = \frac{2}{N} \sum_{k=1}^{N-1} \sin \frac{\pi k}{N} j \sum_{r=1}^{N-1} (z_j^0 \sin \frac{\pi k}{N} r \cos \omega_k t + \frac{z_j^1}{\omega_k} \sin \frac{\pi k}{N} r) \sin \omega_k t, \tag{5}$$
$$j = 1, \dots, N-1,$$

Mathematics Subject Classification 2000: 39A12.
The work is supported by Russian Foundation for Basic Research and Basic Research Foundation of Ministry of Railways of Russian Federation.
Key words: Oscillations of the elastic string with beads, large wave effect, asymptotic properties.

$$\omega_k = 2\sqrt{\frac{c}{m}}\sin\frac{\pi k}{2N}, \quad k = 1, \dots, N-1. \tag{6}$$

In the original Lagrange work [1] the system (1)–(4) was used to describe the behavior of the "elastic chain of beads", i.e., the system of $N+1$ material points of equal mass m oscillating along a straight line due to elastic "springs" of stiffness c connecting neighboring points. For this mechanical problem the function $(j,t) \mapsto z_j(t)$ describes the displacement of the point j from its equilibrium position.

Slightly modified problem (1)–(4) was studied in [2]. In this work the particular initial values

$$z_j^0 = 0, \qquad z_j^1 = 0, \qquad j = 1, \dots, N-1, \tag{7}$$

are chosen and the initial condition (2) is replaced by

$$z_0(t) = F (= \text{const}). \tag{8}$$

The modified problem still describes the oscillations of $N+1$ material points but the displacement of the leftmost point is now equal to F.

From now on we refer to the problem (1)–(4) as to the problem **L**.

Already Lagrange had noted [1] that the oscillations described by problem **L** are non-periodic because the frequencies ω_k are pairwise incommensurable. It was shown in [4] that for N being a prime number or a power of 2 the frequencies ω_k are not only incommensurable but linearly independent over the field of rational numbers $\mathbb{Q}$. We denote by $\mathbb{M}$ the set of such numbers N.

Introduce

$$\zeta_j^N = \sup_{0 \le t < \infty} z_j(t).$$

We have the result from [2]: if $N \in \mathbb{M}$, then

$$\lim_{N\to\infty} \max_j \zeta_j^N = +\infty \tag{9}$$

for the problem **J**.

It was shown in [5] that for N being a power of 2, $s \in (0,1)$ being a diadic number and $j = sN$ the quantity ζ_j^N increases logarithmically as $N \to \infty$.

A popular explanation of the large wave effect is that this effect is produced by the energy pumping due to the constant exterior force. In this article we show that for the problem **L** the asymptotic (9) is true even in the case when the total energy of the system is an independent on N constant. Since $z_j(t)$ is a displacement of material point of number j the expression

$$W = \frac{m}{2}\sum_{j=0}^{N} \dot{z}_j^2 + \frac{c}{2}\sum_{j=1}^{N}(z_j - z_{j-1})^2 \tag{10}$$

is a total energy of the system.

Theorem 1. *Take $1 \le q \le N-1$ and $l > 0$ and consider problem* $\boldsymbol{L}$ *with* $N \in \mathbb{M}$ *and initial conditions*

$$z_j^1 = 0, \ j = 1, \dots, N-1, \tag{11}$$

$$z_j^0 = l, \ \ 1 \le j \le q, \quad z_j^0 = 0, \ \ q < j \le N-1. \tag{12}$$

Then

$$\zeta_j^N = \frac{l}{N} \sum_{k=1}^{N-1} \Big| \sin(\frac{\pi k}{N} j) \frac{\cos \frac{\pi k}{2N} - \cos \frac{\pi k}{2N}(2q+1)}{\sin \frac{\pi k}{2N}} \Big|. \tag{13}$$

P r o o f. Substituting (11) and (12) into (5), we obtain

$$z_j(t) = \frac{2l}{N} \sum_{k=1}^{N-1} \sin \frac{\pi k}{N} j \left(\sum_{r=1}^{q} \sin \frac{\pi k}{N} r \right) \cos \omega_k t, \quad j = 1, \dots, N-1, \tag{14}$$

where ω_k are given by (6).

It follows from the Kronecker theorem (see, for example, [6]) that

$$\zeta_j^N = \frac{2l}{N} \sum_{k=1}^{N-1} | \sin \frac{\pi k}{N} j \sum_{r=1}^{q} \sin \frac{\pi k}{N} r |, \tag{15}$$

if ω_k $(k = 1, \dots, N-1)$ are linear independent over the field of rational numbers $\mathbb{Q}$.

Now one can obtain (13) from (15) by calculating the internal sum in (15). ■

R e m a r k. If $z_j(t)$ is the solution of problem $\mathbf{L}$ with initial conditions (11) and (12) then $W(t) \equiv cl^2$ which does not depend on t and N. ■

Asymptotic properties of the solution ζ_j^N of the problem $\mathbf{L}$ with initial conditions (11) and (12) are given by

Theorem 2. *If q in (12) does not depend on N then ζ_j^N is bounded:*

$$\zeta_j^N < 2q. \tag{16}$$

If $q = N-1$ and $N \in \mathbb{M}$ then

$$\lim_{N \to \infty} \zeta_j^N = \frac{4l}{\pi^2} \ln j + O(1) \tag{17}$$

as $j \to \infty$.

P r o o f. The estimate (16) is a direct consequence of (14) and the bound $|\sin\alpha| \le 1$.

Consider now the case $q = N - 1$ and $N \in \mathbb{M}$. Denote

$$\xi_k = \frac{\pi k}{2N}, \qquad \Delta\xi_k = \frac{\pi}{2N}.$$

Then from Theorem 1 we obtain

$$\zeta_j^N = \frac{2l}{\pi}\sum_{k=1}^{N-1} |\sin(2j\xi_k)\frac{\cos\xi_k - \cos(2N-1)\xi_k}{\sin\xi_k}|\Delta\xi_k$$

$$= \frac{2l}{\pi}\sum_{k=1}^{N-1} |\sin(2j\xi_k)\frac{\cos\xi_k}{\sin\xi_k}(1-(-1)^k)|\Delta\xi_k. \tag{18}$$

The sum in the right hand side of (18) can be taken over odd k only, and we can understand $2\zeta_j^N$ as the Riemann sum for the integral

$$\frac{4l}{\pi}\int_0^{\frac{\pi}{2}} |\sin 2j\xi| \cot\xi \, d\xi.$$

Therefore

$$\zeta_j^N \to \frac{2l}{\pi}\int_0^{\frac{\pi}{2}} |\sin 2j\xi| \cot\xi \, d\xi \tag{19}$$

as $N \to \infty$.

The right hand side of (19) can be represented as

$$\frac{2l}{2j\pi}\int_0^{j\pi} |\sin s| \cot\frac{s}{2j}\, ds = \frac{l}{j\pi}\int_\pi^{j\pi}\Big(|\sin s| - \frac{2}{\pi}\Big)\cot\frac{s}{2j}\, ds$$

$$+\frac{2l}{j\pi^2}\int_\pi^{j\pi} \cot\frac{s}{2j}\, dj + O(1). \tag{20}$$

From the Second theorem on mean value we obtain that the first term in the right hand side of (20) is equal to

$$\frac{l}{j\pi}\cot\frac{\pi}{2j}\int_\pi^{\sigma}\Big(|\sin s| - \frac{2}{\pi}\Big)\, ds, \qquad \pi \le \sigma \le j\pi.$$

For $j \to \infty$ the last expression is bounded because $\frac{2}{\pi}$ is equal to a mean value of the periodic function $s \mapsto |\sin s|$. The second term in the right hand (20) is equal to

$$\frac{2l}{j\pi^2}\cdot(-2j\ln\sin\frac{\pi}{2j}) = \frac{4l}{\pi^2}\ln j + O(1)$$

as $j \to \infty$. This immediately implies (17). ■

Clearly (9) is a consequence of (17). Thus for $q = N - 1$ the expression (9) holds true despite the fact that the energy W is independent on N. Hence the large wave effect is not related to the energy pumping.

We conclude with the discussion of some open problems related to the large wave effect. The first question is how essential is the condition $N \in \mathbb{M}$ for (9). The second question is how fast $\max_j \zeta_j^N$ increases as $N \to \infty$.

In this paper we have studied two cases: $q = \text{const}$ and $q = N - 1$. Also it can be proven that (9) remains true if q increases linearly in N. Therefore it is interesting to understand what is the slowest grows rate $q(N)$ for which (9) remains true.

The next interesting question is a nature of large waves and the time moments of their appearance. We can note the following fact. For $m = \rho N$, $c = E/N$ the system (1) can be considered as an approximation of the wave equation [7]

$$\rho u_{tt} - E u_{xx} = 0.$$

But for the wave equation there is no effect similar to the effect of a large wave. This fact does not contradict to the convergence of the line method [7] because the convergence follows from the estimates of the solution of the wave equation and of system of ordinary differential equations given for the bounded time intervals. Our proof is true for infinite time interval only.

The existence of times when a large wave is observed is a consequence of Kronecker's theorem. The characterization of the set of these times is an open problem.

The effect of large waves is based on the linear independency of the frequencies ω_k over $\mathbb{Q}$. This is the case when the oscillations of the chain are most chaotic [8]. However the relation between chaotic properties and the effect of large waves is an open problem.

For the problem **L** the large wave $\{z_j(t)\}$ is mildly sloping along x-axis and evolves sufficiently slowly because the total energy of the system is constant (10) and the kinetic and potential parts of total energy are not too high. Therefore the large wave should exist for a certain time intervals. The existence and the structure of these intervals is also an open problem.

Acknowledgement. We thank Prof. A.E. Mazel for useful discussion.

References

[1] *G.L. Lagrange*, Mechanique analitique. Paris (1788) (Russian); G.L. Lagrang, Analytical mechanics. V. 1. NTI NKTP SSSR, Moscow, Leningrad (1938).

[2] *A.M. Filimonov, P.F. Kurchanov, and A.D. Myshkis*, Some unexpected results on the classical problem of the string with N beads. — *Comp. Rend. Acad. Sci.* Paris (1991), Ser. 1, t. 313, p. 961–965.

[3] *N.E. Jukovskii*, Action Russian and American mechanism in train. Complect of articles. V. 8. GITTL, Moscow, Leningrad (1937).

[4] *P.F. Kurchanov, A.D. Myshkis, and A.M. Filimonov*, Vibrations of a train and Kronecker's theorem. — *Prikl. mat. i mekhanika* (1991), v. 55, № 6, p. 989–995. (Russian)

[5] *A.M. Filimonov and A.D. Myshkis*, Asymptotic estimate of solution of one mixed difference-differential equation of oscillations theory. — *J. Diff. Eq. and Appl.* (1998), v. 4, p. 13–16.

[6] *B.M. Levitan*, Almost periodic functions. — GITTL, Moscow (1953).

[7] *V.I. Lebedev*, Eqiations and convergence of differential-difference method (linear line method). — *Vestn. MGU, Matematika* (1955), v. 10, p. 47–57.

[8] *A.M. Filimonov, S.A. Maslov, and X. Mao*, Splash effect and ergodic properties of solution of the classic difference-differential equation. — *Diff. Eq. and Appl.* (2000), v. 6, p. 319–328.