Matematicheskaya fizika, analiz, geometriya
2003, v. 10, No. 3, p. 307–325

The Riemann extensions in theory of differential equations and their applications

Valerii Dryuma

Institute of Mathematics and Informatics, AS RM
5 Academiei Str., Kishinev, 2028, Moldova
E-mail:valery@gala.moldova.su

Received March 19, 2003

Some properties of the 4-dim Riemannian spaces with the metrics

$$ds^2 = 2(za_3 - ta_4)dx^2 + 4(za_2 - ta_3)dxdy + 2(za_1 - ta_2)dy^2 + 2dxdz + 2dydt$$

connected with the second order nonlinear differential equations

$$y'' + a_1(x,y)y'^3 + 3a_2(x,y)y'^2 + 3a_3(x,y)y' + a_4(x,y) = 0 \qquad (*)$$

with arbitrary coefficients $a_i(x,y)$ are studied. The properties of dual equations for the equations (*) are considered. The theory of the invariants of second order ODE's for investigation of the nonlinear dynamical systems with parameters is used. The property of the eight dimensional extensions of the four-dimensional Riemannian spaces of General Relativity are discussed.

1. Introduction

The second order ODE's of the type

$$y'' + a_1(x,y)y'^3 + 3a_2(x,y)y'^2 + 3a_3(x,y)y' + a_4(x,y) = 0 \qquad (1)$$

are connected with nonlinear dynamical systems in form

$$\dot{x} = P(x,y,z,\alpha_i), \quad \dot{y} = Q(x,y,z,\alpha_i), \quad \dot{z} = R(x,y,z,\alpha_i),$$

where α_i are parameters.

For example, the Lorenz system

$$\dot{x} = \sigma(y - x), \quad \dot{y} = rx - y - zx, \quad \dot{z} = xy - bz$$

Mathematics Subject Classification 2000: 35Q58.
Work supported in part by MURST, Italy.

having chaotic properties at some values of parameters is equivalent to the equation

$$y'' - \frac{3}{y}y'^2 + (\alpha y - \frac{1}{x})y' + \epsilon x y^4 - \beta x^3 y^4 - \beta x^2 y^3 - \gamma y^3 + \delta \frac{y^2}{x} = 0, \tag{2}$$

where

$$\alpha = \frac{b+\sigma+1}{\sigma}, \quad \beta = \frac{1}{\sigma^2}, \quad \gamma = \frac{b(\sigma+1)}{\sigma^2}, \quad \delta = \frac{(\sigma+1)}{\sigma}, \quad \epsilon = \frac{b(r-1)}{\sigma^2},$$

and for investigation of its properties the theory of invariants was first used in [1–5].

Other example is the third order differential equations like

$$\frac{d^3x}{dt^3} + a\frac{d^2x}{dt^2} - \left(\frac{dx}{dt}\right)^2 + x = 0$$

with parameter a having chaotic properties at the values $2,017 < a < 2,082$ [16]. It can be transformed to the form (1)

$$y'' + \frac{1}{y}y'^2 + \frac{a}{y}y' + \frac{x}{y^2} - 1 = 0 \tag{3}$$

with the help of standard substitution.

According to the Liouville theory [6–9] all equations of type (1) can be divided in two different classes:

I. $\nu_5 = 0$, II. $\nu_5 \neq 0$.

Here the value ν_5 is the expression of the form

$$\nu_5 = L_2(L_1 L_{2x} - L_2 L_{1x}) + L_1(L_2 L_{1y} - L_1 L_{2y}) - a_1 L_1^3 + 3a_2 L_1^2 L_2 - 3a_3 L_1 L_2^2 + a_4 L_2^3,$$

then L_1, L_2 are defined by the formulaes

$$L_1 = \frac{\partial}{\partial y}(a_{4y} + 3a_2 a_4) - \frac{\partial}{\partial x}(2a_{3y} - a_{2x} + a_1 a_4) - 3a_3(2a_{3y} - a_{2x}) - a_4 a_{1x},$$
$$L_2 = \frac{\partial}{\partial x}(a_{1x} - 3a_1 a_3) + \frac{\partial}{\partial y}(a_{3y} - 2a_{2x} + a_1 a_4) - 3a_2(a_{3y} - 2a_{2x}) + a_1 a_{4y}.$$

In case $\nu_5 \neq 0$ the absolute invariants are

$$[5t_m - (m-2)t_7 t_{m-2}]\nu_5^{2/5} = 5\Big(L_1 \frac{\partial t_{m-2}}{\partial y} - L_2 \frac{\partial t_{m-2}}{\partial x}\Big), \tag{4}$$

where

$$t_m = \nu_m \nu_5^{-m/5}, \quad \nu_{m+2} = L_1 \nu_{my} - L_2 \nu_{mx} + m\nu_m(L_{2x} - L_{1y}).$$

From the formulaes (4) follows that some relations between the invariants are important for theory of the second order ODE's.

As example in the simplest case $t_9 = at_7^2$ we have

$$t_{11} = a(2a-1)t_7^3,\ t_{13} = a(2a-1)(3a-2)t_7^4,\ t_{15} = a(2a-1)(3a-2)(4a-3)t_7^5 \ \dots .$$

These relations show that some value of parameters

$$a = 0, \quad 1/2, \quad 2/3, \quad 3/4, \quad 4/5\dots$$

are special for the corresponding second order ODE's.

To take the example of equation in form

$$y'' + a_1(x,y){y'}^3 + a_2(x,y){y'}^2 + 3(-xa_2(x,y) - ya_1(x,y))y'$$
$$+(x^2 - y)a_2(x,y) + xya_1(x,y) - 2/3 = 0.$$

In the case

$$a_1(x,y) = -\frac{2x}{5y^2}, \quad a_2(x,y) = \frac{2}{5y}$$

we get the equation with the following series of absolute invariants:

$$\frac{t_9}{t_7^2} = 2, \quad \frac{t_11}{t_7^3} = 6, \quad \frac{t_{13}}{t_7^4} = 24, \quad \frac{t_{15}}{t_7^5} = 120, \quad \frac{t_{17}}{t_7^6} = 720, \quad \frac{t_{19}}{t_7^7} = 5040,$$

$$\frac{t_{21}}{t_7^8} = 40320, \quad \frac{t_{23}}{t_7^9} = 362880, \quad \frac{t_{25}}{t_7^{10}} = 3628800.$$

So we get the example of the second order ODE with the invariants forming the series

$$2, \quad 6, \quad 24, \quad 120, \quad 720, \quad 5040, \quad 40320, \quad 362880, \quad 3628800, \quad \dots .$$

For the second order differential equation (2) equivalent to the Lorenz system the ν_5-invariant has the form $\nu_5 = Ax^2 + \frac{B}{x^2y^2} + C$, where

$$A = \alpha\beta(10\alpha - \alpha^2 - 6\delta), \quad B = \alpha(\frac{4}{9}\alpha^2 + \frac{2}{3}\alpha\delta - 2\delta^2), \quad C = \alpha(\frac{2}{9}\alpha^4 + 6\epsilon\delta - 4\alpha\epsilon - \alpha^2\gamma).$$

In this case the condition $\nu_5 = 0$ corresponds to the conditions $A = 0$, $B = 0$, $C = 0$ which contains for example the values $\sigma = -1/5$, $b = -16/5$, $r = -7/5$ which have not been previously met in theory of the Lorenz system.

Here we present the expressions for the invariants of the second order equation equivalent to the equation (3).

For that we transform the equation (3) to the form

$$y'' + \frac{1}{y}{y'}^2 + (\frac{4}{x} + \frac{4}{xy})y' + \frac{a}{x^2} - \frac{2}{xy} + \frac{1}{x^2y^2} + \frac{y}{x^2} = 0$$

which is more convenient for calculations. For this equation we get the invariants:

$$\nu_5 = -\frac{1}{9}a^3\frac{(2a^2y + 18xy - 9)}{x^5y^4},$$

$$\nu_7 = \frac{1}{27}a^4\frac{(54xy^2 - 27y - 20a^3y - 180axy + 72a)}{x^7y^{15}},$$

$$\nu_9 = \frac{2}{81}a^6\frac{(702xy^2 - 297y - 140a^3y - 1260axy + 432a)}{x^9y^{19}},$$

$$\nu_{11} = \frac{4}{27}a^8\frac{(990xy^2 - 369y - 140a^3y + 1260axy + 384a)}{x^{11}y^{23}},$$

$$\nu_{13} = \frac{40}{81}a^{10}\frac{(2754xy^2 - 927y - 308a^3y - 2772axy + 768a)}{x^{13}y^{27}},$$

$$\nu_{15} = \frac{80}{243}a^{12}\frac{(42714xy^2 - 13203y - 4004a^3y - 36036axy + 9216a)}{x^{15}y^{31}}.$$

From these expressions we can see that only numerical values of coefficients in the formulaes for invariants are changed at the transition from invariant ν_m to ν_{m+2}. This fact can be of use for studying of the relations between the invariants, when the parameter a is changed. Remark that the starting equation (3) is connected with the Painleve-I equation in the case $a = 0$.

Note that the first applications of the Liouville invariants for the Painleve equations was done in the works of author [1–6]. Last results see in [20].

2. The Riemann spaces in theory of the second order ODE's

Here we present the construction of the Riemann spaces connected with the equations of type (1).

The equations (1) with arbitrary coefficients $a_i(x,y)$ may be considered as equations of geodesics of 2-dimensional space A_2

$$\ddot{x} - a_3\dot{x}^2 - 2a_2\dot{x}\dot{y} - a_1\dot{y}^2 = 0,$$
$$\ddot{y} + a_4\dot{x}^2 + 2a_3\dot{x}\dot{y} + a_2\dot{y}^2 = 0$$

equipped with the projective connection Π^k_{ij} dependent from the coefficients $a_i(x,y)$.

For construction of the Riemannian space connected with the equation of type (1) we use the notice of Riemannian extension D^4 of space A_2 with connection Π^k_{ij} [10] . The corresponding metric is

$$ds^2 = -2\Pi^k_{ij}\xi_k dx^i dx^j + 2d\xi_i dx^i,$$

and in our case it takes the following form ($\xi_1 = z$, $\xi_2 = \tau$):

$$ds^2 = 2(za_3 - \tau a_4)dx^2 + 4(za_2 - \tau a_3)dxdy + 2(za_1 - \tau a_2)dy^2 + 2dxdz + 2dyd\tau. \tag{5}$$

So, it is possible to formulate the following statement

Proposition 1. *For a given equation of type (1) there exists the Riemannian space with metrics (5) having integral curves of equation as part of its geodesics.*

Really, the calculation of geodesics of the space W^4 with the metric (5) lead to the system of equations

$$\frac{d^2x}{ds^2} - a_3\left(\frac{dx}{ds}\right)^2 - 2a_2\frac{dx}{ds}\frac{dy}{ds} - a_1\left(\frac{dy}{ds}\right)^2 = 0,$$

$$\frac{d^2y}{ds^2} + a_4\left(\frac{dx}{ds}\right)^2 + 2a_3\frac{dx}{ds}\frac{dy}{ds} + a_2\left(\frac{dy}{ds}\right)^2 = 0,$$

$$\frac{d^2z}{ds^2} + [z(a_{4y} - \alpha'') - \tau a_{4x}]\left(\frac{dx}{ds}\right)^2 + 2[za_{3y} - \tau(a_{3x} + \alpha'')]\frac{dx}{ds}\frac{dy}{ds}$$
$$+[z(a_{2y} + \alpha) - \tau(a_{2x} + 2\alpha')]\left(\frac{dy}{ds}\right)^2$$
$$+2a_3\frac{dx}{ds}\frac{dz}{ds} - 2a_4\frac{dx}{ds}\frac{d\tau}{ds} + 2a_2\frac{dy}{ds}\frac{dz}{ds} - 2a_3\frac{dy}{ds}\frac{d\tau}{ds} = 0,$$

$$\frac{d^2\tau}{ds^2} + [z(a_{3y} - 2\alpha') - \tau(a_{3x} - \alpha'')]\left(\frac{dx}{ds}\right)^2 + 2[z(a_{2y} - \alpha) - \tau a_{2x}]\frac{dx}{ds}\frac{dy}{ds}$$
$$+[za_{1y} - \tau(a_{1x} + \alpha)]\left(\frac{dy}{ds}\right)^2 + 2a_2\frac{dx}{ds}\frac{dz}{ds} - 2a_3\frac{dx}{ds}\frac{d\tau}{ds} + 2a_1\frac{dy}{ds}\frac{dz}{ds} - 2a_2\frac{dy}{ds}\frac{d\tau}{ds} = 0$$

in which the first two equations of the system for coordinates x, y are equivalent to the equation (1).

In turn two last equations of the system for coordinates $z(s)$ and $t(s)$ have the form of the $2\times$ matrix linear second order differential equations

$$\frac{d^2\Psi}{ds^2} + A(x,y)\frac{d\Psi}{ds} + B(x,y)\Psi = 0, \tag{6}$$

where $\Psi(x,y)$ is two component vector $\Psi_1 = z(s)$, $\Psi_2 = t(s)$ and values $A(x,y)$ and $B(x,y)$ are the 2×2 matrix-functions.

Note that full system of equations has the first integral

$$z\dot{x} + t\dot{y} = \frac{s}{2} + \mu\,,$$

which allow us to use only one linear second order differential equation from the full matrix system (6) in applications.

Thus, we have constructed the four-dimensional Riemannian space with the metric (5).

The curvature tensor of this metric has the components

$$R^1_{112} = -R^3_{312} = -R^2_{212} = R^4_{412} = \alpha', \quad R^1_{212} = -R^4_{312} = \alpha,$$

$$R^2_{112} = -R^3_{412} = -\alpha'', R^1_{312} = R^1_{412} = R^2_{312} = R^2_{412} = 0,$$

$$R^3_{112} = 2z(a_2\alpha'' - a_3\alpha') + 2\tau(a_4\alpha' - a_3\alpha''),$$

$$R^4_{212} = 2z(a_3\alpha' - a_2\alpha) + 2\tau(a_3\alpha - a_2\alpha'),$$

$$R^3_{212} = z(\alpha_x - \alpha'_y + a_1\alpha'' - a_3\alpha) + \tau(\alpha''_y - \alpha'_x + a_4\alpha - a_2\alpha''),$$

$$R^4_{112} = z(\alpha'_y - \alpha_x + a_1\alpha'' - a_3\alpha) + \tau(\alpha'_x - \alpha''_y + a_4\alpha - a_2\alpha'').$$

The Ricci tensor $R_{ik} = R^l_{ilk}$ of the space D^4 has the components

$$R_{11} = 2\alpha'', \quad R_{12} = 2\alpha', \quad R_{22} = 2\alpha,$$

where

$$\alpha = a_{2y} - a_{1x} + 2(a_1a_3 - a_2^2), \quad \alpha' = a_{3y} - a_{2x} + a_1a_4 - a_2a_3,$$

$$\alpha'' = a_{4y} - a_{3x} + 2(a_2a_4 - a_3^2)$$

and the scalar curvature $R = g^{in}g^{km}R_{nm}$ of the space D^4 is $R = 0$.

The Weyl tensor of the space D^4 has only one component $C_{1212} = tL_1 - zL_2$. Note that the values L_1 , L_2 and α^i in this formulaes are the same with the Liouville expressions in theory of invariants of the equations (1).

Using the components of the Riemann tensor, the equation $\mid R_{AB} - \lambda g_{AB} \mid = 0$ for determination of the Petrov type of the spaces D^4 have been considered. Here R_{AB} is symmetric 6×6 matrix constructed from the components of the Riemann tensor R_{ijkl} of the space D^4.

In particular we have checked that all scalar invariants of the space D^4 of such types

$$R_{ij}R^{ij} = 0, \quad R_{ijkl}R^{ijkl} = 0, \dots$$

constructed from the curvature tensor of the space M^4 and its covariant derivatives are equal to zero.

Remark 1. *The spaces with metrics (5) are flat for the equations (1) with the conditions* $\alpha = 0$, $\alpha' = 0$, $\alpha'' = 0$, *on coefficients* $a_i(x, y)$.

The equations with this properties have the components of projective curvature $L_1 = 0$, $L_2 = 0$, *and they are reduced to the form* $y'' = 0$ *with help of the points transformations.*

On the other hand, there are examples of equations (1) with conditions $L_1 = 0$, $L_2 = 0$, *but* $\alpha \neq 0$, $\alpha' \neq 0$, $\alpha'' \neq 0$. *For such type of equations the curvature of corresponding Riemann spaces is not equal to zero.*

As example the equation

$$y'' + 2e^{\varphi}{y'}^3 - \varphi_y {y'}^2 + \varphi_x y' - 2e^{\varphi} = 0,$$

where the function $\varphi(x, y)$ *is solution of the Wilczynski–Tzitzeika nonlinear equation integrable by the Inverse Transform Method*

$$\varphi_{xy} = 4e^{2\varphi} - e^{-\varphi} \tag{7}$$

has the conditions $L_1 = 0$, $L_2 = 0$ *but* $\alpha \neq 0$, $\alpha' \neq 0$, $\alpha'' \neq 0$. *In particular even for the linear second order differential equations the corresponding Riamannian spaces are not flat.*

Remark 2. *The studying of the properties of the Riemann spaces with the metrics (5) for the equations (2) with chaotic behavior at the values of coefficients* ($\sigma = 10$, $b = 8/3$, $r > 24$) *is important problem. The spaces with such values of parameters have a specifical geometric structure. To studying this problem the geodesic deviation equation*

$$\frac{d^2\eta^i}{ds^2} + 2\Gamma^i_{lm}\frac{dx^m}{ds}\frac{d\eta^l}{ds} + \frac{\partial\Gamma^i_{kl}}{\partial x^j}\frac{dx^k}{ds}\frac{dx^l}{ds}\eta^j = 0,$$

where Γ^i_{lm} *are the Christoffell coefficients of the metrics (5) with the coefficients*

$$a_1 = 0, \quad a_2 = -\frac{1}{y}, \quad a_3 = (\frac{\alpha y}{3} - \frac{1}{3x}), \quad a_4 = \epsilon x y^4 - \beta x^3 y^4 - \beta x^2 y^3 - \gamma y^3 + \delta\frac{y^2}{x}$$

may be used.

It is interesting to note that for the Painleve II equation $y'' = 2y^3 + xy + \mu$ *the solutions of the geodesic deviation equations depends from the parameter* μ.

3. On relation with theory of the surfaces

The existence of the Riemann metrics for the equations (1) may be used for construction of the corresponding surfaces.

One possibility concerns the study of two-dimensional subspaces of a given 4-dimensional space which are the generalization of the surfaces of translation. The equations for coordinates $Z^i(u,v)$ of such type of the surfaces are

$$\frac{\partial^2 Z^i}{\partial u \partial v} + \Gamma^i_{jk}\frac{\partial Z^j}{\partial u}\frac{\partial Z^k}{\partial v} = 0, \tag{8}$$

where Γ^{jk}_i are the components of connections of a given space.

From these equations we can see that two last equations of the full system are linear and have the form of the linear 2×2 matrix Laplace equations

$$\frac{\partial^2 \Psi}{\partial u \partial v} + A\frac{\partial \Psi}{\partial u} + B\frac{\partial \Psi}{\partial v} + C\Psi = 0. \tag{9}$$

Let us consider some examples.

The first example concerns the conditions

$$u = x, \quad v = y, \quad z = z(u,v) = z(x,y), \quad t = t(x,y) = t(u,v).$$

From the first equations of the full system we get $a_2(x,y) = 0$, $a_3(x,y) = 0$, and from the next two we get the system of equations

$$\frac{\partial^2 z}{\partial x \partial y} - \frac{1}{a_4}\frac{\partial a_4(x,y)}{\partial y}\frac{\partial z}{\partial x} + a_1(x,y)a_4(x,y)z = 0,$$

$$\frac{\partial^2 t}{\partial x \partial y} - \frac{1}{a_1}\frac{\partial a_1(x,y)}{\partial x}\frac{\partial t}{\partial y} + a_1(x,y)a_4(x,y)t = 0.$$

Any solution of such system of equations give us the examples of the surfaces which corresponds to the second order ODE's in form

$$\frac{d^2 y}{dx^2} + a_1(x,y)\left(\frac{dy}{dx}\right)^3 + a_4(x,y) = 0.$$

4. Symmetry, the Laplace–Beltrami equation, tetradic presentation

Let us consider the system of equations

$$\xi_{i,j} + \xi_{j,i} = 0$$

for the Killing vectors of metrics (5). In particular case $\xi_3 = \xi_4 = 0$, $\xi_i = \xi_i(x,y)$ we get the system of equations

$$\xi_{1x} = -a_3\xi_1 + a_4\xi_2, \quad \xi_{2y} = -a_1\xi_1 + a_2\xi_2,$$

$$\xi_{1y}+\xi_{2x}=2[-a_2\xi_1+a_3\xi_2].$$

1. In case $\xi_1=\Phi_x$, $\xi_2=\Phi_y$ we have the system of equations

$$\Phi_{xx}=a_4\Phi_y-a_3\Phi_x,\quad \Phi_{xy}=a_3\Phi_y-a_2\Phi_x,\quad \Phi_{yy}=a_2\Phi_y-a_1\Phi_x\,, \tag{10}$$

which is compatible at the conditions $\alpha=0$, $\alpha'=0$, $\alpha''=0$.

By analogy way the system of equations for the Killing tensor

$$K_{ij;l}+K_{jl;i}+K_{li;j}=0$$

and the Killing–Yano tensor $Y_{ij}=-Y_{ji}$

$$Y_{ij;l}+Y_{il;i}=0$$

for a spaces with the metrics (5) may be investigated.

Remark 3. *The Laplace–Beltrami operator*

$$\Delta=g^{ij}\Big(\frac{\partial^2}{\partial x^i\partial x^j}-\Gamma^k_{ij}\frac{\partial}{\partial x^k}\Big)$$

and corresponding spectral problem $\Delta\Psi=\lambda\Psi$ can be used for investigation of the properties of the metrics (5).

For example, the equation $\Delta\Psi=0$ has the form

$$(ta_4-za_3)\Psi_{zz}+2(ta_3-za_2)\Psi_{zt}+(ta_2-za_1)\Psi_{tt}+\Psi_{xz}+\Psi_{yt}=0, \tag{11}$$

and at some conditions on coefficients $a_i(x,y)$ it can be integrated by the method of separation of variables.

Another possibility for the studying of the properties of a given Riemann spaces is connected with computation of the heat invariants of the Laplace–Beltrami operator. For that the fundumental solution $K(\tau,x,y$ of the heat equation

$$\frac{\partial\Psi}{\partial\tau}=g^{ij}\Big(\frac{\partial^2\Psi}{\partial x^i\partial x^j}-\Gamma^k_{ij}\frac{\partial\Psi}{\partial x^k}\Big)$$

may be considered.

The function $K(\tau,x,y)$ has the following asymptotic expansion on diagonal as $t\to 0+$

$$K(\tau,x,x)=\sim\sum_{n=0}^{\infty}a_n(x)\tau^{n-2}$$

and the coefficients $a_n(x)$ are local invariants (heat invariants) of the Riemann space D^4 with the metrics (5).

Remark 4. *The metric (5) has a tetradic presentation*

$$g_{ij} = \omega^a_i \omega^b_j \eta_{ab}, \quad \eta_{13} = \eta_{24} = 1, \quad \eta_{ij} = 0.$$

From this we get a particular case of the metrics (5)

$$ds^2 = 2\omega^1\omega^3 + 2\omega^2\omega^4,$$

where

$$\omega^1 = dx + dy, \quad \omega^2 = dx + dy + \frac{1}{t(a_2 - a_4)}(dz - dt),$$

$$\omega^4 = -t(a_4 dx + a_2 dy), \quad \omega^3 = z(a_3 dx + a_1 dy) + \frac{1}{(a_2 - a_4)}(a_2 dz - a_4 dt)$$

and

$$a_1 + a_3 = 2a_2, \quad a_2 + a_4 = 2a_3.$$

Remark 5. *Some of equations on curvature tensors in space D^4 connect with ODE's. For example, the conditions*

$$R_{ij;k} + R_{jk;i} + R_{ki;j} = 0$$

lead to the following relations on coefficients $a_i(x, y)$

$$\begin{gathered} \alpha''_x + 2a_3\alpha'' - 2a_4\alpha' = 0, \quad \alpha_y + 2a_1\alpha' - 2a_2\alpha = 0, \\ \alpha''_y + 2\alpha'_x + 4a_2\alpha'' - 2a_4\alpha - 2a_3\alpha' = 0, \\ \alpha_x + 2\alpha'_y - 4a_3\alpha + 2a_2\alpha' + 2a_1\alpha'' = 0. \end{gathered} \tag{12}$$

The solutions of this system give us the examples of the second order ODE's connected with the space D^4 with a given condition on the Ricci tensor. The simplest one are

$$y'' - \frac{3}{2y}y'^2 + y^3 = 0, \quad y'' - \frac{3}{y}y'^2 + y^4 = 0, \quad y'' + 3(2 + y)y' + y^3 + 6y^2 - 16 = 0.$$

5. The Riemann metrics of zero curvature and the KdV equation

The system of matrix equations in form

$$\Gamma_{2x} - \Gamma_y + [\Gamma_1, \Gamma_2] = 0, \quad \Gamma_{3x} - \Gamma_{1z} + [\Gamma_1, \Gamma_3] = 0, \quad \Gamma_{3y} - \Gamma_{2z} + [\Gamma_2, \Gamma_3] = 0, \tag{13}$$

where $\Gamma_i(x, y, z)$ — the 3×3 matrix functions with conditions $\Gamma^k_{ij} = \Gamma^k_{ji}$ are studied.

This system can be considered as the condition of the zero curvature of the some $3 - dim$ space equipping by the affine connection with the coefficients $\Gamma(x, y, z)$. Let $\Gamma^k_{ij}(x, y, z)$ be in the form

$$\Gamma^k_{ij} = \sum^n y^n A_n(x, z)$$

with some matrix-functions $A_n(x, z)$, depending from the variables x, z. Then after substituting these expressions in formulas (13) we get the system of nonlinear equations for components of affine connection. Some of these equations may be of interested for applications.

As example let us consider the space with the metrics

$$g_{ik} = \begin{pmatrix} y^2 & 0 & y^2 l(x,z)+m(x,z) \\ 0 & 0 & 1 \\ y^2 l(x,z)+m(x,z) & 1 & y^2 l(x,z)^2-2yl_x(x,z)+2l(x,z)m(x,z)+2n(x,z) \end{pmatrix}.$$

Then in particular case $l(x, z) = n(x, z)$ we get the components of the Riemann tensor for a given space

$$R_{1313} = \left(\frac{\partial^3 l}{\partial x^3} - 3l\frac{\partial l}{\partial x} + \frac{\partial l}{\partial z}\right) y^2$$
$$+ \left(\frac{\partial^2 m}{\partial x \partial z} - 2m\frac{\partial^2 l}{\partial x^2} - l\frac{\partial^2 m}{\partial x^2} - 3\frac{\partial m}{\partial x}\frac{\partial m}{\partial x} - \frac{\partial^2 l}{\partial x^2}\right) y$$
$$-m\frac{\partial m}{\partial z} + 2m^2\frac{\partial l}{\partial x} + m\frac{\partial l}{\partial x} + ml\frac{\partial m}{\partial x} - m\frac{\partial m}{\partial z} + 2m^2\frac{\partial l}{\partial x} + m\frac{\partial l}{\partial x} + ml\frac{\partial m}{\partial x}$$

and

$$R_{1323} = \left(-\frac{\partial m}{\partial z} + 2m\frac{\partial l}{\partial x} + l\frac{\partial m}{\partial x} + \frac{\partial l}{\partial x}\right)/y.$$

From the condition $R_{ijkl} = 0$ it follows that the function $l(x, z)$ is the solution of the KdV-equation

$$\frac{\partial l}{\partial z} - 3l\frac{\partial l}{\partial x} + \frac{\partial^3 l}{\partial x^3} = 0$$

and all flat metrics of such type with help of solutions of this equation are determined.

Note that after the Riemannian extension of such space the metrics of the six-dimensional space can be written. The equations of geodesics of such type of 6-dim space contains the linear second order ODE (Schrödinger operator) which well known in theory of the KdV-equation.

6. Antiself-dual-Kahler metrics and the second order ODE's

Here we discuss the relations of the equations (1) with theory of the ASD-Kahler spaces [15].

It is known that all ASD null Kahler metrics are locally given by the

$$ds^2 = -\Theta_{tt}dx^2 + 2\Theta_{zt}dxdy - \Theta_{zz}dy^2 + dxdz + dydt,$$

where the function $\Theta(x, y, z, t)$ is the solution of the equations

$$\Theta_{xz} + \Theta_{yt} + \Theta_{zz}\Theta_{tt} - \Theta_{zt}^2 = \Lambda(x, y, z, t),$$

$$\Lambda_{xz} + \Lambda_{yt} + \Theta_{tt}\Lambda_{zz} + \Theta_{zz}\Lambda_{tt} - 2\Theta_{zt}\Lambda_{tz} = 0.$$

This system of equations has the solution in form

$$\Theta = -\frac{1}{6}a_1(x, y)z^3 + \frac{1}{2}a_1(x, y)z^2t - \frac{1}{2}a_3(x, y)zt^2 + \frac{1}{6}a_4(x, y)t^3$$

and lead to the metrics (5) with geodesics determined by the equation (1).

However in this case the coefficients $a_i(x, y)$ are not arbitrary but satisfy the conditions $L_1 = 0$, $L_2 = 0$. According with the Liouville theory, this means that such type of equations can be transformed to the equation $y'' = 0$ with the help of the points transformations.

Note that the conditions $L_1 = 0$, $L_2 = 0$ are connected with the integrable nonlinear p.d.e. (as the equation (7'), for example) and by this means we can get a many examples of ASD-spaces.

7. The applications to the general relativity

The notice of the Riemann extensions of a given metrics can be used for the studying of general properties of the Riemannian spaces with the Einstein conditions

$$R_{ik} = g^{jl}R_{ijkl} = 0$$

on the Riemann tensor R_{ijkl} and their generalizations [19]. Let us consider some examples.

Let

$$ds^2 = -t^{2p_1}dx^2 - t^{2p_2}dy^2 - t^{2p_3}dz^2 + dt^2 \tag{14}$$

be the metric of the Kasner type which has applications in classical theory of gravitation.

The Ricci tensor of this metrics has the components

$$R_{11} = (p_2 + p_3 + p_1 - 1)t^{2p_1-2}, \quad R_{22} = (p_2 + p_3 + p_1 - 1)t^{2p_2-2},$$

$$R_{33}=p_3(p_2+p_3+p_1-1)t^{2p_3-2},\quad R_{44}=\frac{(p_2+p_3+p_1-p_1^2-p_2^2-p_3^2)}{t^2}\,,\quad (14')$$

and in case $R_{ij}=0$ we get well known the Kasner solution of the vacuum Einsten equations.

Now we apply the construction of Riemann extension for the metrics (14). In result we get the eight-dimensional space with local coordinates (x,y,z,t,P,Q,R,S) and the metrics

$$ds^2=-2\Gamma^k_{ij}\xi_k dx^i dx^j+2dxdP+2dydQ+2dzdR+2dtdS \qquad (15)$$

were Γ^k_{ij} are the Christoffel coefficients of the metrics (14) and $\xi_k=(P,Q,R,S)$.

They are:

$$\Gamma^4_{11}=p_1t^{2p_1-1},\quad \Gamma^4_{22}=p_2t^{2p_2-1},\quad \Gamma^4_{33}=p_3t^{2p_3-1},$$

$$\Gamma^1_{14}=p_1/t,\quad \Gamma^2_{24}=p_2/t,\quad \Gamma^3_{34}=p_3/t.$$

We may use these expressions, and the metrics of the space D^8 can be written in form

$$ds^2=-2p_1t^{2p_1-1}Sdx^2-2p_2t^{2p_2-1}Sdy^2-2p_3t^{2p_3-1}Sdz^2$$

$$-4p_1/tPdxdt-4p_2/tQdydt-4p_3/tRdzdt+2dxdP+2dydQ+2dzdR+2dtdS.$$

The Ricci tensor ${}^8R_{ij}$ of such space have the same components with the Ricci tensor ${}^4R_{ij}$ of the space D^4 (14'). So the condition on the Riemann space to be Ricci-flat is conserved before and after extension.

In turn the equations of geodesics of extended space

$$\frac{d^2t}{ds^2}+p_1t^{2p_1-1}(\frac{dx}{ds})^2+p_2t^{2p_2-1}(\frac{dy}{ds})^2+p_3t^{2p_3-1}(\frac{dz}{ds})^2=0,$$

$$\frac{d^2x}{ds^2}+2\frac{p_1}{t}\frac{dx}{ds}\frac{dt}{ds}=0,\quad \frac{d^2y}{ds^2}+2\frac{p_2}{t}\frac{dy}{ds}\frac{dt}{ds}=0,\quad \frac{d^2z}{ds^2}+2\frac{p_3}{t}\frac{dz}{ds}\frac{dt}{ds}=0,$$

$$\frac{d^2R}{ds^2}-2\frac{p_3}{t}\frac{dt}{ds}\frac{dR}{ds}-2p_3t^{2p_3-1}\frac{dz}{ds}\frac{dS}{ds}$$

$$+\left(2\frac{p_1p_3t^{2p_1-1}}{t}(\frac{dx}{ds})^2+2\frac{p_2p_3t^{2p_2-1}}{t}(\frac{dy}{ds})^2)+2\frac{p_3^2t^{2p_3-1}}{t}(\frac{dz}{ds})^2+2\frac{p_3}{t^2}(\frac{dt}{ds})^2\right)R$$

$$+2\frac{p_3t^{2p_3-1}}{t}\frac{dz}{ds}\frac{dt}{ds}S=0,$$

$$\frac{d^2Q}{ds^2}-2\frac{p_2}{t}\frac{dt}{ds}\frac{dQ}{ds}-2p_2t^{2p_2-1}\frac{dy}{ds}\frac{dS}{ds}$$

$$+\left(2\frac{p_1p_2t^{2p_1-1}}{t}(\frac{dx}{ds})^2+2\frac{p_2p_3t^{2p_3-1}}{t}(\frac{dz}{ds})^2)+2\frac{p_2^2t^{2p_2-1}}{t}(\frac{dy}{ds})^2+2\frac{p_2}{t^2}(\frac{dt}{ds})^2\right)Q$$

$$+2\frac{p_2 t^{2p_2-1}}{t}\frac{dy}{ds}\frac{dt}{ds}S=0,$$

$$\frac{d^2P}{ds^2}-2\frac{p_1}{t}\frac{dt}{ds}\frac{dP}{ds}-2p_1t^{2p_1-1}\frac{dx}{ds}\frac{dS}{ds}$$

$$+\left(2\frac{p_1p_3t^{2p_1-1}}{t}(\frac{dz}{ds})^2+2\frac{p_2p_1t^{2p_2-1}}{t}(\frac{dy}{ds})^2)+2\frac{p_1^2t^{2p_1-1}}{t}(\frac{dx}{ds})^2+2\frac{p_1}{t^2}(\frac{dt}{ds})^2\right)P$$

$$+2\frac{p_1 t^{2p_1-1}}{t}\frac{dx}{ds}\frac{dt}{ds}S=0,$$

$$\frac{d^2S}{ds^2}-2\frac{p_3}{t}\frac{dz}{ds}\frac{dR}{ds}-2\frac{p_2}{t}\frac{dy}{ds}\frac{dQ}{ds}-2\frac{p_1}{t}\frac{dx}{ds}\frac{dP}{ds}+\frac{4p_2^2}{t^2}\frac{dy}{ds}\frac{dt}{ds}Q+\frac{4p_3^2}{t}\frac{dz}{ds}\frac{dt}{ds}R$$

$$+\left(\frac{p_1(2p_1-1)t^{2p_1-1}}{t}(\frac{dx}{ds})^2+\frac{p_2(2p_2-1)t^{2p_2-1}}{t}(\frac{dy}{ds})^2\right.$$

$$\left.+\frac{p_3(2p_3-1)t^{2p_3-1}}{t}(\frac{dz}{ds})^2\right)S=0$$

contain the linear 4×4 matrix system of the second order ODE's for the additional coordinates $\Psi=(P,Q,R,S)$

$$\frac{d^2\Psi}{ds^2}=A(x,y,z,t)\frac{d\Psi}{ds}+B(x,y,z,t)\Psi.$$

Here A, B are the 4×4 matrix-functions depending on the coordinates (x,y,z,t). It is important to note that the relation

$$\dot{x}P+\dot{y}Q+\dot{z}R+\dot{t}S=s/2+\mu$$

must be take into account and using this fact the methods of soliton theory for the integration of the full system of geodesics and the corresponding Einstein equations may be applied.

Note that the signature of the space ${}^8D$ is 0, i.e., it has the form $(++++----)$. From this follows that starting from the Riemann space with the Lorentz signature $(---+)$ we get after the extension the additional subspace with local coordinates P,Q,R,S having the signature $(-+++)$.

For the Schwarzschild metrics

$$ds^2=-1/(1-\frac{m}{x})dx^2-x^2dy^2-x^2\sin^2 y dz^2+(1-\frac{m}{x})dt^2$$

the Christoffell coefficients are

$$\Gamma^1_{11}=\frac{m}{2x(x+m)},\quad \Gamma^1_{22}=-(x+m),$$

$$\Gamma^1_{33} = -(x+m)\sin^2 y, \quad \Gamma^1_{44} = -\frac{(x+m)m}{2x^3},$$

$$\Gamma^2_{12} = \frac{1}{x}, \quad \Gamma^2_{33} = -\sin y\cos y, \quad \Gamma^3_{13} = \frac{1}{x}, \quad \Gamma^3_{23} = \frac{\cos y}{\sin y}, \quad \Gamma^4_{14} = -\frac{m}{2x(x+m)}.$$

After the extension with the help of a new coordinates (P, Q, R, S) we get the D^8 space with the metrics

$$ds^2 = -2\Gamma^1_{11}Pdx^2 - 2\Gamma^1_{22}Pdy^2 - 2\Gamma^1_{33}Pdz^2 - 2\Gamma^1_{44}Pdt^2$$

$$-2\Gamma^2_{33}Qdz^2 - 4\Gamma^2_{12}Qdxdy - 4\Gamma^3_{13}Rdxdz - 4\Gamma^3_{23}Rdydzdx - 4\Gamma^4_{14}Sdxdt.$$

8. Dual equations and the Einstein–Weyl geometry in theory of the second order ODE's

In the theory of the second order ODE's $y'' = f(x, y, y')$ we have the fundamental diagram

$$y'' = f(x, y, y') \Longrightarrow \Phi(x, y, a, b) = 0 \Longleftarrow b'' = g(a, b, b'),$$

which show the relations between a given second order ODE $y'' = f(x, y, y')$, its general integral $\Phi(x, y, a, b) = 0$ and so called dual equation $b'' = g(a, b, b')$ which can be obtained from general integral when variables x and y as the parameters are considered. In particular for the equations of type (1) the dual equation

$$b'' = g(a, b, b') \tag{16}$$

has the function $g(a, b, b')$ satisfying the partial differential equation

$$g_{aacc} + 2cg_{abcc} + 2gg_{accc} + c^2g_{bbcc} + 2cgg_{bccc}$$

$$+g^2g_{cccc} + (g_a + cg_b)g_{ccc} - 4g_{abc} - 4cg_{bbc} - cg_cg_{bcc}$$

$$-3gg_{bcc} - g_cg_{acc} + 4g_cg_{bc} - 3g_bg_{cc} + 6g_{bb} = 0 \tag{16'}$$

Koppish (1905), Kaiser (1914).

The E. Cartan has also shown that the Einstein–Weyl 3-folds parameterize the families of curves of equation (16) which is dual to the equation (1). The theory of the Einstein–Weyl spaces was developed in [11–13].

The association this theory with the problems of dual equations was discussed in [16, 17].

Some examples of solutions of equation (16) were obtained first in [2].

Remark 6. *For more general classes of the form-invariant equations the notice of dual equation is introduced by analogous way.*

For example, for the form-invariant equation of the type

$$P_n(b')b'' - P_{n+3}(b') = 0,$$

where $P_n(b')$ are the polinomial in b' degree n with coefficients depending from the variables a, b the dual equation $b'' = g(a, b, b')$ has right part $g(a, b, b')$ in form of the equation

$$\begin{vmatrix} \psi_{n+4} & \psi_{n+3} & \dots & \psi_4 \\ \psi_{n+5} & \psi_{n+4} & \dots & \psi_5 \\ . & . & \dots & . \\ \psi_{2n+4} & \psi_{2n+3} & \dots & \psi_{n+4} \end{vmatrix} = 0,$$

where the functions ψ_i are determined with help of the relations

$$4!\psi_4 = -\frac{d^2}{da^2}g_{cc} + 4\frac{d}{da}g_{bc} - g_c(4g_{bc} - \frac{d}{da}g_{cc}) + 3g_b g_{cc} - 6g_{bb},$$

$$i\psi_i = \frac{d}{da}\psi_{i-1} - (i-3)g_c\psi_{i-1} + (i-5)g_b\psi_{i-2}, \quad i > 4.$$

As example for equation $2yy'' - y'^4 - y'^2 = 0$ with solution $x = a(t + \sin t) + b$, $y = a(1 - \cos t)$ we get a dual equation in form

$$b'' = -\frac{1}{a}\tan(b'/2).$$

In this case $n = 1$ we get the values

$$4!\psi_4 = \frac{3}{2a^3}\tan\frac{c}{2}(1 + \tan^2\frac{c}{2})^3, \quad 5!\psi_5 = -\frac{15}{4a^4}\tan\frac{c}{2}(1 + \tan^2\frac{c}{2})^4,$$

$$6!\psi_6 = \frac{90}{8a^5}\tan\frac{c}{2}(1 + \tan^2\frac{c}{2})^5,$$

and the relation

$$\begin{vmatrix} \psi_5 & \psi_4 \\ \psi_6 & \psi_5 \end{vmatrix} = \psi_5^2 - \psi_4\psi_6 = 0$$

is satisfied.

9. On the solutions of dual equations

Equation (16') can be written in compact form

$$\frac{d^2 g_{cc}}{da^2} - g_c\frac{dg_{cc}}{da} - 4\frac{dg_{bc}}{da} + 4g_c g_{bc} - 3g_b g_{cc} + 6g_{bb} = 0 \tag{17}$$

with help of the operator $\frac{d}{da} = \partial_a + c\partial_b + g\partial_c$.

It has many types of the reductions and the simplest of them are

$$g = c^{\alpha}\omega[ac^{\alpha-1}], \quad g = c^{\alpha}\omega[bc^{\alpha-2}], \quad g = c^{\alpha}\omega[ac^{\alpha-1}, bc^{\alpha-2}], \quad g = a^{-\alpha}\omega[ca^{\alpha-1}].$$

To integrate a corresponding equations let us consider some particular case $g = g(a, c)$. From the condition (17) we get

$$\frac{d^2 g_{cc}}{da^2} - g_c \frac{dg_{cc}}{da} = 0. \tag{18}$$

After substituting into the equation (18) the relation

$$g_{ac} = -gg_{cc} + \chi(g_c)$$

we get the solutions for $\chi(\xi), \quad \xi = g_c$

$$\chi = \frac{1}{2}\xi^2, \quad \chi = \frac{1}{3}\xi^2.$$

So we have two integrable reductions of the equation (17)

$$g_{ac} + gg_{cc} - \frac{g_c^2}{2} = 0 \quad g_{ac} + gg_{cc} - \frac{g_c^2}{3} = 0.$$

Note that last time the problem of integration of the full dual equation (17) with the right part $g = g(a, b')$ as function of two variables a and b' was solved in work [16].

Remark 7. *In the works of E. Cartan the geometry of the equation*

$$y''' = F(x, y, y', y'')$$

with General Integral in form $\Phi(x, y, a, b, c) = 0$ *has been developed. In particular for the equations with the function* F *satisfying to the system of conditions*

$$\frac{d^2F_2}{dx^2} - 2F_2\frac{dF_2}{dx} - 3\frac{dF_1}{dx} + \frac{4}{9}F_2^3 + 2F_1F_2 + 6F_0 = 0,$$
$$\frac{d^2F_{22}}{dx^2} - \frac{dF_{12}}{dx} + F_{02} = 0, \tag{19}$$

where $\frac{d}{dx} = \partial_x + y'\partial_y + y''\partial y' + F\partial_{y''}$ *the 3-dim Einsten–Weyl geometry in the space of initial values* $(y, y'y''$*, or* $a, b, c)$ *has been realized.*

As example the third order equation

$$y''' = \frac{3y'y''^2}{(1 + y'^2)}$$

of all cycles on the plane give us the simplest but nontrivial example of such type geometry.

It is interesting to note in the case

$$H = H(F_2), \quad \text{and} \quad F = F(x, y'')$$

we get from the system (19) the condition on the function F

$$F_{x2} + F F_{22} - \frac{F_2^2}{3} = 0.$$

The corresponding third-order equation is $y''' = F(x, y'')$*, and it is connected with the second-order equation* $z'' = g(x, z')$ *which has been discussed in the section (9).*

Acknowledgement. The author thanks INTAS-99-01782 Programm and the Royal Swedish Academy of Sciences for financial support.

References

[1] *V. Dryuma*, Application of the E. Cartan method for studyuing of nonlinear dynamical systems. — *Mat. issledovaniya*, Kishinev, Stiintsa (1987), v. 92, p. 49–68.

[2] *V. Dryuma*, Projective duality in theory of the second order differential equations. — *Mat. Researches*, Kishinev, Stiintsa (1990), v. 112, p. 93–103.

[3] *V. Dryuma*, On Initial values problem in theory of the second order ODE's, In: Proc. Workshop on Nonlineariyu, Integrability and all that: Twenty years after NEEDS'79, Gallipoli (Lecce), Italy, July 1–July 10, 1999. (M. Boiti, L. Martina, F. Pempinelli, B. Prinari and G. Soliani, eds.). World Scientific, Singapore (2000), p. 109–116.

[4] *V. Dryuma*, On geometry of the second order differential equations, In: Proc. Conf. Nonlinear Phenomena. (K.V. Frolov, ed.) Nauka, Moskow (1991), p. 41–48.

[5] *V. Dryuma*, Geometrical properties of multidimensional differential equations and the Finsler metrics of dynamical systems. — *Theoret. and Math. Physics* (1994), v. 99, No. 2, p. 241–249.

[6] *V.S. Dryuma*, Geometrical properties of nonlinear dynamical systems, In: Proc. First Workshop on Nonlinear Physics, Le Sirenuse, Gallipoli (Lecce), Italy, June 29–July 7, 1995 (E. Alfinito, M. Boiti, L. Martina, and F. Pempinelli, eds.). World Scientific, Singapore (1996), p. 83–93.

[7] *R. Liouville*, Sur les invariants de certaines équations différentielles et sur leurs applications. — *J. de L'École Polytechnique* (1889), v. 59, p. 7–76.

[8] *A. Tresse*, Détermination des Invariants ponctuels de l'Équation differentielle ordinaire de second ordre: $y'' = w(x, y, y')$. Preisschriften der fürstlichen Jablonowski'schen Gesellschaft XXXII. S. Hirzel, Leipzig (1896).

[9] *E. Cartan*, Sur les variétés a connexion projective. — *Bulletin de la Société Mathémat. de France* (1924), v. 52, p. 205–241.

[10] *E.M. Paterson and A.G. Walker*, Riemann extensions. — *Quart. J. Math. Oxford* (1952), v. 3, p. 19–28.

[11] *E. Cartan*, Sur une classe d'espaces de Weyl. — *Ann. Ec. Norm. Sup.* (1943), v. 14, p. 1–16.

[12] *H. Pedersen and K.P. Tod*, Three dimensional Einsten–Weyl geometry. — *Adv. Math.* (1993), v. 97, p. 71–109.

[13] *K.P. Tod*, Einstein–Weyl spaces and third-order differential equations. — *J. Math. Phys* (2000), v. 10, No. 8, p. 5572–5581.

[14] *J.C. Sprot*, Symplest dissipative chaotic flow. — *Phys. Letters* (1997), v. A228, p. 271–274.

[15] *M. Dunajski*, Antiself-dual four-manifolds with a parallel spinors. arXiv:math. DG/0102225, 18 Oct. 2001.1–18.

[16] *V. Dryuma and M. Pavlov*, On initial value problem in theory of the second order ODE. — *Buletinul AS RM, Matematica* (2003), v. 2(42), p. 51–58.

[17] *V. Dryuma*, Applications of the Riemann and the Einstein–Weyl geometry in theory of the second order ODE. — *Theoret. and Math. Physics* (2001), v. 128, No. 1, p. 15–26.

[18] *J. Hietarinta and V. Dryuma*, Is my ODE a Painleve equation in Disguise?. — *J. Nonlinear Math. Physics* (2002), v. 9, suppl. 1, p. 67–74.

[19] *V. Dryuma*, Integration of the Einstein equations with tools of linear problem in eight-dimensional space. — Abstract, ICM, August, 20–28, China, Bejijng (2002), p. 61.