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Generation of asymptotic solitons in an integrable model of stimulated Raman scattering by periodic boundary data

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We consider an integrable model of stimulated Raman scattering. The corresponding hyperbolic partial differential equations are referred to as SRS nonlinear equations. We study the initial boundary value Goursat problem for these equations in the quarter of (x, t)-plane. The initial function vanishes at infinity while boundary data are local perturbations of a simplest periodic functions. We obtain the representation of the solution of the SRS nonlinear equations in the quarter of (x, t)-plane via functions, satisfying Marchenko integral equations, and, on this basis, we investigate the asymptotic behavior of the solution for large time. We prove that the periodic boundary data generate an unbounded train of solitons running away from the boundary.

1. Introduction

The SRS nonlinear equations as a model of stimulated Raman scattering have appeared in [11, 10, 6]. The phenomenon of stimulated Raman scattering is described [6] by three coupled PDEs which define the pump electric field, the Stokes electric field and the material excitation as functions of distance and time. These equations are integrable, i.e., they admit a Lax pair formulation. These integrable equations are also associated with a model of self-induced transparency [1] in a special case. Since the initial and boundary values for these problems are typically on a finite or semi-infinite interval [7] generally one cannot use the inverse scattering transform in its traditional form of whole line. We will

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consider here the initial boundary values Goursat problem for the SRS nonlinear equations in the quarter xt-plane, i.e., on the semi-infinite interval. Many papers initiated by problems on the half-line deal with nonlinear dynamics of the spectral or scattering data. We prefer an approach [5] where scattering (spectral) data have explicite dynamics: first we input "scattering matrix" for the x-equation (by initial function), then we input "scattering matrix" for the t-equation (by boundary functions), and finally we input "scattering matrix" for the compatible x and t-equations as a product of previous ones. In this case the kernel of the Marchenko integral equations or a jump matrix of the corresponding Riemann–Hilbert problem has an explicit x, t dependance via exponent. It makes possible to study an asymptotic behavior of the solution of the nonlinear problem by using, for example, the powerful steepest descent method of P. Deift and X. Zhou [3] while the nonlinear dynamics of the spectral or scattering data makes almost impossible to obtain an effective asymptotics of the solution. We use below our approach [8, 9] for studying soliton asymptotics based on the Marchenko integral equations. We consider a case when boundary data have a simplest periodic behavior at infinity. Thus, we consider the initial-boundary value problem for the SRS equations:

$$2\mathrm{i}q_t = \mu, \qquad \mu_x = 2\mathrm{i}\nu q, \quad \nu_x = \mathrm{i}(\bar{q}\mu - q\bar{\mu}) \qquad \text{with } x, t \in \mathbb{R}_+, \tag{1.1}$$

$$q(x,0) = u(x), \qquad \text{with } x \in \mathbb{R}_+; \tag{1.2}$$

$$\mu(0,t) = v(t), \quad \nu(0,t) = w(t) \qquad \text{with } t \in \mathbb{R}_+, \tag{1.3}$$

where $u(x)$ vanishes as $x \to \infty$, and the boundary values are perturbations

$$v(t) = \alpha(t) + \hat{v}(t), \qquad \hat{v}(t) \to 0, \quad t \to \infty, \tag{1.4}$$

$$w(t) = \beta(t) + \hat{w}(t), \qquad \hat{w}(t) \to 0, \quad t \to \infty \tag{1.5}$$

of the simplest periodic functions $\alpha(t) = -2a\omega e^{\mathrm{i}\omega t}$ and $\beta(t) \equiv -2b\omega$.

We assume that the solution $q(x,t), \mu(x,t),\ \nu(x,t)$ of the SRS nonlinear equations for $x, t \in \mathbb{R}_+$ is infinitely differentiable, continuous with all its derivatives up to the boundary $\{x = 0; t = 0\}$ of the quarter xt-plane and $q(x,t), \mu(x,t) \in \mathcal{S}(\mathbb{R}_+)$ in x for any fixed $t \in \mathbb{R}_+$, where $\mathcal{S}(\mathbb{R}_+)$ is the space of infinitely differentiable functions on $\mathbb{R}_+$ such that derivatives of any order $n \geq 0$ vanish at infinity faster than any negative power of x.

It is easy to see that equations (1.1) possess the following property:

$$\nu^2(x,t) + |\mu(x,t)|^2 \equiv 1$$

if we put that $4(a^2+b^2)\omega^2 = 1$. This property shows that it is sufficient to find the function $q(x,t)$ because the others unknown functions are defined by formulas

$$\mu(x,t) = 2\mathrm{i}q_t(x,t), \qquad \nu(x,t) = \pm\sqrt{1 - |\mu(x,t)|^2}. \tag{1.6}$$

In what follows we consider only the case when $\nu(\infty, t) = -1$.

To obtain a representation of the solution via Marchenko integral equations and to study of its asymptotic behavior we shall use simultaneous spectral analysis of an eigenvalue problem for the linear x-equation

$$\psi_x + \mathrm{i}k\sigma_3\psi = Q(x,t)\psi, \tag{1.7}$$
$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ -\bar{q}(x,t) & 0 \end{pmatrix}$$

and for the linear t-equation

$$\psi_t = \frac{\mathrm{i}}{4k}\hat{Q}(x,t)\psi, \qquad \hat{Q}(x,t) = \begin{pmatrix} \nu(x,t) & \mathrm{i}\mu(x,t) \\ -\mathrm{i}\bar{\mu}(x,t) & -\nu(x,t) \end{pmatrix}, \tag{1.8}$$

when $x, t \in \mathbb{R}_+$. This system of linear equations are compatible if and only if $q(x,t)$, $\mu(x,t)$ and $\nu(x,t)$ satisfy the SRS nonlinear equations [1, 6].

The main goal of this paper is to obtain the representation of the solution of the SRS nonlinear equations via functions, satisfying Marchenko integral equations, and to study the asymptotic behavior of the solution as $t \to \infty$. We prove that the principal part of the asymptotics is represented by a series of asymptotic solitons. The asymptotics of the solution shows that the periodic regime on the boundary ($x = 0$) generates an infinite train of solitons running away from the boundary.

2. Representation of the solution

First of all we need to input the Floquet–Bloch solution of equation

$$\phi_t = \frac{\mathrm{i}}{4k}\hat{Q}_g(t)\phi, \qquad \hat{Q}_g(t) = \begin{pmatrix} \beta(t) & \mathrm{i}\alpha(t) \\ -\mathrm{i}\bar{\alpha}(t) & -\beta(t) \end{pmatrix} \tag{2.1}$$

with periodic coefficients. For the case $\alpha(t) = -2a\omega e^{\mathrm{i}\omega t}$ and $\beta(t) \equiv -2b\omega$ the Floquet–Bloch solution takes a form

$$\mathcal{E}(t,k) = \begin{pmatrix} \sqrt{\frac{k+b+X(k)}{2X(k)}} & \sqrt{\frac{k+b-X(k)}{2X(k)}}e^{\mathrm{i}\omega t} \\ \sqrt{\frac{k+b-X(k)}{2X(k)}}e^{\mathrm{i}\omega t} & \sqrt{\frac{k+b+X(k)}{2X(k)}} \end{pmatrix} e^{\mathrm{i}\Delta(k)t\sigma_3},$$

where the function $\Delta(k)$ is equal to

$$\Delta(k) = \frac{\omega}{2}\left(1 - \frac{X(k)}{k}\right) \qquad X(k) = \sqrt{(k+b)^2 + a^2}, \quad \omega^2 = \frac{1}{4(a^2+b^2)}.$$

We fix the branch of the square root by relation:

$$X(k) = k + b + O(k^{-1}), \qquad k \to \infty.$$

The vector functions $\mathcal{E}^{\pm}(t,k)$ satisfy equation (2.1) and they are analytic functions away from the point 0 and some set Σ where $\operatorname{Im}\Delta(k) = 0$, $k \in \mathbb{C}$. The simple analysis of the function $\Delta(k)$ shows that the set $\Sigma = \mathbb{R} \cup \gamma_0$, where $\mathbb{R}$ is real axis of the complex k-plane, and γ_0 is a finite arc whose endpoints are branch points $E_0 = -b + \mathrm{i}a$ and $\bar{E}_0 = -b - \mathrm{i}a$ (fig.1). It is easy to prove that the arc γ_0 lies outside the circle of the radius $|E_0|$ (in particular $\kappa_0 = -1/2b\omega^2 < -2b$).

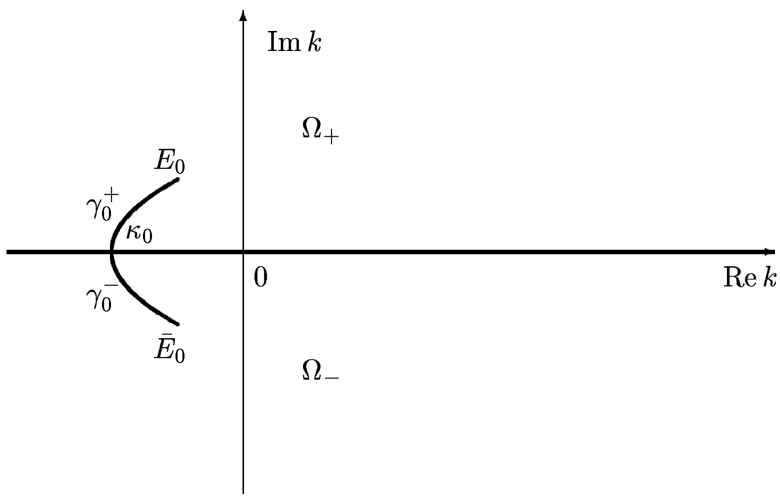

Figure 1: The set Σ $(b > 0)$.

So, we obtain a partition of the complex k-plane: $\Omega_+ \cup \Omega_- \cup \Sigma = \mathbb{C}$, where $\Omega_+ = \{k \in \mathbb{C} \mid \operatorname{Im} k > 0, \operatorname{Im}\Delta(k) > 0\}$, $\Omega_- = \{k \in \mathbb{C} \mid \operatorname{Im} k < 0, \operatorname{Im}\Delta(k) < 0\}$, and $\Sigma = \{k \in \mathbb{C} \mid \operatorname{Im}\Delta(k) = 0\} = \mathbb{R} \cup \gamma_0$.

Let the solution of the problem (1.1)–(1.5) exists, it is sufficiantly smooth and rapidly decreasing. Then the solution $q(x,t)$ can be written as

$$q(x,t) = 2\bar{K}_2(x,x,t), \tag{2.2}$$

where the functions $K_1(x,y,t)$ and $K_2(x,y,t)$ satisfy the Marchenko linear integral equations:

$$K_1(x,y,t) - \int\limits_x^\infty \bar K_2(x,z,t)H(z+y,t)\mathrm{d}z = 0 \text{ for } 0 \le x < y < \infty, \tag{2.3}$$

$$\bar K_2(x,y,t) + \bar H(x+y,t) + \int\limits_x^\infty K_1(x,z,t)\bar H(z+y,t) \tag{2.4}$$

with the kernal

$$H(x,t) = \sum_{z_j\in\Omega_+} m_j \mathrm{e}^{\mathrm{i}z_j x + \mathrm{i}t/2z_j} + \frac{1}{2\pi}\int\limits_{\partial\Omega_+} c(k)\mathrm{e}^{\mathrm{i}kx+\mathrm{i}t/2k}\mathrm{d}k + \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} r(k)\mathrm{e}^{\mathrm{i}kx+\mathrm{i}t/2k}\mathrm{d}k. \tag{2.5}$$

Here $\partial\Omega_+$ is the bondary of the domain Ω_+ which are defined by the spectrum Σ of the t-equation (2.1) with coefficients $\alpha(t) = -2a\omega e^{\mathrm{i}\omega t}$ and $\beta(t) \equiv -2b\omega$. They are depicted on the figure. The functions $r(k)$ and $c(k)$, numbers $z_j \in \Omega_+$ and numbers $m_j \in \mathbb{C}$ $(j = 1, 2, \dots, l)$ are uniquely defining by the initial and boundary functions.

To prove this representation let us introduce basic solutions for compatible x- and t-equations. The first basic solution is the matrix-valued Jost solution of the x-equation (1.7):

$$\Psi(x,t,k) = \left(\mathrm{e}^{-\mathrm{i}kx\sigma_3} + \int\limits_x^\infty K(x,y,t)\mathrm{e}^{-\mathrm{i}ky\sigma_3}\mathrm{d}y\right)\mathrm{e}^{-\frac{\mathrm{i}t\sigma_3}{4k}}, \tag{2.6}$$

which is well-defined for all real k with exception of the two points ∞ and 0 where it has essential singularities. The kernel of this triangular integral representation has the following form (cf. [4]):

$$K(x,y,t) = \begin{pmatrix} K_1(x,y,t) & -\bar K_2(x,y,t) \\ K_2(x,y,t) & \bar K_1(x,y,t) \end{pmatrix}$$

with entries of $\mathcal{C}^\infty(\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}_+)$ functions with respect to x, y, t and of the Schwartz type as $x+y\to\infty$ for any $t\in\mathbb{R}_+$. The matrix $K(x,x,t)$ and matrix $Q(x,t)$ are connected by relation: $\sigma_3K(x,x,t) - K(x,x,t)\sigma_3 = Q(x,t)\sigma_3$. The last equality yields important formula (2.2) for the function $q(x,t)$.

The matrix $\Psi(x,t,k)$ satisfies the x-equation (1.7) and t-equation (1.8). It follows from the next lemma. Let us rewrite the x- and t-equations in the form

$$W_x = U(x,t,k)W, \tag{2.7}$$

$$W_t = V(x,t,k)W \tag{2.8}$$

and let us supposed that this system of equatioins are compatible, i.e., matrices $U(x,t,k)$ and $V(x,t,k)$ satisfy the relation

$$U_t(x,t,k) - V_x(x,t,k) + U(x,t,k)V(x,t,k) - V(x,t,k)U(x,t,k) = 0$$

for all k.

Lemma 2.1. *Let the system* (2.7), (2.8) *be compatible for all* k *and let* $W(x,t,k)$ *satisfy the* x*-equation* (2.7) *for all* t *and* $\det W(x,t,k) \equiv 1$. *Let also* $W(x_0,t,k)$ *satisfy the* t*-equation* (2.8) *for some* $x = x_0$ *(including the case* $x_0 = \infty$*). Then* $W(x,t,k)$ *satisfies the* t*-equation for all* x.

P r o o f. The matrix $\widehat{W}(x,t,k) = W_t - V(x,t,k)W$ solves the equation (2.7). Indeed,

$$\widehat{W}_x = U(x,t,k)\widehat{W} + (U_t - V_x + UV - VU)W = U(x,t,k)\widehat{W}.$$

The matrices W, $\widehat{W}$ are solutions of the same equation. Therefore they are connected by a linear transformation $\widehat{W}(x,t,k) = W(x,t,k)C(t,k)$. Since $\widehat{W}(x_0,t,k) = 0$ and $\det W(x,t,k) \equiv 1$, then $C(t,k)$ and $\widehat{W}(x,t,k)$ are identically equal to zero. Lemma 2.1 is proved. ■

The second basic solution we introduce as follows. Let

$$\varphi(x,t,k) = \mathrm{e}^{-\mathrm{i}kx\sigma_3} + \int\limits_{-x}^{x} A(x,y,t)\mathrm{e}^{-\mathrm{i}ky\sigma_3}\mathrm{d}y \tag{2.9}$$

be the solution of the x-equation and let

$$\hat{\varphi}(t,k) = \mathrm{e}^{-\frac{\mathrm{i}t\sigma_3}{4k}} + \frac{\mathrm{i}}{4k}\int\limits_{-t}^{t} B(t,s)\mathrm{e}^{-\frac{\mathrm{i}s\sigma_3}{4k}}\mathrm{d}s \tag{2.10}$$

be the solution of t-equation with $x = 0$. Then we put

$$\Phi(x,t,k) = \varphi(x,t,k)\hat{\varphi}(t,k).$$

Due to the Lemma (2.1) the matrix $\Phi(x,t,k)$ is the solution of compatible x- and t-equations satisfying the condition: $\Phi(0,0,k) = \sigma_0$, where σ_0 is the identical matrix. The triangular integral representations (2.9) and (2.10) with infinitely smooth kernel $A(x,y,t)$ and $B(t,s)$ can be found in [2] for $\varphi(x,t,k)$ and in [4] for $\hat{\varphi}(t,k)$, because t-equationis the same sort as x-equation for the Heisenberg model of ferromagnets.

In what follows, we use the following notations: the over-bar denotes the complex conjugation; $\mathbb{C}_\pm$ denotes the upper (lower) complex half plane; if $A = (A^- \ A^+)$ denotes a 2×2 matrix, the vectors $A^\mp$ denote the first and second columns of A.

The third basic solution involves the Floquet–Bloch solution $\mathcal{E}(t,k)$ of equation (2.1), which is bounded in $t\in\mathbb{R}$ for $k\in\Sigma$. It is analytic away from essentially singularity point (0) and set Σ. For k outside of Σ the matrix-valued function $\mathcal{E}(t,k)$ is unbounded with respect to $t\in\mathbb{R}$ and $k\in\mathbb{C}$. However, its first column $\mathcal{E}^-(t,k)$ has exponential decay in the domain Ω_+ as $t\to\infty$ and the second column $\mathcal{E}^+(t,k)$ has exponential decay in the domain Ω_-, i.e., $\mathcal{E}^\mp(t,k) = \mathrm{O}(\mathrm{e}^{\mp \mathrm{Im}\Delta(k)t})$, $t\to\infty$. But, they grow exponentially when $k\in\Omega_-$ and $k\in\Omega_+$ respectively. The determinant $\det\mathcal{E}(t,k)\equiv 1$ with ecxeption of the points $k=E_0,\bar{E}_0$ where it vavishes. We introduce the third basic solution as follows. Let $\hat{\Psi}(t,k)$ be a solution of the Volterra integral equation

$$\hat{\Psi}(t,k) = \mathcal{E}(t,k) - \frac{\mathrm{i}}{4k}\int_t^\infty \mathcal{E}(t,k)\ \mathcal{E}^{-1}(\tau,k)[\hat{Q}(0,\tau)-\hat{Q}_g(\tau)]\hat{\Psi}(\tau,k)\mathrm{d}\tau, \qquad (2.11)$$

where $\hat{Q}(0,t)$ is defined by (1.8) with $x=0$, and $\hat{Q}_g(t)$ is defined by (2.1). The matrix $\hat{\Psi}(t,k)$ satisfies the t-equation with $x=0$ under the asymptotic condition $\hat{\Psi}(t,k) = \ \mathcal{E}(t,k)+o(1)$ as $t\to\infty$. We input the matrix

$$Y(x,t,k) = \varphi(x,t,k)\hat{\Psi}(t,k), \qquad k\in\Sigma, \qquad (2.12)$$

where $\varphi(x,t,k)$ is defined by (2.9). Due to the Lemma (2.1) the matrix $Y(x,t,k)$ is a solution of the x- and t-equations with $\det Y(x,t,k)\equiv 1$. For k outside of Σ the function $\hat{\Psi}(t,k)$, hence also $Y(x,t,k)$, is unbounded with respect to $t\in\mathbb{R}_+$. The matrix-valued function $Y(x,t,k)$ has the same analytic properties in $k\in\mathbb{C}$ as $\mathcal{E}(t,k)$, since the Green matrix $\mathcal{E}(t,k)\mathcal{E}^{-1}(\tau,k)$ is an entire function in $k\in\mathbb{C}$, and the integral equation (2.11) is of Volterra type with $\tau\in(t,\infty)$. Therefore, $Y^\pm(x,t,k)$ are analytic in the domains $\Omega_\mp$ while vector functions $\Psi^\pm(x,t,k)$ are analytic in the domains $\mathbb{C}_\pm$, and matrix $\Phi(x,t,k)$ is an entire functioin in the complex k-plane.

The basic solutions we have introduced are clearly linearly dependent

$$\begin{aligned}\Psi(x,t,k) &= \Phi(x,t,k)S(k),\\ Y(x,t,k) &= \Phi(x,t,k)P(k),\\ Y(x,t,k) &= \Psi(x,t,k)R(k).\end{aligned}$$

The matrices $S(k)$, $P(k)$ and $R(k)$ depend neither on x nor on t because by virtue of the x-equation they do not depend on x, and by virtue of the t-equation they do not depend on t. Hence for $k \in \mathbb{R}$ and for $k \in \Sigma$ we have

$$\begin{aligned} S(k) &= \Psi(0,0,k), \quad k \in \mathbb{R}; \\ P(k) &= Y(0,0,k), \quad k \in \Sigma; \\ R(k) &= S^{-1}(k)P(k), \quad k \in \mathbb{R}. \end{aligned} \tag{2.13}$$

Transition matrix $S(k) = \Psi(0,0,k) = \begin{pmatrix} s_1^-(k) & s_1^+(k) \\ s_2^-(k) & s_2^+(k) \end{pmatrix}$ is completely defined by initial function $u(x) \in \mathcal{S}(\mathbb{R}_+)$ by using the scattering problem for the x-equation with $t = 0$. The function $s_2^+(k)$ may vanish at some points $k_j \in \mathbb{C}_+$. Moreover, these zeros can be multiple and there can exist limit points on the real axis $\mathbb{R}$ [4]. To avoid this difficulties we shall consider a subset $\mathcal{S}_0(\mathbb{R}_+)$ of functions $u(x) \in \mathcal{S}(\mathbb{R}_+)$ for which $s_2^+(k)$ has a finite number of simple zeros in $\mathbb{C}_+$ and $s_2^+(k) \neq 0$ for $k \in \mathbb{R}$.

Let us briefly discuss the discrete spectrum of the x-problem. The main relation of the x-scattering problem is

$$\frac{1}{s_2^+(k)}\Phi^-(x,t,k) = \Psi^-(x,t,k) + r(k)\Psi^+(x,t,k) \text{ for } k \in \mathbb{R}, \tag{2.14}$$

where

$$r(k) = -\frac{s_2^-(k)}{s_2^+(k)}.$$

The function $F(x,t,k) = \Phi^-(x,t,k)/s_2^+(k)$ is analytic in $k \in \mathbb{C}_+$ with exception of a discrete set

$$\Sigma_{\mathrm{d}}^{ic} = \{k_j \in \mathbb{C}_+ \mid s_2^+(k_j) = 0, j = 1,2,\dots,n\},$$

where it has poles. If $s_2^+(k_j) = \det[\Phi^-(x,t,k_j), \Psi^+(x,t,k_j)] = 0$, then $\Phi^-(x,t,k_j) = \gamma_j^1\Psi^+(x,t,k_j)$. Hence, $\operatorname{res} F(x,t,k)|_{k=k_j} = c_j^1\Psi^+(x,t,k_j)$ with

$$c_j^1 = \frac{\gamma_j^1}{\dot{s}_2^+(k_j)}, \quad j = 1,2,\dots,n, \quad \text{and} \quad \gamma_j^1 = \frac{1}{s_1^+(k_j)}.$$

The dot indicates differentiation with respect to k. Note that $s_1^+(k_j) \neq 0$ because otherwise we come to a contradiction: $\Psi_+(x,t,k_j) \equiv 0$ since $\Psi_1^+(0,0,k_j) = s_1^+(k_j) = 0$ and $\Psi_2^+(0,0,k_j) = s_2^+(k_j) = 0$.

Transition matrix $P(k)$ is uniquely defined by boundary data $v(t)$ and $w(t)$, and its integral representation derived from the defining relation (2.13), (2.12), (2.11):

$$P(k) = \mathcal{E}(0,k) - \frac{\mathrm{i}}{4k}\int_0^\infty \mathcal{E}(0,k)\mathcal{E}^{-1}(t,k)[\hat{Q}(0,t) - \hat{Q}_g(t)]\hat{\Psi}(t,k)dt,$$

which completely describe analytic properties of the matrix $P(k)$. In particular, we have the following asymptotics:

$$P(k)=\begin{cases}\sigma_0+O(k^{-1}), & k\to\infty,\\ \\ P_0+O(k). & k\to 0\end{cases}$$

with

$$P_0=\frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{1-w(0)} & \dfrac{\mathrm{i}v(0)}{\sqrt{1-w(0)}}\\ \dfrac{\mathrm{i}\bar v(0)}{\sqrt{1-w(0)}} & \sqrt{1-w(0)}\end{pmatrix}.$$

Hence matrix $P(k)$ is well defined for $k=0$.

Transition matrix $R(k)$ of the scattering problem for compatible x- and t-equations is the product $S^{-1}(k)P(k)$ of the transition matrix $S(k)$ of the scattering problem for x-equation defined by initial function $u(x)$ and transition matrix $P(k)$ of the scattering problem for t-equation defined by boundary values $v(t)$ and $w(t)$. The matrix $R(k)$ has the following form:

$$R(k)=\begin{pmatrix}r_1^-(k) & r_1^+(k)\\ r_2^-(k) & r_2^+(k)\end{pmatrix},\quad r_1^+(k)=-\bar r_2^-(k),\quad r_2^+(k)=\bar r_1^-(k)\quad \text{for } k\in\mathbb{R}$$

with $r_1^-(k)=p_1^-(k)s_2^+(k)-p_2^-(k)s_1^+(k)$ analytic in $k\in\Omega_+$ (hence $r_2^+(k)$ is analytic in $k\in\Omega_-$) and $r_2^-(k)=p_2^-(k)s_1^-(k)-p_1^-(k)s_2^-(k)$ nonanalytic and well-defined for $k\in\mathbb{R}$ with exception of the point 0 and self-intersecting point κ_0 of the set Σ (hence $r_1^+(k)$ is not analytic and well-defined for $k\in\mathbb{R}\setminus(\{0\}\cup\{\kappa_0\})$). The analytic properties of the matrices $S(k)$ and $P(k)$ yield that $R(k)$ is of $\mathcal{C}^\infty(\mathbb{R}\setminus\kappa_0)$ matrix. It has the following asymptotics:

$$R(k)=\begin{cases}\sigma_0+\mathrm{O}(k^{-1}), & k\to\infty\\ R_0+\mathrm{O}(k), & k\to 0,\quad \text{where}\quad R_0=S^{-1}(0)P_0.\end{cases}$$

In what follows we suppose that the function $r_1^-(k)$ does not vanish for $k\in\mathbb{R}$, and its zeros can occur at the points $z_j\in\Omega_+$ and all of them are simple.

The second main relation of the compatible scattering problem is:

$$\begin{aligned}G(x,t,k)&=\frac{1}{r_1^-(k)}Y^-(x,t,k)\\ &=\Psi^-(x,t,k)+\rho(k)\Psi^+(x,t,k)\ \text{for } k\in\mathbb{R},\text{where}\quad \rho(k):=r_2^-(k)/r_1^-(k).\end{aligned}\tag{2.15}$$

The function $G(x,t,k)$ is analytic in $k \in \Omega_+$ with exception of the points z_j where $r_1^-(z_j) = 0$. If $r_1^-(z_j) = 0$ then $Y^-(x,t,z_j)$ and $\Psi^+(x,t,z_j)$ are linear dependent

$$Y^-(x,t,z_j) = \gamma_j^2 \Psi^+(x,t,z_j), \qquad j = 1,2,\dots,m,$$

hence $\operatorname{res} G(x,t,z_j) = c_j^2 \Psi^+(x,t,z_j)$, $c_j^2 = \frac{\gamma_j^2}{\dot r_1^-(z_j)}$ (the dot denotes differentiation with respect to k) with $\gamma_j^2 = \dfrac{p_1^-(z_j)}{s_1^+(z_j)} = \dfrac{p_2^-(z_j)}{s_2^+(z_j)}$. Using asymptotics of the function $Y^-(x,t,k)$ in the neighborhood of $k = \infty$ and $k = 0$ for $k \in \Omega_+$, we find

$$G(x,t,k) = \begin{cases} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathrm{O}(|k|^{-1})\right] \mathrm{e}^{-\mathrm{i}(kx + \frac{t}{4k})} \text{ for } |k| \to \infty,\ k \in \Omega_+, \\ \left[\dfrac{1}{r_1^-(0)\sqrt{2(1-\nu(x,t))}} \begin{pmatrix} 1-\nu(x,t) \\ \mathrm{i}\bar\mu(x,t) \end{pmatrix} + \mathrm{O}(|k|)\right] \mathrm{e}^{-\mathrm{i}t/4k} \\ \qquad \text{for } |k| \to 0,\ k \in \Omega_+. \end{cases} \tag{2.16}$$

This asymptotic formula will be used below.

The main relations (2.14) and (2.15) of the compatible scattering problem yield

$$G(x,t,k) - F(x,t,k) = c(k)\Psi^+(x,t,k), \tag{2.17}$$

where

$$c(k) = \rho(k) - r(k) = \frac{p_2^-(k)}{s_2^+(k)\, r_1^-(k)} \qquad \text{for } k \in \mathbb{R}. \tag{2.18}$$

Hence, $c(k)$ has a meromorphic continuation from the real axis $\mathbb{R}$ to the domain Ω_+, because $s_2^+(k)$, $p_2^-(k)$ and $r_1^-(k)$ are analytic in the domain Ω_+. Hence relation (2.17) is true for $k \in \overline{\Omega}_+$ with exception of poles at the points, where $s_2^+(z_j) = r_1^-(z_j) = 0$. Since the zeros of $s_2^+(k)$ and $r_1^-(k)$ are simple and in finite number, all poles of $c(k)$ are simple and their number is finite.

To deduce the integral equations of the inverse scattering problem let us put

$$h(x,t,k) = G(x,t,k) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{e}^{-\mathrm{i}kx - \mathrm{i}t/4k} \text{ for } k \in \mathbb{R}.$$

The function $h(x,t,k)$ has different left and right boundary values on $\gamma_0^+ = \gamma_0 \cap \overline{\Omega}_+$. Indeed,

$$\begin{aligned} h(x,t,k-0) - h(x,t,k+0) &= G(x,t,k-0) - G(x,t,k+0) \\ &= \frac{Y^-(x,t,k-0) r_1^-(k+0) - Y^-(x,t,k+0) r_1^-(k-0)}{r_1^-(k+0) r_1^-(k-0)}, \end{aligned}$$

and since $r_1^-(k) = \det[Y^-(x,t,k), \Psi^+(x,t,k)]$ one can find

$$h(x,t,k-0) - h(x,t,k+0)$$
$$= \frac{\det[\mathcal{E}^-(0,k+0), \mathcal{E}^-(0,k-0)]\Psi^+(x,t,k)}{r_1^-(k+0)r_1^-(k-0)} = \frac{\mathrm{i}\Psi^+(x,t,k)}{r_1^-(k+0)r_1^-(k-0)}.$$

Therefore

$$h(x,t,k-0) - h(x,t,k+0) = f(k)\Psi^+(x,t,k), \quad k \in \gamma_0^+, \tag{2.19}$$

where

$$f(k) = \frac{\mathrm{i}}{r_1^-(k-0)r_1^-(k+0)}.$$

Taking into account analytic continuation of the relation (2.17) into the domain Ω_+, we find that for $k \in \gamma_0^+$

$$[c(k-0) - c(k+0)]\Psi^+(x,t,k) = G(x,t,k-0) - G(x,t,k+0)$$

and by the same reasons as above

$$c(k-0) - c(k+0) = f(k), \quad k \in \gamma_0^+, \tag{2.20}$$

where $c(k \mp 0)$ is the left and the right boundary values of the function $c(k)$ on γ_0^+.

Let us consider the integral

$$J(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(x,t,k)\mathrm{e}^{\mathrm{i}ky+\mathrm{i}t/4k}\mathrm{d}k.$$

Using equations (2.14), (2.15), (2.18), (2.6), we find

$$J(x,y,t) = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}(x,y,t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} F_s(x+y,t) + \int_x^{\infty} \begin{pmatrix} -\bar{K}_2 \\ \bar{K}_1 \end{pmatrix}(x,z,t)F_s(z+y,t)\mathrm{d}z,$$

where

$$F_s(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \rho(k)\mathrm{e}^{\mathrm{i}k(x+y)+\mathrm{i}t/2k}\mathrm{d}k$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} c(k)\mathrm{e}^{\mathrm{i}k(x+y)+\mathrm{i}t/2k}\mathrm{d}k + \frac{1}{2\pi}\int_{-\infty}^{\infty} r(k)\mathrm{e}^{\mathrm{i}k(x+y)+4\mathrm{i}t/2k}\mathrm{d}k.$$

On the other hand, using estimate (2.16) of $G(x,t,z)$ for large and small k, taking into account (2.18) and (2.19), applying the Jordan lemma, we find

$$J(x,y,t) = \mathrm{i} \sum_{\substack{z_j \in \Omega_+ \\ r_1^-(z_j)=0}} \mathrm{res}\,[h(x,t,k)\mathrm{e}^{\mathrm{i}ky+\mathrm{i}t/4k}]$$
$$- \frac{1}{2\pi} \int_{\gamma_0^+} [h(x,t,k-0) - h(x,t,k+0)]\mathrm{e}^{\mathrm{i}ky+\mathrm{i}t/4k}\mathrm{d}k$$
$$= - \sum_{z_j \in \Omega_+} m_j \mathrm{e}^{\mathrm{i}z_j y + \mathrm{i}t/2z_j} \Psi^+(x,t,z_j)$$
$$- \frac{1}{2\pi} \int_{\gamma_0^+} f(k)\mathrm{e}^{\mathrm{i}ky+\mathrm{i}t/4k} \Psi^+(x,t,k)\mathrm{d}k.$$

Finally, we have the integral equations (2.3), (2.4) of the inverse scattering problem with kernal:

$$H(x,t) = \sum_{z_j \in \Omega_+} m_j \mathrm{e}^{\mathrm{i}z_j x + \mathrm{i}t/2z_j} + \frac{1}{2\pi} \int_{\gamma_0^+} f(k)\mathrm{e}^{\mathrm{i}kx+\mathrm{i}t/2k}\mathrm{d}k$$
$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} c(k)\mathrm{e}^{\mathrm{i}kx+4\mathrm{i}t/2k}\mathrm{d}k + \frac{1}{2\pi} \int_{-\infty}^{\infty} r(k)\mathrm{e}^{\mathrm{i}kx+\mathrm{i}t/k}\mathrm{d}k. \tag{2.21}$$

The coefficients m_j are given by

$$m_j = p_1^-(z_j)[\mathrm{i}s_1^+(z_j)\dot{r}_1^-(z_j)]^{-1} = p_2^-(z_j)[\mathrm{i}s_2^+(z_j)\dot{r}_1^-(z_j)]^{-1} = -\mathrm{i}\,\mathrm{res}_{\,z=z_j} c(k). \tag{2.22}$$

Taking into account jump relation (2.20) for the function $c(k)$ on arcs γ_0^+, the kernel $H(x,t)$ can be written in the form (2.5).

Now it is natural to introduce the set

$$\mathcal{R} = \{k_1, k_2, \ldots, k_n \in \Omega_+; z_1, z_2, \ldots, z_m \in \Omega_+; r(k), \rho(k), r_1^-(k), k \in \mathbb{R}\}$$

and to call it as the scattering data for compatible differential equations (1.7), (1.8) with $q(x,t), \nu(x,t)$ and $\mu(x,t)$ satisfying (1.1)–(1.5). These scattering data possess the following properties:

Condition A

- $r(k) \in \mathcal{C}^\infty(\mathbb{R}) \qquad r(k) = \mathrm{O}(k^{-1}) \quad k \to \infty;$

- $r(k)s_2^+(k)$ is analytic in $k \in \mathbb{C}_-$, where

$$s_2^+(k) = \prod_{j=1}^{n} \frac{k-k_j}{k-\bar{k}_j} \exp\left[\frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty} \frac{\log(1+|r(s)|^2)\mathrm{d}s}{s-k}\right], \qquad k \in \mathbb{C}_+;$$

  is analytic for $k \in \mathbb{C}_+$.

Condition B

- $\rho(k), r_1^-(k) \in \mathcal{C}^\infty(\mathbb{R} \setminus \{\kappa_0\})$;
- the function $r_1^-(k)$ is analytic in $k \in \Omega_+$, where it has finite number of simple zeroes z_j $(j = 1, 2, \ldots, m)$
- the function $\rho(k)$ and $r_1^-(k)$ are not independent:

$$1 + |\rho(k)|^2 = |r_1^-(k)|^{-2}, \qquad k \in \mathbb{R}.$$

- The function $\rho(k)$ and all its derivatives have jumps at the real point κ_0:

$$\rho^{(l)}(\kappa_0 - 0) - \rho^{(l)}(\kappa_0 + 0) = f^{(l)}(\kappa_0), \quad l = 0, 1, \ldots,$$

  where $f(k) = \mathrm{i}(r_1^-(k-0)r_1^-(k+0))^{-1}$, $k \in \gamma_0^+$ and $\lim\limits_{k\to E_0}(k-E_0)^{-1/2}f(k) = f_0 \neq 0$.

Condition C

- the function $c(k) = \rho(k) - r(k)$ extends analytically to the domain Ω_+, where it has finite number of simple poles at points z_j $(j = 1, 2, \ldots, m)$ and satisfies the jump relations:

$$c(k-0) - c(k+0) = f(k), \qquad k \in \gamma_j^+.$$

The kernel $H(x,t)$ of the Marchenko equations is completely defining by the scattering data $\mathcal{R}$ because deficient coefficients m_j (2.22) and functions $f(k)$, $c(k)$ evaluated via known from scattering data values. Thus, for example, for the case when initial function $u(x) \equiv 0$ and boundary values are pure periodic ($v(t) = -2a\omega e^{\mathrm{i}\omega t}$ and $w(t) \equiv -2b\omega$) the solution $q(x,t)$, $\mu(x,t)$ and $\nu(x,t)$ of the SRS nonlinear problem is defined by Marchenko integral equations with following special kernel:

$$H(x,t) = \frac{1}{2\pi}\int\limits_{\partial\Omega_+} \left(k + b - \sqrt{(k+b)^2 + a^2}\right) \mathrm{e}^{\mathrm{i}kx + \mathrm{i}t/2k}\mathrm{d}k.$$

3. Generation of asymptotic solitons by boundary data

Let a data $\mathcal{R} = \{k_1, k_2, \ldots, k_n \in \Omega_+; z_1, z_2, \ldots, z_m \in \Omega_+; r(k), \rho(k), r_1^-(k), k \in \mathbb{R}\}$ satisfy the conditions A, B, C. Then the following statement are true.

Statements. 1. *The xt-integral equation*

$$K(x,y,t) + \mathcal{H}(x+y,t) + \int_x^\infty K(x,z,t)\mathcal{H}(z+y,t)\mathrm{d}z = 0, \qquad (3.1)$$

$$0 \le x < y < \infty, \quad 0 \le t < \infty$$

with the 2×2 *matrix kernel*

$$\mathcal{H} = \begin{pmatrix} 0 & H(x,t) \\ -\bar{H}(x,t) & 0 \end{pmatrix},$$

where scalar function $H(x,t)$ *given by* (2.21), *is uniquely solvable in the space* $\mathcal{L}^1(x,\infty)$ *for any* $x \ge 0$ *and* $t \ge 0$.

2. *The solution* $K(x,y,t)$ *belongs to* $\mathcal{C}^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+)$, *it and all its derivatives decrease faster than any negative power of* $x+y$, *for* $x+y \to \infty$, *and* t *fixed.*

3. *The matrix*

$$\Psi(x,t,k) = \left[\mathrm{e}^{-\mathrm{i}kx\sigma_3} + \int_x^\infty K(x,y,t)\mathrm{e}^{-\mathrm{i}ky\sigma_3}dy\right]\mathrm{e}^{-\mathrm{i}t\sigma_3/4k}$$

satisfies the symmetry condition

$$\Psi(x,t,k) = \sigma_2\bar{\Psi}(x,t,k)\sigma_2 \ for\ k \in \mathbb{R}$$

and is a solution of the x*-equation (1.7) with* $Q(x,t)$ *given by*

$$Q(x,t) = \sigma_3 K(x,x,t)\sigma_3 - K(x,x,t). \qquad (3.2)$$

4. $\Psi(x,t,k)$ *is a solution of the both* x- *and* t*-equations constructed from the matrices* $Q(x,t)$ *and* $\hat{Q}(x,t)$, *using equations* (3.2), (1.6), (1.7) *and* (1.8).

Statement 1 follows from the following lemma about the solvability of the xt-integral equations.

Lemma 3.1. *Let the data* $\mathcal{R}$ *satisfying properties* A, B, C. *Then, the* xt*-integral equation* (3.1) *is uniquely solvable in the space* $\mathcal{L}^1(x,\infty)$.

P r o o f. Under conditions A–C the integral operator of the xt-integral equation is compact in the space $\mathcal{L}^1(x,\infty)$. Then, by Fredholm theory the xt-integral equation has a unique solution if the homogeneous equation has no nonzero solution. If a nonzero solution does exist in $\mathcal{L}^1(x,\infty)$, in view of the homogeneity of the integral equation, it is bounded, hence it belongs to $\mathcal{L}^2(x,\infty)$. The integral operator is clearly skew-Hermitian in $\mathcal{L}^2(x,\infty)$, so we obtain a contradiction, because the only solution in this case is zero. ■

We omit proofs of the rest statements because they are almost the same that in [4]. Now let us describe the main result of the paper.

Let us put $C = |2E_0|^{-2} = \omega^2$ and for $N \in \mathbb{N}$ let us define the asymptotic domain $G_N(t)$ by

$$G_N(t) = \{x \in \mathbb{R}_+ \mid x > Ct - \frac{1}{2a}\log t^{N+1}\}.$$

Theorem 3.1. *Let* $\min\limits_{1\le j\le m}|z_j| > |E_0|$. *Then the solution of the nonlinear problem* (1.1)–(1.5) *has in* $G_N(t)$ *the following asymptotics for* $t\to\infty$*:*

$$q(x,t) = \sum_{j=1}^{[\frac{N+1}{2}]} 2\mathrm{i}a\frac{\exp[-2\mathrm{i}b(x+Ct)-\mathrm{i}\delta_j]}{\cosh[2a(x-Ct-x_j)+\log t^{2j-1/2}]} + o(1),\ \ t\to\infty\ \ x\in G_N(t),$$

and

$$\mu(x,t) = 2\mathrm{i}q_t(x,t), \qquad \nu(x,t) = -\sqrt{1-4|q_t(x,t)|^2},$$

where

$$\delta_j = \delta_j^{(0)} + \frac{1}{\pi}\int\limits_{-|E_0|}^{|E_0|}\frac{(\lambda+b)\log[1+|\rho(\lambda)|^2]}{(\lambda+b)^2+a^2}\mathrm{d}\lambda,$$

$$x_j = x_j^{(0)} + \frac{1}{2\pi}\int\limits_{-|E_0|}^{|E_0|}\frac{\log[1+|\rho(\lambda)|^2]}{(\lambda+b)^2+a^2}\mathrm{d}\lambda, \quad \rho(\lambda) = r(\lambda)+c(\lambda),$$

and numbers δ_j^0 *and* $x_j^{(0)}$ *are given formulas involving* $h(E_0)$, *where* $h(k) = (k-E_0)^{-1/2}f(k)$. *The functions* $r(\lambda)$ *and* $c(\lambda)$ *are defined by initial and boundary data according to the Section 2.*

Remark 1. *If there exist numbers z_j with $|z_j| < |E_0|$ then the asymptotics of the function $q(x,t)$ will contain additionally a finite number of ordinary solitons*

$$\sum_{(j:|z_j|<|E_0|)} 2\mathrm{i}\,\mathrm{Im}\, z_j \frac{\exp[2\mathrm{i}\,\mathrm{Re}\, z_j(x+t/4|z_j|^2 - \mathrm{i}\eta_j)]}{\cosh[2\,\mathrm{Im}\, z_j(x-t/4|z_j|^2 - y_j)]},$$

which move faster than asymptotic solitons away from the boundary in the domain $x > Ct$ and correspond to those eigenvalues $z_j \in \mathbb{C}_+$ for which $|z_j| < |E_0|$.

Remark 2. *The main result given by Theorem 3.1 is also true for the case when $\alpha(t)$ and $\beta(t)$ are arbitrary periodic or quasi-periodic finite-gap functions with respect to differential equations (2.1).*

It is easy to see that asymptotic solitons given by Theorem 1.1 are similar to ordinary solitons but their velocities depend on t. In contrast with ordinary solitons they are not exact solutions of nonlinear equation, however they satisfy it with increasing accuracy when $t \to \infty$. For this reason such objects are called asymptotic solitons. The number of these asymptotic solitons increases to infinity when $t \to \infty$ if the observation domain in the neighborhood of the the solution front is extended correspondingly.

Qualitatively these results do not depend on whether the initial function $u(x)$ and vanishing part $\hat{v}(t)$ and $\hat{w}(t)$ of boundary functions are identically zeroes or not. In the case $u(x) \equiv 0$ the scattering function $r(k)$ is also identically zero, and $\rho(k)$ depends on the boundary data only. Thus we come to the following conclusion: *any periodic or quasi-periodic boundary data $\alpha(t)$ and $\beta(t)$ generates an unbounded train of asymptotic solitons which run away from the boundary. This phenomenon can not be destroy by any local or vanishing perturbations of the periodic boundary data as well as by vanishing initial functions. An ordinary solitons (in finite number) may appear if there exist such number z_j that $|z_j| < |E_0|$.*

P r o o f. [Sketch of a proof of the Theorem 3.1.] For study of the asymptotic behavior of the solution $q(x,t)$ we use the integral equation (3.1), which is uniquely solvable in the space $\mathcal{L}^1(x,\infty)$ and, due to the statements above, represent infinitely smooth and rapidly decreasing function $q(x,t)$. We carry out the asymptotic analysis of the integral equations by reducing the problem to degenerated integral equations, obtaining a determinant formula for the solution and studying its asymptotics as $t \to \infty$.

Taking into account properties A, B, C and using the method of steepest descent and integration by parts we come to the following decomposition of the scalar kernel $H(x,t)$ as $t \to \infty$:

$$H(x,t) = H_N(x,t) + H_1(x,t) + R_0(x,t) + R_1(x,t),$$

where $H_N(x,t)$ is degenerate one:

$$H_N(x+y,t) = e^{-(a+\mathrm{i}b)(\xi+\eta)} \sum_{m=0}^{N-1} \frac{d_m(t)}{m!t^{m+3/2}}(\xi+\eta)^m,$$

$$\xi = x - Ct, \quad \eta = y - Ct, \quad C = |2E|^{-2}.$$

The function $R_0(x+y,t)$ has the explicit form in term of function $\rho(k)$:

$$R_0(x+y,t) = \frac{1}{\sqrt{t}}\left[\hat{\rho}(X+Y)e^{it\sqrt{X+Y}} - \bar{\hat{\rho}}(X+Y)e^{-it\sqrt{X+Y}}\right],$$
$$\hat{\rho}(Y) = \frac{e^{i\pi/4}}{\sqrt{2\pi}}Y^{-3/4}\rho\Big(\frac{1}{Y}\Big),$$

where $X = 2x/t$ and $Y = 2y/t$. $H_1(x+y,t)$ and $R_1(x+y,t)$ admit the estimates

$$|H_1(x+y,t)| \le C_1 \ |\xi+\eta|^N \ t^{-N-3/2}\mathrm{e}^{-a(\xi+\eta)},$$
$$|R_1(x+y,t)| \le C_2 \ t^{-3/2}(X+Y)^{-3/2}\Big[|\rho''((X+Y)^{1/2})| + |\rho''(-(X+Y)^{1/2})|\Big].$$

Let $\hat{H}$ be an integral operator which acts in space $\mathcal{L}^2(x,\infty)$ by the formula

$$\hat{H}f(y) = \int_x^\infty \begin{pmatrix} 0 & H(y+z,t) \\ -\bar{H}(y+z,t) & 0 \end{pmatrix} \begin{pmatrix} f_1(y) \\ f_2(y) \end{pmatrix} dz.$$

Then the Marchenko integral equations take the form

$$(I+\hat{H})K = G, \qquad K = (K_1,K_2), \qquad G = (0,\bar{H}). \tag{3.3}$$

Under conditions A, B, C equation (3.3) has unique solution in the space $\mathcal{L}^2(x,\infty)$ and

$$\| (I+\hat{H})^{-1} \| \le 1.$$

Let $\hat{H}_N$, $\hat{R}_0$, $\hat{H}_1$, $\hat{R}_1$ be corresponding integral operators in the space $\mathcal{L}^2(x,\infty)$ given by kernals H_N, R_0, H_1, R_1 . We look for the solution of equation (3.3) in the form

$$K = \tilde{K} + \psi,$$

where vector-function $\tilde{K}$ satisfies equation

$$(I+\hat{H}_N+\hat{R}_0)\tilde{K} = G_N + G_0 \tag{3.4}$$

with $G_N = (0, \bar{H}_N)$ and $G_0 = (0, \bar{R}_0)$. Then difference $\psi = K - \tilde{K}$ satisfies an equation

$$(I + \hat{H})\psi = G_1 - \hat{H}_1\tilde{K} - \hat{R}_1\tilde{K}$$

with $G_1 = (0, \bar{H}_1 + \bar{R}_1)$. The last equation yields the estimate

$$\| \psi \| = \| K - \tilde{K} \| \leq C(N)t^{-1/2+\epsilon}$$

in the domain $\Omega_N(t)$ with $0 < \epsilon < 1/2$. This estimate allow us to consider below integral equation (3.4) instead of the equation (3.3).

The next step is as follows. Let $L = (L_1(x, y, t),\ L_2(x, y, t))$ be a solution of the equation

$$(I + \hat{R}_0)L = G_R, \qquad G_R = (0, \bar{R}_0). \tag{3.5}$$

Then solution $\tilde{K}$ of the equation (3.4) can be written in the form

$$\tilde{K} = L + (I + \hat{Q})M, \tag{3.6}$$

where $I + \hat{Q} = (I + \hat{R}_0)^{-1}$, and $\hat{Q}$ is an integral operator. The kernal $Q(y, z)$ of this operator has an explicite representation via vector-function $L(y, z)$. The substitution (3.6) into (3.5) yields a degenerate integral equation

$$M + \hat{H}_N(I + \hat{Q})M = G_N - \hat{H}_N L, \tag{3.7}$$

The next decisive step is that the equation (3.7) can be explicitely solved in the limit $t \to \infty$. Namely, it is possible to show [9] that for $t \to \infty$

$$\| L_1(x, y, t,) - L_1^{(\infty)}(x, y, t,) \|_{C[X,\infty)} = o(1),$$
$$\| L_2(x, y, t,) - L_2^{(\infty)}(x, y, t,) \|_{\mathcal{L}^2(X,\infty)} = o(1),\ t \to \infty,$$

where

$$L_1^0(x, y, t) = -2\int_X^\infty \left[N(X, Z)\hat{\rho}(Z + Y)e^{it(\sqrt{X+Z}+\sqrt{Z+Y})} + c.c\right] dZ$$
$$+2\int_X^\infty \left[N(X, Z)\bar{\hat{\rho}}(Z + Y)e^{it(\sqrt{X+Z}-\sqrt{Z+Y})} + c.c.\right] dZ,$$

$$L_2^0(x, y, t) = \frac{1}{\sqrt{t}}\left[N(X, Y)e^{it\sqrt{X+Y}} - \bar{N}(X, Y)e^{-it\sqrt{X+Y}}\right]$$

with function $N(X,Y)$ given in an explicite form

$$N(X,Y) = \hat{\rho}(X+Y)\exp\left(\frac{\mathrm{i}}{2\pi}\int\limits_X^\infty \Big(\frac{X+Y}{X+S}\Big)^{1/2}\frac{\ln[1+|\rho((X+S)^{-1/2})|^2]}{S-Y-i0}\,dS\right).$$

This explicite formula and degenerate integral equation (2.18) allow us to complete the proof of the Theorem 1 by the same way that in [9]. ■

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