Matematicheskaya fizika, analiz, geometriya
2003, v. 10, No. 3, p. 385–410

The Stokes structure and connection coefficients for the Airy equation

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Received July 5, 2003

Following the methodology outlined in the series of papers "The Stokes structure in asymptotic analysis", [8, 9], we present some features of the Airy equation which are not generally available in the classical literature on transcendental functions and asymptotic analysis.

1. Airy's equation

Airy's equation has the form

$$y''(z) - zy(z) = 0. \tag{1.1}$$

All functions, $y(z)$, satisfying (1.1) are analytic and single-valued in $\mathbb{C}$. We will regard these functions as *regular solutions* as opposed to *formal solutions* which we define below.

The general regular solution of Airy's equation can be expressed in term of Bessel functions of order $\pm\frac{1}{3}$, see [21, 1]:

$$y(z) = C_1\sqrt{z}J_{\frac{1}{3}}\left(\frac{2}{3}iz^{\frac{3}{2}}\right) + C_2\sqrt{z}J_{-\frac{1}{3}}\left(\frac{2}{3}iz^{\frac{3}{2}}\right). \tag{1.2}$$

Since the Airy equation plays an important role in various problems of physics and mechanics, and an exceptional role in asymptotic analysis, its regular solutions form a separate class of special functions, the Airy functions. Quite apart from this, the Airy equation is a good testing ground for ideas in asymptotic analysis. We use it to illustrate an approach which has been described in [8] and [9] and which can be used more generally.

Mathematics Subject Classification 2000: 34M25, 34M30, 34M37, 34M40.

The regular solutions of Airy's equation in the complex plane have the following properties:

(i) Every regular solution $y(z)$ of (1.1) is an entire function represented by the power series

$$\begin{aligned} y(z) &= y(0)\,(1+\frac{z^3}{2\cdot 3}+\frac{z^6}{(2\cdot 3)(5\cdot 6)}+\ldots)+ \\ &\quad + y'(0)\,(z+\frac{z^4}{3\cdot 4}+\frac{z^7}{(3\cdot 4)(6\cdot 7)}\ldots), \end{aligned} \tag{1.3}$$

which is convergent for any z (see [15, p. 54, (8.03)]). In particular if $y(z)$ is a solution then $y(ze^{2\pi i})$ is also a solution and in fact

$$y(ze^{2\pi i}) = y(z). \tag{1.4}$$

(ii) If $y(z)$ is a solution of (1.1) then the functions $y(\omega^{-1}z)$ and $y(\omega z)$, where $\omega = e^{\frac{2\pi i}{3}}$, are also solutions of (1.1) and any two of these solutions are linearly independent. Moreover, the following identity is valid:

$$y(z) + \omega y(\omega z) + \omega^{-1} y(\omega^{-1} z) \equiv 0 \tag{1.5}$$

(see [15, p. 55, (8.06)]).

Although the Airy equation was introduced by Airy in 1838, the systematic study of its solutions was not pursued until almost a hundred years later by Fock [7], and Miller [14].

2. Airy's functions

Introduce the angular domain, or sector, of $\mathbb{C}$ for $-\infty < \alpha < \beta < \infty$

$$S(\alpha,\beta) = \{z : \alpha < \arg z < \beta, 0 < |z| < \infty\} \tag{2.1}$$

and the following ray

$$l_\theta = \{z : \arg z = \theta, 0 < |z| < \infty\}\,. \tag{2.2}$$

The *Airy functions*, $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$, are regular solutions of (1.1) defined by

$$\mathrm{Ai}(z) = \frac{1}{2\pi i}\int\limits_{\gamma_0} \exp(zt - \frac{t^3}{3})dt, \tag{2.3}$$

where the path γ_0 runs from ∞ along a ray inside the sector $S(-\frac{5\pi}{6}, -\frac{\pi}{2})$ to the origin and then returns to ∞ along any ray inside the conjugate sector $S(\frac{\pi}{2}, \frac{5\pi}{6})$; and

$$\mathrm{Bi}(z) = i\omega^2 \mathrm{Ai}(\omega^2 z) - i\omega \mathrm{Ai}(\omega z) \tag{2.4}$$

(see [4, p. 9, (2.5.12)]) and [15, p. 53, (8.02); p. 393 (1.10)]).

Traditionally the paths above are taken along the rays $l_{-\frac{2\pi}{3}}$ and $l_{\frac{2\pi}{3}}$. These rays, together with the positive real axis l_0, play an important role in asymptotic analysis and are called *Stokes rays* for the Airy equation. The integral (2.3) is absolutely convergent for any finite z and obviously satisfies (1.5). Of course, it is possible to deform any finite part of the path of integration (see Fig. 1). Solving (1.1) and using the Laplace transform yields, in fact, three contours $\gamma_k, k = -1, 0, 1$, and three corresponding solutions which, following [15, p. 413, (8.02)], we represent as

$$\mathrm{Ai}_0(z) = \mathrm{Ai}(z),\ \mathrm{Ai}_1(z) = \mathrm{Ai}(\omega^{-1}z), \mathrm{Ai}_{-1}(z) = \mathrm{Ai}(\omega z). \tag{2.5}$$

Any two of them are linearly independent.

We have the integral representation

$$\mathrm{Ai}_k(z) = \frac{\omega^k}{2\pi i}\int_{\gamma_k} \exp(zt - \frac{t^3}{3})dt, k = -1, 0, 1. \tag{2.6}$$

(See Fig. 1).

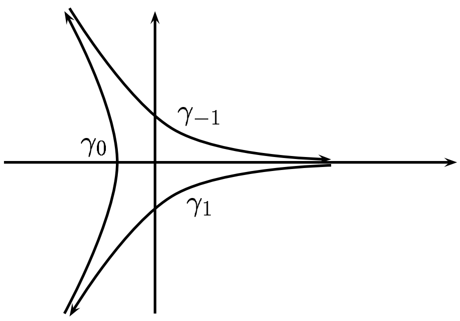

Figure 1. Contours for integral representation of Airy functions.

Moreover, one can prove, by the method of steepest descent that as $z \to \infty$,

$$\mathrm{Ai}_k(z) = -\frac{1}{2\sqrt{\pi}} e^{\frac{k\pi i}{6}} z^{-\frac{1}{4}} e^{-(-1)^k \frac{2}{3} z^{\frac{3}{2}}}\ (1 + o(1)), z \notin l_{\pi + \frac{2k\pi}{3}}, k = -1, 0, 1. \tag{2.7}$$

Thus $\mathrm{Ai}_k(z)$ is bounded inside the sector $\omega^k S\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$ excluding a neighborhood of the origin. (See [15, p. 116, (4.02)]).

In particular,

$$\mathrm{Ai}(z) = \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\frac{2}{3} z^{\frac{3}{2}}} (1 + o(1)), z \notin l_{\pm\pi}, z \to \infty. \tag{2.8}$$

and relation (2.8) describes the behavior of $\mathrm{Ai}(z)$ throughout the complex plane with the significant exception of the negative real axis $l_{\pm\pi}$. For the negative real axis the behavior is different and is given by

$$\mathrm{Ai}(-z) = \pi^{-\frac{1}{2}} z^{-\frac{1}{4}} \cos\left(\frac{2}{3} z^{\frac{3}{2}} - \frac{\pi}{4}\right)(1 + o(1)), z \to +\infty. \tag{2.9}$$

which is also obtained by the method of steepest descent (see [15, p. 102–103, 13.4]). Relations (2.8) and (2.9) show that the Airy function is rapidly decreasing along the positive axis and rapidly oscillating along the negative axis. The first ray is a Stokes ray, and the second is a separation ray. (See Fig. 2).

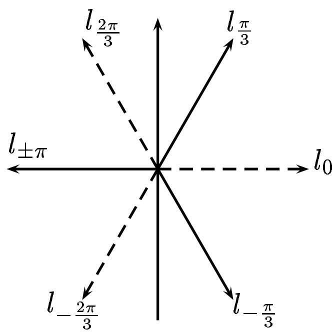

Figure 2. Stokes rays – dashed, separation rays – black.

Unfortunately integral representations such as (2.3) are not generally available for other equations so other more widely applicable approaches are desirable.

3. Airy functions and hypergeometric functions

We now introduce another integral representation of the Airy function in terms of Gauss' hypergeometric function. We will show that the phase amplitude of the Airy function can be expressed as the Laplace transform of Gauss' hypergeometric function.

Let us begin with the *formal solutions* of the Airy equation. Set firstly

$$\zeta = \frac{2}{3} z^{\frac{3}{2}} \tag{3.1}$$

and recall the Pochhammer symbol

$$(a)_k = a(a+1)\dots(a+k-1) = \frac{\Gamma(k+a)}{\Gamma(a)}. \tag{3.2}$$

The expressions

$$\hat{y}_0(z) = z^{-\frac{1}{4}}\ e^{-\zeta} \sum_{k=0}^{\infty} \frac{(-1)^k\ \left(\frac{5}{6}\right)_k\ \left(\frac{1}{6}\right)_k}{2^k\ (1)_k\ \zeta^k} \tag{3.3}$$

and

$$\hat{y}_1(z) = z^{-\frac{1}{4}}\ e^{\zeta} \sum_{k=0}^{\infty} \frac{\left(\frac{5}{6}\right)_k\ \left(\frac{1}{6}\right)_k}{2^k\ (1)_k\ \zeta^k} \tag{3.4}$$

satisfy (1.1) formally and thus represent a pair $\hat{y}_0(z), \hat{y}_1(z)$ of linearly independent formal solutions. We can write these as

$$\hat{y}_0(z) = z^{-\frac{1}{4}} e^{-\zeta} \hat{P}_0(z), \tag{3.5}$$

$$\hat{y}_1(z) = z^{-\frac{1}{4}} e^{\zeta} \hat{P}_1(z), \tag{3.6}$$

where

$$\hat{P}_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k\ \left(\frac{5}{6}\right)_k\ \left(\frac{1}{6}\right)_k}{2^k\ (1)_k\ \zeta^k}, \tag{3.7}$$

$$\hat{P}_1(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{5}{6}\right)_k\ \left(\frac{1}{6}\right)_k}{2^k\ (1)_k\ \zeta^k}. \tag{3.8}$$

We regard $\hat{P}_0(z)$ and $\hat{P}_1(z)$ as the *formal phase amplitudes* of the formal solutions.

There is a striking similarity between the coefficients in $\hat{P}_0(z), \hat{P}_1(z)$ and those of the Gauss hypergeometric functions

$$F\left(\frac{1}{6}, \frac{5}{6}, 1; -\frac{\xi}{2}\right) = \sum_{k=0}^{\infty} \frac{\left(\frac{5}{6}\right)_k\ \left(\frac{1}{6}\right)_k}{(1)_k\ k!} \left(-\frac{\xi}{2}\right)^k, \tag{3.9}$$

$$F\left(\frac{1}{6}, \frac{5}{6}, 1; \frac{\xi}{2}\right) = \sum_{k=0}^{\infty} \frac{\left(\frac{5}{6}\right)_k\ \left(\frac{1}{6}\right)_k}{(1)_k\ k!} \left(\frac{\xi}{2}\right)^k, \tag{3.10}$$

which suggests a relationship between them. Indeed, using

$$\frac{1}{\zeta^n} = \frac{\zeta}{n!} \int_0^{\infty} e^{-\xi\zeta} \xi^n d\xi,$$

which is the key to *Borel summation*, we can express $\hat{P}_0(z)$ formally as*

$$\hat{P}_0(z) \overset{\text{f.p.s.}}{=\!=} \zeta \int_0^{\infty} e^{-\xi\zeta} \hat{F}\left(\frac{1}{6}, \frac{5}{6}, 1; -\frac{\xi}{2}\right) d\xi \tag{3.11}$$

* The symbol $\overset{\text{f.p.s.}}{=\!=}$ means that the equality is understood in the sense of formal power series: we have to substitute the formal expansion (3.9) into the integral in (3.11), and then integrate the formal expansion term-by-term. The resulting formal series will be a formal power series (with respect to $\frac{1}{\zeta}$).

and hence

$$\hat{y}_0(z) \overset{\text{f.p.s.}}{=\!=} z^{-\frac{1}{4}} e^{-\zeta}\zeta \int\limits_0^{\infty} e^{-\xi\zeta} \hat{F}\left(\frac{1}{6}, \frac{5}{6}, 1; -\frac{\xi}{2}\right) d\xi, \tag{3.12}$$

where $\hat{F}(\frac{1}{6}, \frac{5}{6}, 1; -\frac{\xi}{2})$ is Gauss' hypergeometric series,

$$\hat{F}\left(\frac{1}{6}, \frac{5}{6}, 1; -\frac{\xi}{2}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{5}{6}\right)_k \left(\frac{1}{6}\right)_k}{2^k (1)_k k!} \xi^k. \tag{3.13}$$

Similarly we have

$$\hat{P}_1(z) \overset{\text{f.p.s.}}{=\!=} \zeta \int\limits_0^{\infty} e^{-\xi\zeta} \hat{F}\left(\frac{1}{6}, \frac{5}{6}, 1; \frac{\xi}{2}\right) d\xi, \tag{3.14}$$

$$\hat{y}_1(z) \overset{\text{f.p.s.}}{=\!=} z^{-\frac{1}{4}} e^{\zeta}\zeta \int\limits_0^{\infty} e^{-\xi\zeta} \hat{F}\left(\frac{1}{6}, \frac{5}{6}, 1; \frac{\xi}{2}\right) d\xi, \tag{3.15}$$

where $\hat{F}\left(\frac{1}{6}, \frac{5}{6}, 1; \frac{\xi}{2}\right)$ is also defined by (3.13) with the change of variable $\xi \to -\xi$.

Replacing $\hat{F}(\frac{1}{6}, \frac{5}{6}, 1; -\frac{\xi}{2})$ by $F(\frac{1}{6}, \frac{5}{6}, 1; -\frac{\xi}{2})$ in (3.11), we obtain an absolutely convergent integral for $\Re(\zeta) > 0$. This integral represents an analytic function

$$P_0(z) = \zeta \int\limits_0^{\infty} e^{-\xi\zeta} F\left(\frac{1}{6}, \frac{5}{6}, 1; -\frac{\xi}{2}\right) d\xi. \tag{3.16}$$

Here $F(\frac{1}{6}, \frac{5}{6}, 1; -\frac{\xi}{2})$ is an *element* of Gauss' hypergeometric function obtained by analytic continuation of the series (3.9) from the disk $|\xi| < 2$ to the annulus $2 < \|\xi\| < \infty$ cut along the negative ray. We take that element which is real when $\xi > 0$, and single-valued in the whole ξ-complex plane cut along the negative ray. We can prove that $F\left(\frac{1}{6}, \frac{5}{6}, 1; -\frac{\xi}{2}\right) = O\left(|\xi|^{\frac{5}{6}}\right), \xi \to +\infty$. It can be shown from (3.16) that the function $P_0(z)$ is bounded in the sector $S\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$ of the z-plane.

Given θ, $-\pi < \theta < \pi$, introduce

$$P_{0,\theta}(z) = \zeta \int\limits_0^{\infty \cdot e^{i\theta}} e^{-\xi\zeta} F\left(\frac{1}{6}, \frac{5}{6}, 1; -\frac{\xi}{2}\right) d\xi, \tag{3.17}$$

where the path of integration is taken along l_θ, so that $P_{0,0}(z) = P_0(z)$. The function $P_{0,\theta}(z)$ is analytic and bounded in the sector $S\left(-\frac{\pi+2\theta}{3}, \frac{\pi-2\theta}{3}\right)$. Further analysis of the integral (3.17) shows that the functions $P_{0,\theta}(z)$ are elements of the same analytic function $P_0(z)$ bounded in the wider sector $S(-\pi,\pi)$. ($P_0(z)$ is here used to denote the analytic continuation of the function defined by (3.16)). Thus we can regard $P_{0,\theta}(z)$ as a restriction of $P_0(z)$ to $S\left(-\frac{\pi+2\theta}{3},\frac{\pi-2\theta}{3}\right)$. Moreover, one can verify that

$$P_0(z)=1+o(1),\quad z\to\infty, \tag{3.18}$$

inside any closed subsector of $S(-\pi,\pi)$.

On the other hand, we can prove that the function

$$z^{-\frac{1}{4}}e^{-\zeta}P_0(z) \tag{3.19}$$

is a solution of Airy's equation. Comparison of (3.18) and (3.19) with (2.8) shows that (3.19) is simply a constant multiple of $\mathrm{Ai}(z)$.

However in attempting to make a similar replacement in the integral (3.14) it needs to be noted that the point $\xi=2$ is a branch point of the hypergeometric function $F(\frac{1}{6},\frac{5}{6},1;\frac{\xi}{2})$. Thus on the interval $(2,+\infty)$ the formal expression $\hat{F}(\frac{1}{6},\frac{5}{6},1;\frac{\xi}{2})$ must be replaced by a branch of $F(\frac{1}{6},\frac{5}{6},1;\frac{\xi}{2})$. Cutting the complex plane along the positive real axis, we consider two possibilities

$$P_{-1}(z)=\zeta\int\limits_{C_1}e^{-\xi\zeta}F\left(\frac{1}{6},\frac{5}{6},1;\frac{\xi}{2}\right)d\xi, \tag{3.20}$$

$$P_1(z)=\zeta\int\limits_{C_2}e^{-\xi\zeta}F\left(\frac{1}{6},\frac{5}{6},1;\frac{\xi}{2}\right)d\xi, \tag{3.21}$$

where the first path C_1 is along the upper edge of the cut, and the second C_2 along the lower edge. Both integrals are absolutely convergent, as $F\left(\frac{1}{6},\frac{5}{6},1;\frac{\xi}{2}\right)=O\left(|\log|2-\xi||\right)$, $\xi\to 2$, and, as observed above, $F\left(\frac{1}{6},\frac{5}{6},1;\frac{\xi}{2}\right)=O\left(|\xi|^{5/6}\right),\xi\to\infty$, on both edges of the cut. We thus obtain a pair of corresponding functions

$$z^{-\frac{1}{4}}e^{\zeta}P_{-1}(z),z^{-\frac{1}{4}}e^{\zeta}P_1(z).$$

Similar analysis to that above shows that these functions are proportional to $\mathrm{Ai}_{-1}(z)$ and $\mathrm{Ai}_1(z)$, respectively. We thus have the theorem.

Theorem 1. *For θ in the intervals $-\pi<\theta<\pi, 0<\theta<2\pi$ and $-2\pi<\theta<0$, respectively, and for $z\in S\left(-\frac{\pi+2\theta}{3},\frac{\pi-2\theta}{3}\right)$ the following integral representations are valid*

$$\mathrm{Ai}(z)=\frac{1}{2\sqrt{\pi}}z^{-\frac{1}{4}}e^{-\zeta}\zeta\int\limits_{l_\theta}e^{-\xi\zeta}F\left(\frac{1}{6},\frac{5}{6},1;-\frac{\xi}{2}\right)d\xi, \tag{3.22}$$

$$-\pi<\theta<\pi,$$

$$\mathrm{Ai}_{-1}(z) = \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{\zeta} \zeta \int\limits_{l_\theta} e^{-\xi\zeta} F\left(\frac{1}{6}, \frac{5}{6}, 1; \frac{\xi}{2}\right) d\xi, \tag{3.23}$$

$$0 < \theta < 2\pi,$$

$$\mathrm{Ai}_{1}(z) = \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{\zeta} \zeta \int\limits_{l_\theta} e^{-\xi\zeta} F\left(\frac{1}{6}, \frac{5}{6}, 1; \frac{\xi}{2}\right) d\xi, \tag{3.24}$$

$$-2\pi < \theta < 0.$$

Remark 1. *These formulae can also be obtained from formulae (11) or (12) of [8] for the Hankel functions setting $\nu = \frac{1}{3}$ which is not surprising in view of (1.2).*

Writing

$$\mathrm{Ai}_k(z) = \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{(-1)^{k+1}\frac{2}{3}z^{\frac{3}{2}}} P_k(z), \tag{3.25}$$

we call $P_k(z)$ *the phase amplitude* of $\mathrm{Ai}_k(z)$. Thus,

$$P_k(z) = 1 + o(1), z \in \omega^k S\left(-\pi, \pi\right), z \to \infty. \tag{3.26}$$

Remark 2. *Comparing (3.22) and (3.12), shows that* $\mathrm{Ai}(z) = \mathrm{Ai}_0(z)$ *and* $\hat{y}_0(z)$ *are generated in the same way by the same elements of Gauss' hypergeometric function and series, respectively. Further analysis, see (3.12) and (3.15), shows that different formal, as well as regular solutions of the Airy equation, are generated similarly by different elements of the same Gauss hypergeometric function. These observations are very important and can be considered as a particular case of a much more comprehensive theory (see [18] and the article of Ramis in [5]).*

Remark 3. *It is remarkable that the simple formula (3.22) seems not to have appeared in the classical literature. Moreover, the formal solution of the Airy equation, has been presented in the classical literature (see, for example, Olver, [15, p. 116]), Antosiewicz in [1, p. 448, (10.4.59)], as*

$$\frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\zeta} \sum_{k=0}^{\infty} \frac{(-1)^k 2^k \Gamma(3k + \frac{1}{2})}{3^{3k}(2k)!\Gamma(\frac{1}{2})\zeta^k}. \tag{3.27}$$

And the relationship to the hypergeometric function is obscure, while (3.27) is identical to

$$\frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\zeta} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{5}{6}\right)_k \left(\frac{1}{6}\right)_k}{2^k (1)_k \zeta^k} \tag{3.28}$$

(see Berry in [19] and Ramis [18]), which is obviously closer to the hypergeometric series $\hat{F}\left(\frac{1}{6}, \frac{5}{6}, 1; -\frac{\xi}{2}\right)$.

We now compare the two integral representations for the Airy function (2.3) and (3.22). The relation (2.3) is more elementary and is valid everywhere in the complex plane. However (3.22) has a definite advantage over (2.3). To extract the asymptotic behavior of the Airy function from (2.3) requires a calculation using the method of steepest descent. To obtain the asymptotic behavior from (3.22) in the complex plane, except for a small sector including the negative real axis, we observe that the main contribution is from a small interval of integration $(0, \delta)$ on the positive real axis. Substituting the hypergeometric series for $F(\frac{1}{6}, \frac{5}{6}, 1; -\frac{\xi}{2})$ in (3.22) and integrating term by term yields, by *Watson's lemma*, the full asymptotic expansion of the Airy function. In addition, the second integral representation (3.22) may be used to obtain the optimal estimate for the remainder in the complex plane. Such an estimate would be based on the behavior of $F\left(\frac{1}{6}, \frac{5}{6}, 1; -\frac{\xi}{2}\right)$ at its branch point $\xi = -2$, as opposed to Olver's method shown in [16, 17].

Probably (3.22) can be used to provide numerical evaluation of the Airy function which could be compared with recent results in [20].

There is another more important advantage of (3.22) which we now demonstrate, and which, in particular, leads naturally to "asymptotics beyond all orders" (see the article of Berry in [19] and [3]).

The Gauss hypergeometric functions $F(a, b, c; \xi)$ are analytic in the complex plane with the points $0, 1, \infty$ deleted. There is a branch of $F(a, b, c; \xi)$ which is analytic inside the unit circle with Taylor series

$$F(a, b, c; \xi) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} \xi^k \tag{3.29}$$

convergent for $|\xi| < 1$. This function and its analytical continuations satisfy the well known hypergeometric differential equation. If we consider the analytic continuation

$$F^*(a, b, c; \xi) = F(a, b, c; 1 + (\xi - 1)e^{\pm 2i\pi}) \tag{3.30}$$

its value, after encircling the point 1, will differ from $F(a, b, c; \xi)$. In fact, the following *monodromic relations* are valid (see [12, (4.2 eq. (25)]):

$$F^*(a, b, c; \xi)$$

$$= F(a, b, c; \xi) \mp T\,(a, b, c)\,(1 - \xi)^{c-a-b} F(c - a, c - b, 1 + c - a - b; 1 - \xi), \tag{3.31}$$

where

$$T\,(a, b, c) = \frac{2i\pi\Gamma(c)e^{\pm i\pi(c-a-b)}}{\Gamma(a)\Gamma(b)\Gamma(1 + c - a - b)}. \tag{3.32}$$

For our particular case $a = \frac{1}{6}, b = \frac{5}{6}, c = 1$, (3.31) can be written as

$$F\left(\frac{1}{6},\frac{5}{6},1;1+(\frac{\xi}{2}-1)e^{\pm 2i\pi}\right) = F\left(\frac{1}{6},\frac{5}{6},1;\frac{\xi}{2}\right) \mp TF\left(\frac{5}{6},\frac{1}{6},1;1-\frac{\xi}{2}\right) \quad (3.33)$$

with $T = i$.

The monodromic property (3.33) must be inherited by solutions of the Airy equation. Using integral representations (3.22), (3.23) and (3.24), we could derive the corresponding monodromic relations for the phase amplitudes of the Airy functions. For example, using (3.22), (3.25), (3.33), we can derive the following relation:

$$P_0(z) = P_{-1}(ze^{-2\pi i}) + Te^{\frac{4}{3}z^{\frac{3}{2}}}P_1(z) \quad (3.34)$$

and similar relations for $P_{-1}(ze^{2\pi i}), P_0(ze^{\pm 2\pi i}), P_1(ze^{\pm 2\pi i})$ where the T's are complex constants, connection coefficients, which can be calculated from (3.32). In our cases $T = \pm i$, and for (3.34), $T = i$.

We can also consider the phase-amplitudes as functions of the variable ζ

$$P_k(z) = \mathcal{P}_k(\zeta). \quad (3.35)$$

Thus we can seek monodromic relations among $\mathcal{P}_k\left(\zeta e^{\pm 2\pi i}\right)$ and $\mathcal{P}_j(\zeta)$. For example, we can derive from (3.22) the following relation:

$$\mathcal{P}_0(\zeta e^{2\pi i}) = \mathcal{P}_0(\zeta) + \mathcal{T}_0 e^{2\zeta}\mathcal{P}_1(\zeta e^{2\pi i}) \quad (3.36)$$

and the similar relation for $\mathcal{P}_1(\zeta e^{2\pi i})$

$$\mathcal{P}_1(\zeta e^{2\pi i}) = \mathcal{P}_1(\zeta) + \mathcal{T}_1 e^{-2\zeta}\mathcal{P}_0(\zeta). \quad (3.37)$$

These relations agree with relations (30), (35) of [8] with $\nu = \frac{1}{3}$ and are also in agreement with (1.2). It is possible to express these connection coefficients in terms of those in the three element Stokes structure.

It is worth noting that the relation (3.34), because of (3.25), can be rewritten in terms of Airy functions as

$$\mathrm{Ai}_0(ze^{\pi i}) = -i\mathrm{Ai}_{-1}(ze^{-\pi i}) + i\mathrm{Ai}_1(ze^{\pi i}),$$

which immediately provides another derivation of (2.9).

However it is more important for us that the above monodromic relations for the phase-amplitudes suggest the definition of the Stokes structure.

Definition 1. *A set of functions* $\{p_{-1}(z), p_0(z), p_1(z)\}$

(i) *analytic on the Riemann surface of* $\log z$ *and for any* $\varepsilon > 0$ *satisfying the estimate*

$$|p(z)| < M_\varepsilon \exp((\frac{2}{3}+\varepsilon)\,|z|^{\frac{3}{2}})$$

for some $M_\varepsilon > 0$,

(ii) *bounded in the sectors*

$$\begin{aligned} S(-1) &= S(-5\pi/3, \pi/3), \\ S(0) &= S(-\pi, \pi), \\ S(1) &= S(-\pi/3, 5\pi/3), \end{aligned} \tag{3.38}$$

respectively,

(iii) *satisfying the monodromic relations with connection coefficients* $T_{-1},\ T_0,\ T_1$

$$p_{-1}(z) = p_1(z) + T_{-1}e^{-2\zeta}p_0(z), \tag{3.39}$$

$$p_0(ze^{2\pi i}) = p_{-1}(z) + T_0 e^{-2\zeta}p_1(ze^{2\pi i}), \tag{3.40}$$

$$p_1(ze^{2\pi i}) = p_0(z) + T_1 e^{2\zeta}p_{-1}(z) \tag{3.41}$$

forms the three-element Stokes structure

$$\mathfrak{S}(3) = \mathfrak{S}\{p_{-1}(z), p_0(z), p_1(z)\} \tag{3.42}$$

generated by $e^{2\zeta}$, *with* ζ *defined by (3.1).*

Definition 2. *A set of functions* $\{p_1(\zeta), p_2(\zeta)\}$

(i) *analytic on the Riemann surface of* $\log\zeta$ *and for any* $\varepsilon > 0$ *satisfying the estimate*

$$|p(z)| < M_\varepsilon \exp((2+\varepsilon)\,|\zeta|)$$

with $M_\varepsilon > 0$,

(ii) *bounded in the sectors*

$$S(1) = S(-\frac{3\pi}{2}, \frac{3\pi}{2}),$$

$$S(2) = S(-\frac{\pi}{2}, \frac{5\pi}{2}),$$

respectively,

(iii) *satisfying the monodromic relations with connection coefficients* T_1, T_2

$$p_1(\zeta e^{2\pi i}) = p_1(\zeta) + T_1 e^{2\zeta}p_2(\zeta e^{2\pi i}), \tag{3.43}$$

$$p_2(\zeta e^{2\pi i}) = p_2(\zeta) + T_2 e^{-2\zeta}p_1(\zeta) \tag{3.44}$$

forms the two-element Stokes structure

$$\mathfrak{S}(2) = \mathfrak{S}\{p_1(z), p_2(z)\}$$

generated by $e^{2\zeta}$.

Remark 4. *As explained above it is possible to show that the phase amplitudes $P_k(z), k = -1, 0, 1,$ of the Airy functions form a three element Stokes structure with $T_{-1} = T_0 = T_1 = i$. However in terms of ζ defined by (3.1) the phase amplitudes $\mathcal{P}_k(\zeta), k = 0, 1,$ defined by (3.35), form a two element Stokes structure. This can be understood from the following observation. The change of variable*

$$\zeta = \frac{2}{3} z^{\frac{3}{2}}, y(z) = z^{-\frac{1}{4}} w(\zeta) \tag{3.45}$$

constitutes the Liouville transformation, *see [6]. This transforms the Airy equation to the form*

$$w''(\zeta) - w(\zeta) = -\frac{5}{36\zeta^2} w(\zeta). \tag{3.46}$$

The equation (3.46) can be regarded as a canonical form *for the Airy equation which "separates" the exponential terms and the phase amplitudes. We will show in Section 5, using the Airy equation as a model, how to extend this idea of a canonical form to matrix equations. It is worth noting that, unlike the Airy functions, for solutions of (3.46) we have $w(\zeta e^{2\pi i}) \neq w(\zeta)$. In fact, the two-element Stokes structure above is the Stokes structure for the equation (3.46).*

In the next section we will show how to derive the Stokes structure directly from the differential equation without any reference to Gauss' hypergeometric function. The principal aim of this paper is to obtain formulae for the connection coefficients of the Stokes structure. Another aim, which will be the subject of subsequent papers, is to derive the integral representations of Theorem 1 starting from the Stokes structure.

Unfortunately, the method used above to obtain the integral representation of Theorem 1 is, again, not generally available for other equations.

4. The Stokes structure for the Airy equation

We consider three adjoining sectors $s(-1) = S(-\pi, -\frac{\pi}{3}), s(0) = S(-\frac{\pi}{3}, \frac{\pi}{3})$, and $S(\frac{\pi}{3}, \pi)$.

As in [9] the *separation rays* for the Airy equation are defined as

$$l_{\theta_k}, \theta_k = -\frac{\pi}{3} + \frac{2\pi}{3} k, k = 0, \pm 1. \tag{4.1}$$

The sectors $s(k)$, together with the separation rays $l_{\theta_k}, k = 0, \pm 1$, comprise the whole complex plane $\mathbb{C}$ with the origin deleted (see Fig. 2).

Proposition 1. *Given $k, k = 0, \pm 1$, there exists a solution of (1.1) which is bounded in $s(k)$. Moreover, this solution is uniquely defined up to a constant factor.*

Proposition 2. *Given a ray l any solution $y(z)$ of (1.1) can be presented in the form*

$$y(z) = (Az^{-\frac{1}{4}}e^{-\frac{2}{3}z^{\frac{3}{2}}} + Bz^{-\frac{1}{4}}e^{\frac{2}{3}z^{\frac{3}{2}}})(1 + o(1)), z \in l, z \to \infty,$$

with complex constants A and B.

Using these propositions we can introduce the *principal* normalized solution $y_k(z)$ such that

$$y_k(z) = z^{-\frac{1}{4}}e^{-\frac{2}{3}(-1)^k z^{\frac{3}{2}}}\left(1 + o\left(1\right)\right), z \in s(k), z \to \infty.$$

Thus, we obtain another definition of Airy functions. Writing

$$y_k(z) = z^{-\frac{1}{4}}e^{-\frac{2}{3}(-1)^k z^{\frac{3}{2}}} P_k(z), k = 0, \pm 1, \tag{4.2}$$

we can regard $y_k(z), k = 0, \pm 1$, as normalized *Airy functions* and we will regard $P_k(z)$ as their phase-amplitudes.

Although these propositions follow from any simple version of the B–H–T (Birkhoff–Hukuhara–Turrittin) theorem, they are in fact more elementary. For example, we can derive Proposition 1, using (3.46) and the corresponding integral equation (see [6, p. 79 (5)]). This equation can be written in the form

$$w = w_0 + Kw,$$

where K is the Volterra integral operator. Let S be a sector in the complex plane and $\mathfrak{S}(S)$ the space of functions w, analytic and bounded inside S, such that $w(z) \to 0$ as $z \to \infty$, and with uniform norm $\|w\| = \sup|w|$. We consider the operator K acting in $\mathfrak{S}(S)$. Then our proposition follows simply from the fact that $\|K\| < q < 1$ for some q.

For $z \in s(k)$ we have respectively

$$P_k(z) = 1 + o(1), z \to \infty, k = 0, \pm 1. \tag{4.3}$$

We now introduce the extended sectors

$$S(k) = \left\{z : -\pi + \frac{2\pi}{3}k < \arg z < \pi + \frac{2\pi}{3}k, 0 < |z| < \infty, k = 0, \pm 1.\right\}. \tag{4.4}$$

Note that $S(k)$ subtends an angle three times as large as that of $s(k)$.

Proposition 3. *The phase-amplitude $P_k(z)$ of the Airy function $y_k(z)$, $k = 0, \pm 1$, satisfies (4.3) inside any closed sub-sector of the extended open sector $S(k)$.*

This can be proved following the steps of the proof of Proposition 3 ([9]).

We can extend what has been said above to any perturbation of the Airy equation of the form

$$y''(z) - \left(z + \frac{b_2}{z^2} + \cdots\right) y(z) = 0, \tag{4.5}$$

where $b_2, \ldots$ are complex constants.

If $b_2 \neq 0$ then any regular solution of this equation is a multi-valued analytic function which can be considered as single-valued on the Riemann surface of $\log z$. We can preserve the preceding notation $y_k(z)$ for the normalized solutions of (4.5) with the behavior

$$y_k(z) = z^{-(-1)^k \frac{1}{4}} e^{-\frac{2}{3} z^{\frac{3}{2}}} (1 + o(1)), z \to \infty,$$

inside $S(k)$. After crossing a separation ray $l_{\theta_m}, \theta_m = -\frac{\pi}{3} + \frac{2\pi}{3} m, m \in \mathbb{Z}, m \neq 0$, the behavior of, say, $y_0(z)$ takes the form

$$y_0(z) = (A_m z^{-\frac{1}{4}} e^{-\zeta} + B_m z^{-\frac{1}{4}} e^{\zeta})(1 + o(1)), z \to \infty,$$

with complex constants A_m, B_m.

Our main aim is to introduce methods for evaluating the coefficients A_m, B_m in terms of the coefficients of the Stokes structure.

Let us consider again the system of sectors $S(-1), S(0), S(1)$ and the system of normalized Airy functions. We can extend this system to a system of sectors $S(k-1), S(k), S(k+1)$ on the Riemann surface of $\log z$ defined by (4.4) as previously but for all integers k. We also consider the corresponding systems of normalized Airy functions $\{y_{k-1}(z), y_k(z), y_{k+1}(z)\}$ satisfying (4.2) and (4.3) for all integers k. We preserve the same notation $\{y_{k-1}(z), y_k(z), y_{k+1}(z)\}$ for normalized solutions of the perturbed Airy equation (4.5) satisfying (4.2) and (4.3) for all integer k. It is clear that neighboring functions $y_k(z), y_{k+1}(z)$ are linearly independent solutions of (1.1) or (4.5), respectively. Moreover the following fact is true.

Proposition 4. *1. If $y_k(z), y_{k+1}(z), y_{k+2}(z)$ are neighboring normalized solutions of (1.1) or (4.5), then for $k \in \mathbb{Z}$*

$$y_k(z) = y_{k+2}(z) + T_k y_{k+1}(z), \tag{4.6}$$

where T_k are complex constants.

2. For solutions of (1.1) or (4.5) we have

$$y_{k+3}(z) = -i y_k(z e^{-2\pi i}). \tag{4.7}$$

Applying this proposition to the principal normalized solutions $y_k(z), k = 0, \pm 1$, of (1.1) or (4.5), yields

$$\begin{aligned} y_{-1}(z) &= y_1(z) + T_{-1} y_0(z), && (4.8)\\ y_0(z) &= -iy_{-1}(ze^{-2\pi i}) + T_0 y_1(z), && (4.9)\\ y_1(z) &= -iy_0(ze^{-2\pi i}) - iT_1 y_{-1}(ze^{-2\pi i}). && (4.10) \end{aligned}$$

Using (4.2), we will rewrite these relations (4.8–4.10) in terms of phase amplitudes of the normalized solutions

$$\begin{aligned} P_{-1}(z) &= P_1(z) + T_{-1} e^{-2\zeta} P_0(z), && (4.11)\\ P_0(ze^{2\pi i}) &= P_{-1}(z) + T_0 e^{-2\zeta} P_1(ze^{2\pi i}), && (4.12)\\ P_1(ze^{2\pi i}) &= P_0(z) + T_1 e^{2\zeta} P_{-1}(z). && (4.13) \end{aligned}$$

These monodromic relations form a basis for the three element Stokes structure. In fact, together with (4.3) they establish

Theorem 2. *If $P_{-1}(z), P_0(z), P_1(z)$ are the phase-amplitudes of the normalized Airy functions, then they form a three-element Stokes structure (3.42).*

Thus we can regard the Stokes structures (3.42) and (3.44) as a set of algebraic relations in certain classes of analytic functions.

The Stokes structure can be studied abstractly, independently of the Airy equation. Applying an appropriate Fourier (Borel) transform to the elements of the Stokes structure and studying the properties of these transforms, we can prove, for example, that the Stokes structure generates the three formal power series which serve as the asymptotic expansions for those elements.

The Stokes structure of Theorem 2 serves not only for the Airy equation, but also for all of its perturbations (4.5), where now the connection coefficients depend on the parameters b_2, etc. of the equations. Our goal is to indicate how to obtain formulae for these coefficients. However for the particular case of the Airy equation these coefficients can be found immediately without any calculation.

Theorem 3. *Let $P_{-1}(z), P_0(z), P_1(z)$ form a three-element Stokes structure (3.42). Assume in addition that they are the phase-amplitudes of the normalized Airy functions, then we have*

$$T_{-1} = T_0 = T_1 = i. \tag{4.14}$$

P r o o f. Using (4.2), the relations (4.11–4.13) can be rewritten in terms of $y_{-1}(z), y_0(z), y_1(z)$ as (4.8–4.8). Since the Airy functions are single-valued in the complex plane, $y_k(ze^{2\pi i}) = y_k(z)$, and we can rewrite (4.8–4.10) as

$$\begin{aligned} y_0(z) &= T_{-1}^{-1} y_{-1}(z) - T_{-1}^{-1} y_1(z), && (4.15)\\ y_0(z) &= -iy_{-1}(z) + T_0 y_1(z), && (4.16)\\ y_0(z) &= -T_1 y_{-1}(z) + iy_1(z). && (4.17) \end{aligned}$$

which immediately yields (4.14). ∎

Corollary 1. *The relation (2.9) is valid.*

P r o o f. Using (1.4) and (4.14), we can rewrite the relation (4.8) as

$$y_0(ze^{\pi i}) = -iy_{-1}(ze^{-\pi i}) + iy_1(ze^{\pi i}). \qquad (4.18)$$

Then by (4.2), (4.3) and Proposition 3, we have, as $z \to +\infty$,

$$y_{-1}(z) = z^{-\frac{1}{4}} e^{\frac{2}{3} z^{\frac{3}{2}}} (1 + o(1)), z \in S(-\frac{5\pi}{3}, \frac{\pi}{3}),$$

$$y_1(z) = z^{-\frac{1}{4}} e^{\frac{2}{3} z^{\frac{3}{2}}} (1 + o(1)), z \in S(-\frac{\pi}{3}, \frac{5\pi}{3}).$$

Remembering that all solutions are single-valued in $\mathbb{C}$, these expressions for $y_{-1}(z)$ and $y_1(z)$, when substituted in (4.18,) yield immediately the behavior of $y_0(z)$ on the negative real axis as

$$2z^{-\frac{1}{4}} \frac{e^{i(\frac{2}{3} z^{\frac{3}{2}} + \frac{\pi}{4})} - e^{-i(\frac{2}{3} z^{\frac{3}{2}} + \frac{\pi}{4})}}{2i} (1 + o(1)), z \to +\infty,$$

identical to (2.9) apart from a normalizing factor. ∎

Remark 5. *We have established (2.9) using merely substitution without any calculation whatsoever.*

In fact, Theorem 3, as well as the notion of the Stokes structure represent our realization of an old idea of Zwaan, shown in [11]. From the Historical Survey of [11]: "Zwaan [23], (1929), appears to have been the first to use the complex $z-$plane to investigate transition points....". In our context Theorem 3 follows immediately from the Stokes structure.

Although (4.5) has the same Stokes structure as the Airy equation, the technique of Theorem 3 cannot be applied to it. Nevertheless we can extend this theorem with appropriate modification to a much larger class of equations, for example, to the Kummer equation

$$y^{(n)}(z) - (-1)^n z^m y(z) = 0, n, m \in \mathbb{Z}^+, \tag{4.19}$$

or to the equation

$$y^{(n)}(z) + a_1 y^{(n-1)}(z) + \ldots + a_{n-1} y'(z) + a_n z y(z) = 0, \tag{4.20}$$

where $a_1, \ldots, a_{n-1}, a_n$ are complex numbers.

The first class of equations, (4.19), was introduced by Kummer (see [13]). The detailed analysis of this equation and a matrix version of it is given in a series of papers by J. Heading (see [10]), who also provides explicit formulae for the connection coefficients.

As for the second equation , it appears in many areas, from parabolic partial differential equations to harmonic analysis. Solutions of these equation can be represented as the Fourier transform of the exponential of a polynomial

$$y(z) = \int_{-\infty}^{\infty} e^{iz\xi} e^{\mathcal{P}(\xi)} d\xi,$$

where $\mathcal{P}$ is a polynomial, and the path of integration can be replaced by an appropriate contour as in (2.6).

In what follows we show how to evaluate connection coefficients for a much larger set of ODE's continuing to use the Airy equation as an example.

5. The canonical form of the Airy equation and the Liouville transformation

We introduce now another way for deriving a *canonical form* for matrix equations which separates the *matrix exponential* from the *matrix phase-amplitude.*

We will rewrite the Airy equation in matrix form in the standard way. Set

$$\mathbf{Y} = \begin{bmatrix} y \\ y' \end{bmatrix}, \tag{5.1}$$

then the Airy equation takes the form

$$\mathbf{Y}' = \begin{bmatrix} 0 & 1 \\ z & 0 \end{bmatrix} \mathbf{Y}. \tag{5.2}$$

Indeed, we may treat $\mathbf{Y}(z)$ as a square matrix whose columns are linearly independent vector solutions of (5.2) and we then consider matrix solutions $\mathbf{Y}(z)$ of the Airy equation, with $\det \mathbf{Y} \neq 0$.

The change of variable

$$\mathbf{Y} = \begin{bmatrix} 1 & 1 \\ \sqrt{z} & -\sqrt{z} \end{bmatrix} \mathbf{U} \tag{5.3}$$

reduces (5.2) to the form

$$\frac{d\mathbf{U}(z)}{dz} = \left(\sqrt{z} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \frac{1}{4z} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right) \mathbf{U}(z). \tag{5.4}$$

Following (3.1) and setting

$$\mathbf{U}(z) = \mathbf{V}(\zeta), \zeta = \frac{2}{3} z^{\frac{3}{2}}, \tag{5.5}$$

yields

$$\frac{d\mathbf{V}(\zeta)}{d\zeta} = \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \frac{1}{6\zeta} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right) \mathbf{V}(\zeta). \tag{5.6}$$

Let us introduce a pair of complex constants σ_1 and σ_2. Using a transformation of the form

$$\mathbf{V}(\zeta) = \begin{bmatrix} 1 & \frac{\sigma_1}{\zeta} \\ \frac{\sigma_2}{\zeta} & 1 \end{bmatrix} \mathbf{W}(\zeta), \tag{5.7}$$

we can eliminate the off-diagonal terms of the second matrix in (5.6) by a suitable choice of σ_1 and σ_2, at the cost of an additional term with behaviour $O\left(\frac{1}{\zeta^2}\right)$ as $\zeta \to \infty$. The resulting form can be regarded as a canonical form.

On the other hand, using a transformation of the form

$$\mathbf{V}(\zeta) = \begin{bmatrix} 0 & S_1(\zeta) \\ S_2(\zeta) & 0 \end{bmatrix} \mathbf{W}(\zeta), \tag{5.8}$$

where $S_1(\zeta)$ and $S_2(\zeta)$ are complex functions, we can eliminate the diagonal terms of the second matrix in (5.6), after proper choice of the functions $S_1(\zeta)$ and $S_2(\zeta)$, thus reducing the equation to another canonical form.

There are various benefits of these canonical forms. Using them we can (i) obtain the formal solutions, (ii) prove a version of the B–H–T theorem, (iii) find optimal estimates for the remainder following, for example, Olver's method presented in [16, 17], (iv) establish integral representations for the connection coefficients.

For our present purposes it is advantageous to use (5.8).

The change of variables

$$\mathbf{V}(\zeta) = \begin{bmatrix} 0 & \zeta^{-\frac{1}{6}} \\ \zeta^{-\frac{1}{6}} & 0 \end{bmatrix} \mathbf{W}(\zeta) \tag{5.9}$$

takes (5.6) to the form

$$\frac{d\mathbf{W}(\zeta)}{d\zeta} = \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{1}{6\zeta} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \mathbf{W}(\zeta), \tag{5.10}$$

which can be regarded as a *canonical form of the Airy equation.* This form allows us to separate exponentials and phase-amplitudes. Indeed, the change of variables

$$\mathbf{W}(\zeta) = \mathbf{P}(\zeta) \begin{bmatrix} e^{\zeta} & 0 \\ 0 & e^{-\zeta} \end{bmatrix} \tag{5.11}$$

transforms (5.10) to

$$\mathbf{P}'(\zeta) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{P}(\zeta) - \mathbf{P}(\zeta) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{1}{6\zeta} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{P}(\zeta). \tag{5.12}$$

This can be considered as a matrix equation

$$\begin{bmatrix} P'_{11} & P'_{12} \\ P'_{21} & P'_{22} \end{bmatrix} = \begin{bmatrix} 0 & 2P_{12} \\ -2P_{21} & 0 \end{bmatrix} + \frac{1}{6\zeta} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}. \tag{5.13}$$

or as a system of vector equations

$$\begin{bmatrix} P'_{11} \\ P'_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ -2P_{21} \end{bmatrix} + \frac{1}{6\zeta} \begin{bmatrix} P_{21} \\ P_{11} \end{bmatrix} \tag{5.14}$$

and

$$\begin{bmatrix} P'_{12} \\ P'_{22} \end{bmatrix} = \begin{bmatrix} 2P_{12} \\ 0 \end{bmatrix} + \frac{1}{6\zeta} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_{22} \\ P_{12} \end{bmatrix}. \tag{5.15}$$

We can regard the first factor in (5.11) as the *matrix phase-amplitude* and the second exponential factor as the *matrix "phase"*.

The transformations (5.5) and (5.9) are simply the Liouville transformations. However this presentation shows that it is possible to extend this transformation to higher order scalar or matrix equations (see [10]), while the classical Liouville transformation is limited to the case of second order scalar equations.

Consider the vector phase amplitudes

$$\mathbf{P}_1(\zeta) = \begin{bmatrix} P_{11}(\zeta) \\ P_{21}(\zeta) \end{bmatrix}, \mathbf{P}_2(\zeta) = \begin{bmatrix} P_{12}(\zeta) \\ P_{22}(\zeta) \end{bmatrix}. \tag{5.16}$$

An obvious analysis shows these vector phase amplitudes form a two-element vector Stokes structure (compare with Definition 2, permuting indices 1 and 2)

$$\begin{aligned} \mathbf{P}_2(\zeta e^{2\pi i}) &= \mathbf{P}_2(\zeta) + \mathcal{T}_1 e^{2\zeta} \mathbf{P}_1(\zeta e^{2\pi i}) \\ \mathbf{P}_1(\zeta e^{2\pi i}) &= \mathbf{P}_1(\zeta) + \mathcal{T}_2 e^{-2\zeta} \mathbf{P}_2(\zeta). \end{aligned} \tag{5.17}$$

6. Formal solutions and regular solutions

It is clear now that we can seek a formal solution of (5.12), or equivalently of (5.14) and (5.15), in the form of a formal power series in ζ^{-1} :

$$\hat{\mathbf{P}}(\zeta) = \sum_{k=0}^{\infty} \mathbf{P}^{(k)} \zeta^{-k}. \tag{6.1}$$

Substituting (6.1) into (5.12) yields

$$\hat{P}_{11}(\zeta) = \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{6}\right)_k \left(\frac{1}{6}\right)_k}{2^k k! \zeta^k}, \tag{6.2}$$

$$\hat{P}_{21}(\zeta) = -6 \sum_{k=0}^{\infty} k \frac{\left(-\frac{1}{6}\right)_k \left(\frac{1}{6}\right)_k}{2^k k! \zeta^k}, \tag{6.3}$$

$$\hat{P}_{12}(\zeta) = -6 \sum_{k=0}^{\infty} (-1)^k k \frac{\left(-\frac{1}{6}\right)_k \left(\frac{1}{6}\right)_k}{2^k k!} \frac{1}{\zeta^k}, \tag{6.4}$$

$$\hat{P}_{22}(\zeta) = \sum_{k=0}^{\infty} (-1)^k \frac{\left(-\frac{1}{6}\right)_k \left(\frac{1}{6}\right)_k}{2^k k!} \frac{1}{\zeta^k}. \tag{6.5}$$

Returning to the initial variables (5.9) and using (5.11) and $U(z) = V(\zeta)$ yields

$$\begin{aligned} U_{11} &= \zeta^{-\frac{1}{6}} e^{\zeta} P_{21}, \\ U_{21} &= \zeta^{-\frac{1}{6}} e^{\zeta} P_{11}, \end{aligned} \tag{6.6}$$

$$\begin{aligned} U_{12} &= \zeta^{-\frac{1}{6}} e^{-\zeta} P_{22}, \\ U_{22} &= \zeta^{-\frac{1}{6}} e^{-\zeta} P_{12}. \end{aligned} \tag{6.7}$$

It follows from

$$Y = \begin{bmatrix} 1 & 1 \\ \sqrt{z} & -\sqrt{z} \end{bmatrix} U \tag{6.8}$$

that

$$y_1 = Y_{11} = U_{11} + U_{21}, \quad y_1' = Y_{21} = \sqrt{z}(U_{11} - U_{21}),$$
$$y_0 = Y_{12} = U_{12} + U_{22}, \quad y_0' = Y_{22} = \sqrt{z}(U_{12} - U_{22}),$$

which finally yields

$$\begin{aligned} y_1 &= \zeta^{-\frac{1}{6}} e^{\zeta} (P_{11} + P_{21}), \\ y_0 &= \zeta^{-\frac{1}{6}} e^{-\zeta} (P_{12} + P_{22}). \end{aligned} \tag{6.9}$$

and

$$\begin{aligned} P_{22} &= 2^{-1} \zeta^{\frac{1}{6}} e^{\zeta} (y_0 + \tfrac{y_0'}{\sqrt{z}}) \\ P_{11} &= 2^{-1} \zeta^{\frac{1}{6}} e^{-\zeta} (y_1 - \tfrac{y_1'}{\sqrt{z}}). \end{aligned} \tag{6.10}$$

Using (6.9) and (6.2), (6.3), (6.4), (6.5), yields the formal solutions of the Airy equation shown in (3.3) and (3.4).

The system of scalar equations obtained from (5.14) and (5.15)

$$\begin{aligned} P_{11}'(\zeta) &= \tfrac{1}{6\zeta}P_{21}(\zeta), \\ P_{21}'(\zeta) &= -2P_{21}(\zeta) + \tfrac{1}{6\zeta}P_{11}(\zeta), \end{aligned} \tag{6.11}$$

and

$$\begin{aligned} P_{12}'(\zeta) &= 2P_{12}(\zeta) + \tfrac{1}{6\zeta}P_{22}(\zeta), \\ P_{22}'(\zeta) &= \tfrac{1}{6\zeta}P_{12}(\zeta) \end{aligned} \tag{6.12}$$

yields immediately an equivalent system of Volterra integral equations which is very convenient for successive iterations. Using this system it is possible to prove a version of the B–H–T theorem.

We will show below how to derive from this system an integral representation for the connection coefficients.

7. Connection coefficients

Theorem 4. *Let γ be the path running from ∞ along the positive axis, encircling the origin in a counterclockwise direction, and returning to ∞ along the positive axis (see Fig. 3). Let $\mathcal{T}_1, \mathcal{T}_2$ be the connection coefficients of (5.17). Then*

$$\mathcal{T}_1 = \int_\gamma e^{-2\zeta}\frac{1}{6\zeta}P_{22}(\zeta)d\zeta, \tag{7.1}$$

$$\mathcal{T}_2 = \int_{-\gamma} e^{2\zeta}\frac{1}{6\zeta}P_{11}(\zeta)d\zeta, \tag{7.2}$$

where $P_{22}(\zeta), P_{11}(\zeta)$ are defined by (6.10).

P r o o f. Consider the right-hand side of (7.1). Using the first equation

$$P_{12}'(\zeta) = 2P_{12}(\zeta) + \frac{1}{6\zeta}P_{22}(\zeta)$$

of (6.12), we have

$$\int_\gamma e^{-2\zeta}\frac{1}{6\zeta}P_{22}(\zeta)d\zeta = \int_\gamma e^{-2\zeta}\left(P_{12}'(\zeta) - 2P_{12}(\zeta)\right)d\zeta. \tag{7.3}$$

This integral can be expressed as the limit

$$\int_\gamma = \lim \int_a^b,$$

where a is positive, $b = ae^{2\pi i}$ and $a \to +\infty$. Thus a belongs to the upper branch of the path γ, $\arg a = 0$, and b, $\arg b = 2\pi$, belongs to the lower branch of γ (see Fig. 3).

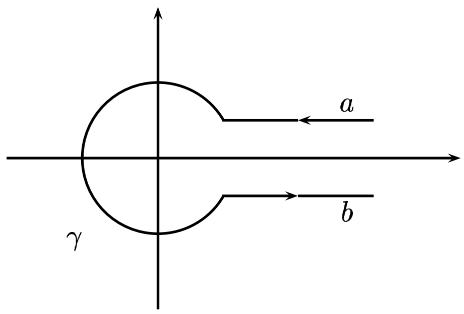

Figure 3. The path γ for the integral representation of the connection coefficients $\mathcal{T}_1$.

Representing the right-hand side of (7.3) as

$$\int_\gamma e^{-2\zeta} \left(P'_{12}(\zeta) - 2P_{12}(\zeta)\right) d\zeta = \int_a^b e^{-2\zeta} \left(P'_{12}(\zeta) - 2P_{12}(\zeta)\right) d\zeta + o\,(1)\,, a \to +\infty,$$

and integrating by parts gives

$$e^{-2\zeta} P_{12}(\zeta)\Big]_a^b + o\,(1)\,. \tag{7.4}$$

Thus

$$\int_\gamma e^{-2\zeta} \frac{1}{6\zeta} P_{22}(\zeta) d\zeta = e^{-2b} P_{12}(b) - e^{-2a} P_{12}(a) + o\,(1)\,, a \to +\infty. \tag{7.5}$$

which can be re-written as

$$e^{-2a}\,(P_{12}(b) - P_{12}(a)) + o\,(1)\,, a \to +\infty. \tag{7.6}$$

It follows from the first co-ordinate of the first vector of (5.17)

$$P_{12}(\zeta e^{2\pi i}) - P_{12}(\zeta) = \mathcal{T}_1 e^{2\zeta} P_{11}(\zeta e^{2\pi i})$$

that

$$P_{12}(b) - P_{12}(a) = \mathcal{T}_1 e^{2a} P_{11}(b). \tag{7.7}$$

Combining (7.5), (7.6) and (7.7), we have

$$\int\limits_{\gamma} e^{-2\zeta}\frac{1}{6\zeta}P_{22}(\zeta)d\zeta = e^{-2a}\mathcal{T}_1 e^{2a}P_{11}(b) + o\,(1)\,, a \to +\infty.$$

Then the integral representation (7.1) follows from the fact that

$$P_{11}(b) \to 1,$$

as $b \to \infty$ along the lower branch of the positive ray. As follows from Proposition 3, this ray, $\arg b = 2\pi$, is inside the extended sector

$$S(1) = \left\{z : -\pi + \frac{2\pi}{3} < \arg z < \pi + \frac{2\pi}{3}, 0 < |z| < \infty\right\},$$

shown by (4.4), which in terms of ζ can be written as

$$S(1) = \left\{z : -\frac{\pi}{2} < \arg\zeta < \frac{5}{2}\pi, 0 < |z| < \infty.\right\}$$

■

The second relation is proved similarly.

It is quite clear how to alter formulae (7.1), (7.2) to be valid for the perturbed Airy equation (4.5).

8. Concluding remarks

To derive the values of $\mathcal{T}_1$ and $\mathcal{T}_2$ from (7.1) and (7.2) we substitute the formal phase amplitudes $\hat{P}_{22}$ and $\hat{P}_{11}$ from (6.5) and (6.2) for P_{22} and P_{11}. Interchanging formally summation and integration yields a convergent series which, for the Airy equation, can be expressed in terms of Gauss' hypergeometric functions at their singular points. Then, the connection coefficients will be the values of the corresponding hypergeometric functions at their finite branch points. These also yield i, which is verified by the earlier result in Theorem 3.

To justify this we are forced to introduce a version of the Fourier (Borel) transforms

$$\mathcal{T}_1(\xi) = \int\limits_{\gamma} e^{-2\zeta\xi}\frac{1}{6\zeta}P_{22}(\zeta)d\zeta \tag{8.1}$$

and

$$\mathcal{T}_2(\xi) = \int\limits_{-\gamma} e^{2\zeta\xi}\frac{1}{6\zeta}P_{11}(\zeta)d\zeta \tag{8.2}$$

and to study their analytical properties.

We obtain $\mathcal{T}_1(\xi)$ using (6.5) and the formal substitution described above

$$\int_\gamma e^{-2\zeta\xi}\frac{1}{6\zeta}P_{22}(\zeta)d\zeta \overset{\text{f.p.s.}}{=\!=} \int_\gamma e^{-2\zeta\xi}\frac{1}{6\zeta}\sum_{k=0}^{\infty}(-1)^k\frac{\left(-\frac{1}{6}\right)_k\left(\frac{1}{6}\right)_k}{2^k k!}\frac{1}{\zeta^k}d\zeta. \tag{8.3}$$

Interchanging summation and integration formally yields

$$\sum_{k=0}^{\infty}(-1)^k\frac{\left(-\frac{1}{6}\right)_k\left(\frac{1}{6}\right)_k}{6\cdot 2^k k!}\int_\gamma e^{-2\zeta\xi}\frac{1}{\zeta^{k+1}}d\zeta, \tag{8.4}$$

which can be written as

$$-\sum_{k=0}^{\infty}\frac{\left(-\frac{1}{6}\right)_k\left(\frac{1}{6}\right)_k\xi^k}{6\cdot k!}\int_\gamma e^{-t}\frac{1}{(-t)^{k+1}}dt. \tag{8.5}$$

Using Hankel's contour integral (see [1, 6.1.4]), yields

$$\frac{\pi i}{3}\sum_{k=0}^{\infty}\frac{\left(-\frac{1}{6}\right)_k\left(\frac{1}{6}\right)_k\xi^k}{k!\,(1)_k}. \tag{8.6}$$

Thus

$$\mathcal{T}_1(\xi) = \frac{\pi i}{3}F\left(-\frac{1}{6},\frac{1}{6},1;\xi\right). \tag{8.7}$$

Using $\hat{P}_{11}(-\zeta) = \hat{P}_{22}(\zeta)$, we obtain

$$\mathcal{T}_2(\xi) = \frac{\pi i}{3}F\left(-\frac{1}{6},\frac{1}{6},1;\xi\right). \tag{8.8}$$

We now use Gauss' formula (see, e.g. [2, p. 66]) to check that

$$\mathcal{T}_1 = \mathcal{T}_2 = \frac{\pi i}{3}F\left(-\frac{1}{6},\frac{1}{6},1;1\right) = i.$$

Of course, using (6.10) and integrating by parts, we can rewrite $\mathcal{T}_1$, $\mathcal{T}_2$ from (7.1), (7.2) and $\mathcal{T}_1(\xi)$, $\mathcal{T}_2(\xi)$ from (8.1), (8.2) in terms of the Airy functions $y_0(z), y_{-1}(z)$.

To obtain (7.1), (7.2) we only used the corresponding relations of (5.17) together with a canonical form of the Airy equation.

To justify the method of evaluation of $\mathcal{T}_1$ and $\mathcal{T}_2$ shown above we require the Stokes structure in its entirety. This will be the subject of a subsequent paper.

Acknowledgment. The authors wish to express their gratitude to David Gillam for his careful reading and valuable suggestions.

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