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The spectrum of Schrödinger operators with quasi-periodic algebro-geometric KdV potentials

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In this announcement we report on a recent characterization of the spectrum of one-dimensional Schrödinger operators $H = -d^2/dx^2 + V$ in $L^2(\mathbb{R}; dx)$ with quasi-periodic complex-valued algebro-geometric potentials V (i.e., potentials V which satisfy one (and hence infinitely many) equation(s) of the stationary Korteweg–de Vries (KdV) hierarchy) associated with non-singular hyperelliptic curves in [1]. It turns out the spectrum of H coincides with the conditional stability set of H and that it can explicitly be described in terms of the mean value of the inverse of the diagonal Green's function of H. As a result, the spectrum of H consists of finitely many simple analytic arcs and one semi-infinite simple analytic arc in the complex plane. Crossings as well as confluences of spectral arcs are possible and discussed as well. These results extend to the $L^p(\mathbb{R}; dx)$-setting for $p \in [1, \infty)$.

Dedicated with great pleasure to Vladimir A. Marchenko on the occasion of his 80th birthday

1. Introduction

It is well-known since the work of Novikov [34], Marchenko [29, 30], Dubrovin [9], Dubrovin, Matveev, and Novikov [10], Flaschka [14], Its and Matveev [22], Lax [28], McKean and van Moerbeke [33] (see also [4, Sects. 3.4, 3.5], [17, p.

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111–112, App. J], [31, Sect. 4.4], [35, Sects. II.6–II.10] and the references therein) that the self-adjoint Schrödinger operator

$$H = -\frac{d^2}{dx^2} + V, \quad \mathrm{dom}(H) = H^{2,2}(\mathbb{R}) \tag{1.1}$$

in $L^2(\mathbb{R}; dx)$ with a real-valued periodic, or more generally, *quasi-periodic* and *real-valued* potential V, that satisfies one (and hence infinitely many) equation(s) of the stationary Korteweg–de Vries (KdV) equations, leads to a finite-gap, or perhaps more appropriately, to a finite-band spectrum $\sigma(H)$ of the form

$$\sigma(H) = \bigcup_{m=0}^{n-1} [E_{2m}, E_{2m+1}] \cup [E_{2n}, \infty). \tag{1.2}$$

It is also well-known, due to work of Serov [39] and Rofe-Beketov [38] in 1960 and 1963, respectively (see also [40]), that if V is *periodic* and *complex-valued* then the spectrum of the non-self-adjoint Schrödinger operator H defined as in (1.1) consists either of infinitely many simple analytic arcs, or else, of a finite number of simple analytic arcs and one semi-infinite simple analytic arc tending to infinity. It seems plausible that the latter case is again connected with (complex-valued) stationary solutions of equations of the KdV hierarchy, but to the best of our knowledge, this has not been studied in the literature. In particular, the next scenario in line, the determination of the spectrum of H in the case of *quasi-periodic* and *complex-valued* solutions of the stationary KdV equation apparently has never been clarified. The latter problem is open since the mid-seventies and it is the purpose of this announcement to provide a comprehensive solution of it.

Dedication. It is with great pleasure that we dedicate this paper to Vladimir A. Marchenko on the occasion of his 80th birthday. His enormous influence on the subject at hand is universally admired. In particular, he has been a most influental source of inspiration for this investigation.

2. The KdV hierarchy, hyperelliptic curves, and the Its–Matveev formula

In this section we briefly review the recursive construction of the KdV hierarchy and associated Lax pairs following [18] and especially [17, Ch. 1]. Moreover, we discuss the class of algebro-geometric solutions of the KdV hierarchy corresponding to the underlying hyperelliptic curve and recall the Its–Matveev formula for such solutions. The material in this preparatory section is known and detailed accounts with proofs can be found, for instance, in [17, Ch. 1].

Throughout this section we suppose the hypothesis

$$V \in C^\infty(\mathbb{R}) \tag{2.1}$$

and consider the one-dimensional Schrödinger differential expression

$$L = -\frac{d^2}{dx^2} + V. \tag{2.2}$$

To construct the KdV hierarchy we need a second differential expression P_{2n+1} of order $2n+1$, $n \in \mathbb{N}_0$, defined recursively as follows.

Define $\{f_\ell\}_{\ell \in \mathbb{N}_0}$ recursively by

$$f_0 = 1, \quad f_{\ell,x} = -(1/4)f_{\ell-1,xxx} + Vf_{\ell-1,x} + (1/2)V_x f_{\ell-1}, \quad \ell \in \mathbb{N}, \tag{2.3}$$

where $\{c_k\}_{k\in\mathbb{N}} \subset \mathbb{C}$ denote integration constants which naturally arise when solving (2.3). The differential expressions P_{2n+1} of order $2n+1$ are then defined by

$$P_{2n+1} = \sum_{\ell=0}^{n} \Big(f_{n-\ell}\frac{d}{dx} - \frac{1}{2} f_{n-\ell,x} \Big) L^\ell, \quad n \in \mathbb{N}_0. \tag{2.4}$$

Using the recursion (2.3), the commutator of P_{2n+1} and L can be explicitly computed and one obtains

$$[P_{2n+1}, L] = 2f_{n+1,x}, \quad n \in \mathbb{N}_0. \tag{2.5}$$

In particular, (L, P_{2n+1}) represents the celebrated *Lax pair* of the KdV hierarchy. Varying $n \in \mathbb{N}_0$, the stationary KdV hierarchy is then defined in terms of the vanishing of the commutator of P_{2n+1} and L in (2.5) by*,

$$-[P_{2n+1}, L] = -2f_{n+1,x}(V) = \text{s-KdV}_n(V) = 0, \quad n \in \mathbb{N}_0. \tag{2.6}$$

By definition, the set of solutions of (2.6), with n ranging in $\mathbb{N}_0$ and c_k in $\mathbb{C}$, $k \in \mathbb{N}$, represents the class of algebro-geometric KdV solutions. At times it will be convenient to abbreviate algebro-geometric stationary KdV solutions V simply as KdV *potentials*.

In the following we will frequently assume that V satisfies the nth stationary KdV equation. By this we mean it satisfies one of the nth stationary KdV equations after a particular choice of integration constants $c_k \in \mathbb{C}$, $k = 1, \dots, n$, $n \in \mathbb{N}$, has been made.

Next, we introduce a polynomial F_n of degree n with respect to the spectral parameter $z \in \mathbb{C}$ by

$$F_n(z,x) = \sum_{\ell=0}^{n} f_{n-\ell}(x) z^\ell. \tag{2.7}$$

* In a slight abuse of notation we will occasionally stress the functional dependence of f_ℓ on V, writing $f_\ell(V)$.

The recursion relation (2.3) and equation (2.6) imply that

$$F_{n,xxx} - 4(V - z)F_{n,x} - 2V_x F_n = 0. \tag{2.8}$$

Multiplying (2.8) by F_n, a subsequent integration with respect to x results in

$$(1/2)F_{n,xx}F_n - (1/4)F_{n,x}^2 - (V - z)F_n^2 = R_{2n+1}, \tag{2.9}$$

where R_{2n+1} is a monic polynomial of degree $2n+1$. We denote its roots by $\{E_m\}_{m=0}^{2n}$, and hence write

$$R_{2n+1}(z) = \prod_{m=0}^{2n}(z - E_m), \quad \{E_m\}_{m=0}^{2n} \subset \mathbb{C}. \tag{2.10}$$

One can show that equation (2.9) leads to an explicit determination of the integration constants $c_1, \dots, c_n$ in

$$\text{s-KdV}_n(V) = -2f_{n+1,x}(V) = 0 \tag{2.11}$$

in terms of the zeros $E_0, \dots, E_{2n}$ of the associated polynomial R_{2n+1} in (2.10). In fact, one can prove

$$c_k = c_k(\underline{E}), \quad k = 1, \dots, n, \tag{2.12}$$

where

$$c_k(\underline{E}) = -\sum_{\substack{j_0,\dots,j_{2n}=0 \\ j_0+\cdots+j_{2n}=k}}^{k} \frac{(2j_0)! \cdots (2j_{2n})!}{2^{2k}(j_0!)^2 \cdots (j_{2n}!)^2(2j_0 - 1)\cdots(2j_{2n}-1)} E_0^{j_0} \cdots E_{2n}^{j_{2n}},$$
$$k = 1, \dots, n. \tag{2.13}$$

R e m a r k 2.1. Suppose $V \in C^{2n+1}(\mathbb{R})$ satisfies the nth stationary KdV equation $\text{s-KdV}_n(V) = -2f_{n+1,x}(V) = 0$ for a given set of integration constants c_k, $k = 1, \dots, n$. Introducing F_n as in (2.7) then yields equation (2.8) and hence (2.9). The latter equation in turn, as shown inductively in [20, Prop. 2.1], yields

$$V \in C^\infty(\mathbb{R}) \text{ and } f_\ell \in C^\infty(\mathbb{R}), \ \ell = 0, \dots, n. \tag{2.14}$$

Thus, without loss of generality, we may assume in the following that solutions of $\text{s-KdV}_n(V) = 0$ satisfy $V \in C^\infty(\mathbb{R})$.

Next, we study the restriction of the differential expression P_{2n+1} to the two-dimensional kernel (i.e., the formal null space in an algebraic sense as opposed to the functional analytic one) of $(L - z)$. More precisely, let

$$\ker(L - z) = \{\psi \colon \mathbb{R} \to \mathbb{C}_\infty \text{ meromorphic} \mid (L - z)\psi = 0\}, \quad z \in \mathbb{C}. \tag{2.15}$$

Then (2.4) implies

$$P_{2n+1}\big|_{\ker(L-z)} = \Big(F_n(z)\frac{d}{dx} - \frac{1}{2}F_{n,x}(z)\Big)\Big|_{\ker(L-z)}. \tag{2.16}$$

We emphasize that the result (2.16) is valid independently of whether or not P_{2n+1} and L commute. However, if one makes the additional assumption that P_{2n+1} and L commute, one can prove that this implies an algebraic relationship between P_{2n+1} and L.

Theorem 2.2. *Fix $n \in \mathbb{N}_0$ and assume that P_{2n+1} and L commute, $[P_{2n+1}, L] = 0$, or equivalently, suppose* s-KdV$_n(V) = -2f_{n+1,x}(V) = 0$. *Then L and P_{2n+1} satisfy an algebraic relationship of the type* (*cf.* (2.10))

$$\begin{aligned} &\mathcal{F}_n(L, -iP_{2n+1}) = -P_{2n+1}^2 - R_{2n+1}(L) = 0, \\ &R_{2n+1}(z) = \prod_{m=0}^{2n}(z - E_m), \quad z \in \mathbb{C}. \end{aligned} \tag{2.17}$$

The expression $\mathcal{F}_n(L, -iP_{2n+1})$ is called the Burchnall–Chaundy polynomial of the pair (L, P_{2n+1}). Equation (2.17) naturally leads to the hyperelliptic curve $\mathcal{K}_n$ of (arithmetic) genus $n \in \mathbb{N}_0$ (possibly with a singular affine part), where

$$\begin{aligned} &\mathcal{K}_n \colon \mathcal{F}_n(z, y) = y^2 - R_{2n+1}(z) = 0, \\ &R_{2n+1}(z) = \prod_{m=0}^{2n}(z - E_m), \quad \{E_m\}_{m=0}^{2n} \subset \mathbb{C}. \end{aligned} \tag{2.18}$$

The curve $\mathcal{K}_n$ is compactified by joining the point P_∞ but for notational simplicity the compactification is also denoted by $\mathcal{K}_n$. Points P on $\mathcal{K}_n \backslash \{P_\infty\}$ are represented as pairs $P = (z, y)$, where $y(\cdot)$ is the meromorphic function on $\mathcal{K}_n$ satisfying $\mathcal{F}_n(z, y) = 0$. The complex structure on $\mathcal{K}_n$ is then defined in the usual way, see Appendix B in [17]. Hence, $\mathcal{K}_n$ becomes a two-sheeted hyperelliptic Riemann surface of (arithmetic) genus $n \in \mathbb{N}_0$ (possibly with a singular affine part) in a standard manner.

In the special case where the affine part of $\mathcal{K}_n$ is nonsingular, that is,

$$E_m \neq E_{m'} \text{ for } m \neq m', \quad m, m' = 0, 1, \dots, 2n, \tag{2.19}$$

we introduce an appropriate set of (nonintersecting) cuts $\mathcal{C}_j$ joining $E_{m(j)}$ and $E_{m'(j)}$, $j = 1, \dots, n$, and $\mathcal{C}_{n+1}$, joining E_{2n} and ∞ and we denote

$$\mathcal{C} = \bigcup_{j=1}^{n+1} \mathcal{C}_j, \quad \mathcal{C}_j \cap \mathcal{C}_k = \emptyset, \quad j \neq k. \tag{2.20}$$

For subsequent purposes, the corresponding cut plane Π is defined by $\Pi = \mathbb{C}\backslash\mathcal{C}$ and one introduces the holomorphic function

$$R_{2n+1}(\cdot)^{1/2} \colon \Pi \to \mathbb{C}, \quad z \mapsto \left(\prod_{m=0}^{2n}(z - E_m)\right)^{1/2} \tag{2.21}$$

on Π with an appropriate choice of the square root branch in (2.21).

We also emphasize that by fixing the curve $\mathcal{K}_n$ (i.e., by fixing $E_0, \dots, E_{2n}$), the integration constants $c_1, \dots, c_n$ in $f_{n+1,x}$ (and hence in the corresponding stationary KdV_n equation) are uniquely determined as is clear from (2.12) and (2.13), which establish the integration constants c_k as symmetric functions of $E_0, \dots, E_{2n}$.

For notational simplicity we will usually tacitly assume that $n \in \mathbb{N}$. The trivial case $n = 0$ which leads to $V(x) = E_0$ is of no interest to us in this paper.

In the following, the zeros of the polynomial $F_n(\cdot, x)$ (cf. (2.7)) will play a special role. We denote them by $\{\mu_j(x)\}_{j=1}^n$ and hence write

$$F_n(z, x) = \prod_{j=1}^{n}[z - \mu_j(x)]. \tag{2.22}$$

From (2.9) we see that

$$R_{2n+1} + (1/4)F_{n,x}^2 = F_n H_{n+1}, \tag{2.23}$$

where

$$H_{n+1}(z, x) = (1/2)F_{n,xx}(z, x) + (z - V(x))F_n(z, x) \tag{2.24}$$

is a monic polynomial of degree $n + 1$. We introduce the corresponding roots $\{\nu_\ell(x)\}_{\ell=0}^n$ of $H_{n+1}(\cdot, x)$ by

$$H_{n+1}(z, x) = \prod_{\ell=0}^{n}[z - \nu_\ell(x)]. \tag{2.25}$$

The next step is crucial; it permits us to "lift" the zeros μ_j and ν_ℓ of F_n and H_{n+1} from $\mathbb{C}$ to the curve $\mathcal{K}_n$. From (2.23) one infers

$$R_{2n+1}(z) + (1/4)F_{n,x}(z)^2 = 0, \quad z \in \{\mu_j, \nu_\ell\}_{j=1,\dots,n,\ell=0,\dots,n}. \tag{2.26}$$

We now introduce $\{\hat{\mu}_j(x)\}_{j=1,\dots,n} \subset \mathcal{K}_n$ and $\{\hat{\nu}_\ell(x)\}_{\ell=0,\dots,n} \subset \mathcal{K}_n$ by

$$\hat{\mu}_j(x) = (\mu_j(x), -(i/2)F_{n,x}(\mu_j(x), x)), \quad j = 1, \dots, n, \ x \in \mathbb{R} \tag{2.27}$$

and

$$\hat{\nu}_\ell(x) = (\nu_\ell(x), (i/2)F_{n,x}(\nu_\ell(x), x)), \quad \ell = 0, \dots, n,\, x \in \mathbb{R}. \tag{2.28}$$

Due to the $C^\infty(\mathbb{R})$ assumption (2.1) on V, $F_n(z, \cdot) \in C^\infty(\mathbb{R})$ by (2.3) and (2.7), and hence also $H_{n+1}(z, \cdot) \in C^\infty(\mathbb{R})$ by (2.24). Thus, one concludes

$$\mu_j, \nu_\ell \in C(\mathbb{R}),\ \ j = 1, \dots, n,\ \ell = 0, \dots, n, \tag{2.29}$$

taking multiplicities (and appropriate renumbering) of the zeros of F_n and H_{n+1} into account. (Away from collisions of zeros, μ_j and ν_ℓ are of course C^∞.)

Next, we define the fundamental meromorphic function $\phi(\cdot, x)$ on $\mathcal{K}_n$,

$$\begin{aligned} \phi(P, x) &= \frac{iy + (1/2)F_{n,x}(z, x)}{F_n(z, x)} && (2.30) \\ &= \frac{-H_{n+1}(z, x)}{iy - (1/2)F_{n,x}(z, x)}, && (2.31) \\ & P = (z, y) \in \mathcal{K}_n,\ x \in \mathbb{R} \end{aligned}$$

with divisor $(\phi(\cdot, x))$ of $\phi(\cdot, x)$ given by

$$(\phi(\cdot, x)) = \mathcal{D}_{\hat{\nu}_0(x)\underline{\hat{\nu}}(x)} - \mathcal{D}_{P_\infty \underline{\hat{\mu}}(x)}, \tag{2.32}$$

using (2.22), (2.25), and (2.29). Here we abbreviated

$$\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\},\ \underline{\hat{\nu}} = \{\hat{\nu}_1, \dots, \hat{\nu}_n\} \in \operatorname{Sym}^n(\mathcal{K}_n) \tag{2.33}$$

and used the following (additive) notation for divisors $\mathcal{D}\colon \mathcal{K}_n \to \mathbb{Z}$ on $\mathcal{K}_n$:

$$\begin{aligned} &\mathcal{D}_{Q_0\underline{Q}} = \mathcal{D}_{Q_0} + \mathcal{D}_{\underline{Q}}, \quad \mathcal{D}_{\underline{Q}} = \mathcal{D}_{Q_1} + \cdots + \mathcal{D}_{Q_m}, && (2.34) \\ &\underline{Q} = \{Q_1, \dots, Q_m\} \in \operatorname{Sym}^m \mathcal{K}_n, \quad Q_0 \in \mathcal{K}_n,\ m \in \mathbb{N}, \end{aligned}$$

where for any $Q \in \mathcal{K}_n$,

$$\mathcal{D}_Q\colon \mathcal{K}_n \to \mathbb{N}_0, \quad P \mapsto \mathcal{D}_Q(P) = \begin{cases} 1 & \text{for } P = Q, \\ 0 & \text{for } P \in \mathcal{K}_n \backslash \{Q\}, \end{cases} \tag{2.35}$$

and $\operatorname{Sym}^m \mathcal{K}_n$ denotes the mth symmetric product of $\mathcal{K}_n$. In particular, $\operatorname{Sym}^m \mathcal{K}_n$ can be identified with the set of nonnegative divisors $0 \le \mathcal{D} \in \operatorname{Div}(\mathcal{K}_n)$ of degree $m \in \mathbb{N}$.

The stationary Baker–Akhiezer function $\psi(\cdot, x, x_0)$ on $\mathcal{K}_n \backslash \{P_\infty\}$ is then defined in terms of $\phi(\cdot, x)$ by

$$\psi(P, x, x_0) = \exp\left(\int_{x_0}^{x} dx'\, \phi(P, x')\right), \quad P \in \mathcal{K}_n \backslash \{P_\infty\},\ (x, x_0) \in \mathbb{R}^2. \tag{2.36}$$

From this point on we assume that the affine part of $\mathcal{K}_n$ is nonsingular (cf. (2.19)).

Since nonspecial divisors play a fundamental role in this context we also recall the following fact.

Lemma 2.3. *Suppose that the affine part of $\mathcal{K}_n$ is nonsingular and assume that $V \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R}; dx)$ satisfies the nth stationary KdV equation* (2.6)*. Let $\mathcal{D}_{\underline{\hat{\mu}}}$, $\underline{\hat{\mu}} = (\hat{\mu}_1, \dots, \hat{\mu}_n)$ be the Dirichlet divisor of degree n associated with V defined according to* (2.27)*, that is,*

$$\hat{\mu}_j(x) = (\mu_j(x), -(i/2)F_{n,x}(\mu_j(x), x)), \quad j = 1, \dots, n, \ x \in \mathbb{R}. \tag{2.37}$$

Then $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ is nonspecial for all $x \in \mathbb{R}$. Moreover, there exists a constant $C > 0$ such that

$$|\mu_j(x)| \le C, \quad j = 1, \dots, n, \ x \in \mathbb{R}. \tag{2.38}$$

R e m a r k 2.4. Assume that $V \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R}; dx)$ satisfies the nth stationary KdV equation (2.6). We recall that $f_\ell \in C^\infty(\mathbb{R})$, $\ell \in \mathbb{N}_0$, by (2.14) since f_ℓ are differential polynomials in V. Moreover, we note that (2.38) implies that $f_\ell \in L^\infty(\mathbb{R}; dx)$, $\ell = 0, \dots, n$, employing the fact that f_ℓ, $\ell = 0, \dots, n$, are elementary symmetric functions of $\mu_1, \dots, \mu_n$ (cf. (2.7) and (2.22)). Since $f_{n+1,x} = 0$, one can use the recursion relation (2.3) to reduce f_k for $k \ge n+2$ to a linear combination of $f_1, \dots, f_n$. Thus,

$$f_\ell \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R}; dx), \quad \ell \in \mathbb{N}_0. \tag{2.39}$$

Using the fact that for fixed $1 \le p \le \infty$,

$$h, h^{(k)} \in L^p(\mathbb{R}; dx) \ \text{ imply } \ h^{(\ell)} \in L^p(\mathbb{R}; dx), \ \ \ell = 1, \dots, k-1 \tag{2.40}$$

(cf., e.g., [3, p. 168–170]), one then infers

$$V^{(\ell)} \in L^\infty(\mathbb{R}; dx), \quad \ell \in \mathbb{N}_0, \tag{2.41}$$

applying (2.40) with $p = \infty$.

We continue with the theta function representation for ψ and V but first we need to introduce some notation.

Using the local chart near P_∞, one verifies that dz/y is a holomorphic differential on $\mathcal{K}_n$ with zeros of order $2(n-1)$ at P_∞ and hence

$$\eta_j = \frac{z^{j-1}dz}{y}, \quad j = 1, \dots, n, \tag{2.42}$$

form a basis for the space of holomorphic differentials on $\mathcal{K}_n$. Normalized differentials ω_j for $j = 1, \ldots, n$, are then of the form

$$\omega_j = \sum_{\ell=1}^{n} c_j(\ell)\eta_\ell, \quad \int_{a_k} \omega_j = \delta_{j,k}, \quad j,k = 1, \ldots, n. \tag{2.43}$$

Next, define the matrix $\tau = \left(\tau_{j,\ell}\right)_{j,\ell=1}^{n}$ by

$$\tau_{j,\ell} = \int_{b_j} \omega_\ell, \quad j,\ell = 1, \ldots, n. \tag{2.44}$$

Associated with τ one introduces the period lattice

$$L_n = \{\underline{z} \in \mathbb{C}^n \mid \underline{z} = \underline{m} + \underline{n}\tau, \ \underline{m}, \underline{n} \in \mathbb{Z}^n\} \tag{2.45}$$

and the Riemann theta function associated with $\mathcal{K}_n$ and the given homology basis $\{a_j, b_j\}_{j=1,\ldots,n}$,

$$\theta(\underline{z}) = \sum_{\underline{n} \in \mathbb{Z}^n} \exp\left(2\pi i(\underline{n}, \underline{z}) + \pi i(\underline{n}, \underline{n}\tau)\right), \quad \underline{z} \in \mathbb{C}^n, \tag{2.46}$$

where $(\underline{u}, \underline{v}) = \overline{\underline{u}}\,\underline{v}^\top = \sum_{j=1}^{n} \overline{u}_j v_j$ denotes the scalar product in $\mathbb{C}^n$. Choosing a base point $Q_0 \in \mathcal{K}_n \backslash \{P_\infty\}$, one denotes by $J(\mathcal{K}_n) = \mathbb{C}^n / L_n$ the Jacobi variety of $\mathcal{K}_n$, and defines the Abel map $\underline{A}_{Q_0}$ by

$$\underline{A}_{Q_0} \colon \mathcal{K}_n \to J(\mathcal{K}_n), \quad \underline{A}_{Q_0}(P) = \left(\int_{Q_0}^{P} \omega_1, \ldots, \int_{Q_0}^{P} \omega_n \right) \pmod{L_n}, \quad P \in \mathcal{K}_n. \tag{2.47}$$

Similarly, we introduce

$$\underline{\alpha}_{Q_0} \colon \operatorname{Div}(\mathcal{K}_n) \to J(\mathcal{K}_n), \quad \mathcal{D} \mapsto \underline{\alpha}_{Q_0}(\mathcal{D}) = \sum_{P \in \mathcal{K}_n} \mathcal{D}(P) \underline{A}_{Q_0}(P), \tag{2.48}$$

where $\operatorname{Div}(\mathcal{K}_n)$ denotes the set of divisors on $\mathcal{K}_n$. The vector of Riemann constants, $\underline{\Xi}_{Q_0} = (\Xi_{Q_{0,1}}, \ldots, \Xi_{Q_{0,n}})$, is then given by

$$\Xi_{Q_{0,j}} = \frac{1}{2}(1 + \tau_{j,j}) - \sum_{\substack{\ell=1 \\ \ell \neq j}}^{n} \int_{a_\ell} \omega_\ell(P) \int_{Q_0}^{P} \omega_j, \quad j = 1, \ldots, n. \tag{2.49}$$

Next, let $\omega^{(2)}_{P_\infty,0}$ denote the normalized differential of the second kind defined by

$$\omega^{(2)}_{P_\infty,0} = -\frac{1}{2y}\prod_{j=1}^{n}(z-\lambda_j)dz \underset{\zeta\to 0}{=} \big(\zeta^{-2}+O(1)\big)d\zeta \text{ as } P\to P_\infty, \tag{2.50}$$

$$\zeta = \sigma/z^{1/2},\ \sigma\in\{1,-1\},$$

where the constants $\lambda_j \in \mathbb{C}$, $j=1,\dots,n$, are determined by employing the normalization

$$\int_{a_j} \omega^{(2)}_{P_\infty,0} = 0, \quad j=1,\dots,n. \tag{2.51}$$

One then infers

$$\int_{Q_0}^{P} \omega^{(2)}_{P_\infty,0} \underset{\zeta\to 0}{=} -\zeta^{-1} + e^{(2)}_0(Q_0) + O(\zeta) \text{ as } P\to P_\infty \tag{2.52}$$

for some constant $e^{(2)}_0(Q_0)\in\mathbb{C}$. The vector of b-periods of $\omega^{(2)}_{P_\infty,0}/(2\pi i)$ is denoted by

$$\underline{U}^{(2)}_0 = (U^{(2)}_{0,1},\dots,U^{(2)}_{0,n}), \quad U^{(2)}_{0,j} = \frac{1}{2\pi i}\int_{b_j}\omega^{(2)}_{P_\infty,0},\ j=1,\dots,n. \tag{2.53}$$

In the following it will be convenient to introduce the abbreviation

$$\underline{z}(P,\underline{Q}) = \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}}),\ P\in\mathcal{K}_n,\ \underline{Q}=\{Q_1,\dots,Q_n\}\in \mathrm{Sym}^n(\mathcal{K}_n). \tag{2.54}$$

We note that $\underline{z}(\cdot,\underline{Q})$ is independent of the choice of base point Q_0.

Theorem 2.5. *Suppose that $V\in C^\infty(\mathbb{R})\cap L^\infty(\mathbb{R};dx)$ satisfies the nth stationary KdV equation (2.6) on $\mathbb{R}$. In addition, assume the affine part of $\mathcal{K}_n$ to be nonsingular and let $P\in\mathcal{K}_n\backslash\{P_\infty\}$ and $x,x_0\in\mathbb{R}$. Then $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(x)}$ are nonspecial for $x\in\mathbb{R}$. Moreover,*

$$\psi(P,x,x_0) = \frac{\theta(\underline{z}(P_\infty,\underline{\hat{\mu}}(x_0)))\theta(\underline{z}(P,\underline{\hat{\mu}}(x)))}{\theta(\underline{z}(P_\infty,\underline{\hat{\mu}}(x)))\theta(\underline{z}(P,\underline{\hat{\mu}}(x_0)))} \times\exp\bigg[-i(x-x_0)\bigg(\int_{Q_0}^{P}\omega^{(2)}_{P_\infty,0} - e^{(2)}_0(Q_0)\bigg)\bigg], \tag{2.55}$$

with the linearizing property of the Abel map,

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)}) = \Big(\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0)}) + i\underline{U}^{(2)}_0(x-x_0)\Big) \pmod{L_n}. \tag{2.56}$$

The Its–Matveev formula for V reads

$$V(x) = E_0 + \sum_{j=1}^{n}(E_{2j-1} + E_{2j} - 2\lambda_j) - 2\partial_x^2 \ln\big(\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_\infty) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)}))\big). \tag{2.57}$$

Combining (2.56) and (2.57) shows the remarkable linearity of the theta function with respect to x in the Its–Matveev formula for V. In fact, one can rewrite (2.57) as

$$V(x) = \Lambda_0 - 2\partial_x^2 \ln(\theta(\underline{A} + \underline{B}x)), \tag{2.58}$$

where

$$\underline{A} = \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_\infty) - i\underline{U}_0^{(2)}x_0 + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0)}), \tag{2.59}$$

$$\underline{B} = i\underline{U}_0^{(2)}, \tag{2.60}$$

$$\Lambda_0 = E_0 + \sum_{j=1}^{n}(E_{2j-1} + E_{2j} - 2\lambda_j). \tag{2.61}$$

Hence the constants $\Lambda_0 \in \mathbb{C}$ and $\underline{B} \in \mathbb{C}^n$ are uniquely determined by $\mathcal{K}_n$ (and its homology basis), and the constant $\underline{A} \in \mathbb{C}^n$ is in one-to-one correspondence with the Dirichlet data $\underline{\hat{\mu}}(x_0) = (\hat{\mu}_1(x_0), \dots, \hat{\mu}_n(x_0)) \in \mathrm{Sym}^n(\mathcal{K}_n)$ at the point x_0.

Theorem 2.6. *Assume V in* (2.57) (*or* (2.58)) *to be quasi-periodic* (*cf.* (3.10) *and* (3.11)). *Then there exists a homology basis $\{\tilde{a}_j, \tilde{b}_j\}_{j=1}^n$ on $\mathcal{K}_n$ such that $\underline{\widetilde{B}} = i\underline{\widetilde{U}}_0^{(2)}$ satisfies the constraint*

$$\underline{\widetilde{B}} = i\underline{\widetilde{U}}_0^{(2)} \in \mathbb{R}^n. \tag{2.62}$$

This is a key result. It follows from a careful study of (vector) periods of meromorphic functions $F\colon \mathbb{C}^n \to \mathbb{C} \cup \{\infty\}$ (i.e., the ratio of two entire functions of n variables, $n \in \mathbb{N}$) to be found, for instance in [32, Ch. 2], and its proper application to the Its–Matveev formula (2.58).

3. The diagonal Green's function of H

In this section we focus on the diagonal Green's function of H and derive a variety of results to be used in our principal Section 4.

We denote by

$$W(f,g)(x) = f(x)g_x(x) - f_x(x)g(x) \quad \text{for a.e. } x \in \mathbb{R} \tag{3.1}$$

the Wronskian of $f, g \in AC_{\mathrm{loc}}(\mathbb{R})$ (with $AC_{\mathrm{loc}}(\mathbb{R})$ the set of locally absolutely continuous functions on $\mathbb{R}$).

Assume $q \in L^1_{\mathrm{loc}}(\mathbb{R})$, define $\tau = -d^2/dx^2 + q$, and let $u_j(z)$, $j = 1, 2$ be two (not necessarily distinct) distributional solutions of $\tau u = zu$ for some $z \in \mathbb{C}$. Introducing

$$\mathfrak{g}(z,x) = u_1(z,x)u_2(z,x)/W(u_1(z), u_2(z)), \quad z \in \mathbb{C}, \ x \in \mathbb{R}, \tag{3.2}$$

one can proof the following result.

Lemma 3.1. *Assume that $q \in L^1_{\mathrm{loc}}(\mathbb{R})$ and $(z,x) \in \mathbb{C} \times \mathbb{R}$. Then,*

$$2\mathfrak{g}_{xx}\mathfrak{g} - \mathfrak{g}_x^2 - 4(q-z)\mathfrak{g}^2 = -1, \tag{3.3}$$

$$-\left(\mathfrak{g}^{-1}\right)_z = 2\mathfrak{g} - \mathfrak{g}_{xxz} + \left[\mathfrak{g}^{-1}\mathfrak{g}_x\mathfrak{g}_z\right]_x. \tag{3.4}$$

If in addition $q_x \in L^1_{\mathrm{loc}}(\mathbb{R})$, then

$$\mathfrak{g}_{xxx} - 4(q-z)\mathfrak{g}_x - 2q_x\mathfrak{g} = 0. \tag{3.5}$$

Equation (3.4) is known and can be found, for instance, in [15].

Next, we turn to the analog of $\mathfrak{g}$ in connection with the algebro-geometric potential V in (2.57). Introducing

$$g(P,x) = \frac{\psi(P,x,x_0)\psi(P^*,x,x_0)}{W(\psi(P,\cdot,x_0),\psi(P^*,\cdot,x_0))}, \quad P \in \mathcal{K}_n \backslash \{P_\infty\}, \ x, x_0 \in \mathbb{R}, \tag{3.6}$$

where $P = (z,y)$, $P^* = (z,-y)$, one can show that

$$g(P,x) = \frac{iF_n(z,x)}{2y}, \quad P = (z,y) \in \mathcal{K}_n \backslash \{P_\infty\}, \ x \in \mathbb{R}. \tag{3.7}$$

Together with $g(P,x)$ we also introduce its two branches $g_\pm(z,x)$ defined on the upper and lower sheets $\Pi_\pm$ of $\mathcal{K}_n$

$$g_\pm(z,x) = \pm\frac{iF_n(z,x)}{2R_{2n+1}(z)^{1/2}}, \quad z \in \Pi, \ x \in \mathbb{R}, \tag{3.8}$$

with $\Pi = \mathbb{C}\backslash\mathcal{C}$ the cut plane introduced after (2.20). A comparison of (3.2), (3.6)–(3.8), then shows that $g_\pm(z,\cdot)$ satisfy (3.3)–(3.5).

For convenience we will subsequently focus on g_+ whenever possible and then use the simplified notation

$$g(z,x) = g_+(z,x), \quad z \in \Pi, \ x \in \mathbb{R}. \tag{3.9}$$

Next, we assume that V is quasi-periodic and compute the mean value of $g(z,\cdot)^{-1}$ using (3.4). Before embarking on this task we briefly review a few properties of quasi-periodic functions.

We denote by $CP(\mathbb{R})$ and $QP(\mathbb{R})$, the sets of continuous periodic and quasi-periodic functions on $\mathbb{R}$, respectively. In particular, f is called quasi-periodic with fundamental periods $(\Omega_1,\dots,\Omega_N)\in(0,\infty)^N$ if the frequencies $2\pi/\Omega_1,\dots,2\pi/\Omega_N$ are linearly independent over $\mathbb{Q}$ and if there exists a continuous function $F\in C(\mathbb{R}^N)$, periodic of period 1 in each of its arguments

$$F(x_1,\dots,x_j+1,\dots,x_N)=F(x_1,\dots,x_N),\quad x_j\in\mathbb{R},\ j=1,\dots,N, \tag{3.10}$$

such that

$$f(x)=F(\Omega_1^{-1}x,\dots,\Omega_N^{-1}x),\quad x\in\mathbb{R}. \tag{3.11}$$

For any quasi-periodic (in fact, Bohr (uniformly) almost periodic) function f, the mean value $\langle f\rangle$ of f, defined by

$$\langle f\rangle=\lim_{R\to\infty}\frac{1}{2R}\int_{x_0-R}^{x_0+R}dx\,f(x), \tag{3.12}$$

exists and is independent of $x_0\in\mathbb{R}$.

For the rest of this section and the next it will be convenient to introduce the following hypothesis:

Hypothesis 3.2. Assume the affine part of $\mathcal{K}_n$ to be nonsingular. Moreover, suppose that $V\in C^\infty(\mathbb{R})\cap QP(\mathbb{R})$ satisfies the nth stationary KdV equation (2.6) on $\mathbb{R}$.

Next, we note the following result.

Lemma 3.3. *Assume Hypothesis 3.2. Then $V^{(k)}$, $k\in\mathbb{N}$, and f_ℓ, $\ell\in\mathbb{N}$, and hence all x and z-derivatives of $F_n(z,\cdot)$, $z\in\mathbb{C}$, and $g(z,\cdot)$, $z\in\Pi$, are quasi-periodic. Moreover, taking limits to points on $\mathcal{C}$, the last result extends to either side of the cuts in the set $\mathcal{C}\backslash\{E_m\}_{m=0}^{2n}$ (cf. (2.20)) by continuity with respect to z.*

For future purposes we introduce the set

$$\Pi_C=\Pi\Big\backslash\Big\{\{z\in\mathbb{C}\,|\,|z|\le C+1\}\cup\{z\in\mathbb{C}\,|\,\mathrm{Re}(z)\ge\min_{m=0,\dots,2n}[\mathrm{Re}(E_m)]-1,$$
$$\min_{m=0,\dots,2n}[\mathrm{Im}(E_m)]-1\le\mathrm{Im}(z)\le\max_{m=0,\dots,2n}[\mathrm{Im}(E_m)]+1\}\Big\}, \tag{3.13}$$

where $C>0$ is the constant in (2.38). Moreover, without loss of generality, we may assume Π_C contains no cuts, that is,

$$\Pi_C\cap\mathcal{C}=\emptyset. \tag{3.14}$$

Lemma 3.4. *Assume Hypothesis 3.2 and let $z, z_0 \in \Pi$. Then*

$$\langle g(z,\cdot)^{-1}\rangle = -2\int_{z_0}^{z} dz' \, \langle g(z',\cdot)\rangle + \langle g(z_0,\cdot)^{-1}\rangle, \tag{3.15}$$

where the path connecting z_0 and z is assumed to lie in the cut plane Π. Moreover, by taking limits to points on $\mathcal{C}$ in (3.15)*, the result* (3.15) *extends to either side of the cuts in the set $\mathcal{C}$ by continuity with respect to z.*

R e m a r k 3.5. For $z \in \Pi_C$, $g(z,\cdot)^{-1}$ is quasi-periodic and hence $\langle g(z,\cdot)^{-1}\rangle$ is well-defined. If one analytically continues $g(z,x)$ with respect to z, $g(z,x)$ will acquire zeros for some $x \in \mathbb{R}$ and hence $g(z,\cdot)^{-1} \notin QP(\mathbb{R})$. Nevertheless, as shown by the right-hand side of (3.15), $\langle g(z,\cdot)^{-1}\rangle$ admits an analytic continuation in z from Π_C to all of Π, and from now on, $\langle g(z,\cdot)^{-1}\rangle$, $z \in \Pi$, always denotes that analytic continuation (cf. also (3.17)).

Next, invoking the Baker–Akhiezer function $\psi(P,x,x_0)$, permits one to analyze the expression $\langle g(z,\cdot)^{-1}\rangle$ in more detail:

Theorem 3.6. *Assume Hypothesis 3.2, let $P = (z,y) \in \Pi_\pm$, and $x, x_0 \in \mathbb{R}$. Moreover, select a homology basis $\{\tilde a_j, \tilde b_j\}_{j=1}^n$ on $\mathcal{K}_n$ such that $\underline{\widetilde B} = i\underline{\widetilde U}_0^{(2)}$, with $\underline{\widetilde U}_0^{(2)}$ the vector of $\tilde b$-periods of the normalized differential of the second kind, $\widetilde\omega_{P_\infty,0}^{(2)}$, satisfies the constraint*

$$\underline{\widetilde B} = i\underline{\widetilde U}_0^{(2)} \in \mathbb{R}^n \tag{3.16}$$

(*cf. Theorem 2.6*)*. Then,*

$$\mathrm{Re}\big(\langle g(P,\cdot)^{-1}\rangle\big) = -2\mathrm{Im}\big(y\langle F_n(z,\cdot)^{-1}\rangle\big) = 2\mathrm{Im}\bigg(\int_{Q_0}^{P} \widetilde\omega_{P_\infty,0}^{(2)} - \tilde e_0^{(2)}(Q_0)\bigg). \tag{3.17}$$

4. Spectra of Schrödinger operators with quasi-periodic algebro-geometric KdV potentials

In this section we describe the connection between the algebro-geometric formalism of Section 2 and the spectral theoretic description of Schrödinger operators H in $L^2(\mathbb{R}; dx)$ with quasi-periodic algebro-geometric KdV potentials. In particular, we introduce the conditional stability set of H and state the principal result of [1], the characterization of the spectrum of H. We conclude with a qualitative description of the spectrum of H in terms of analytic spectral arcs.

Suppose that $V \in C^\infty(\mathbb{R}) \cap QP(\mathbb{R})$ satisfies the nth stationary KdV equation (2.6) on $\mathbb{R}$. The corresponding Schrödinger operator H in $L^2(\mathbb{R}; dx)$ is then introduced by

$$H = -\frac{d^2}{dx^2} + V, \quad \operatorname{dom}(H) = H^{2,2}(\mathbb{R}). \tag{4.1}$$

Thus, H is a densely defined closed operator in $L^2(\mathbb{R}; dx)$.

Assuming Hypothesis 3.2 we now introduce the set $\Sigma \subset \mathbb{C}$ by

$$\Sigma = \{\lambda \in \mathbb{C} \,|\, \operatorname{Re}(\langle g(\lambda, \cdot)^{-1} \rangle) = 0\}. \tag{4.2}$$

Below we will show that Σ plays the role of the conditional stability set of H, familiar from the spectral theory of one-dimensional periodic Schrödinger operators (cf. [12, Sect. 5.3], [38, 41, 42]).

Lemma 4.1. *Assume Hypothesis 3.2. Then Σ coincides with the conditional stability set of H, that is,*

$$\begin{aligned}\Sigma = \{\lambda \in \mathbb{C} \,|\, &\text{there exists at least one bounded distributional solution} \\ &0 \neq \psi \in L^\infty(\mathbb{R}; dx) \text{ of } H\psi = \lambda\psi.\}\end{aligned} \tag{4.3}$$

This is an elementary consequence of (2.55) (in the special homology basis discussed in Theorem 3.6) and (3.17).

R e m a r k 4.2. At first sight our *a priori* choice of cuts $\mathcal{C}$ for $R_{2n+1}(\cdot)^{1/2}$ might seem unnatural as they completely ignore the actual spectrum of H. However, the spectrum of H is not known from the outset, and in the case of complex-valued periodic potentials, spectral arcs of H may actually cross each other (cf. [19, 36], and Theorem 4.6 (iv)) which renders them unsuitable for cuts of $R_{2n+1}(\cdot)^{1/2}$.

Before we state our first principal result on the spectrum of H, we find it convenient to recall a number of basic facts in connection with the spectral theory of non-self-adjoint operators (we refer to [13, Chs. I, III, IX], [21, Sects. 1, 21–23], [23, Sects. IV.5.6, V.3.2], and [37, p. 178–179] for more details). Let S be a densely defined closed operator in a complex separable Hilbert space $\mathcal{H}$. We denote by $\ker(T)$ the kernel (null space) of a linear operator T in $\mathcal{H}$. Moreover, we introduce the following abbreviations associated with S: the spectrum, $\sigma(S)$, the point spectrum (i.e., the set of eigenvalues), $\sigma_{\mathrm{p}}(S)$, the continuous spectrum, $\sigma_{\mathrm{c}}(S)$, the residual spectrum, $\sigma_{\mathrm{r}}(S)$, the approximate point spectrum, $\sigma_{\mathrm{ap}}(S)$, and the numerical range $\Theta(S)$. Moreover, two kinds of essential spectra, $\sigma_{\mathrm{e}}(S)$, and $\widetilde{\sigma}_{\mathrm{e}}(S)$, and the sets $\Delta(S)$ and $\widetilde{\Delta}(S)$ are defined as follows:

$$\Delta(S) = \{z \in \mathbb{C} \,|\, \dim(\ker(S - zI)) < \infty \text{ and } \operatorname{ran}(S - zI) \text{ is closed}\}, \tag{4.4}$$

$$\sigma_{\mathrm{e}}(S) = \mathbb{C} \backslash \Delta(S), \tag{4.5}$$

$$\widetilde{\Delta}(S) = \{z \in \mathbb{C} \,|\, \dim(\ker(S - zI)) < \infty \text{ or } \dim(\ker(S^* - \overline{z}I)) < \infty\}, \tag{4.6}$$

$$\widetilde{\sigma}_{\mathrm{e}}(S) = \mathbb{C} \backslash \widetilde{\Delta}(S), \tag{4.7}$$

We start with the following elementary result.

Lemma 4.3. *Let H be defined as in* (4.1). *Then,*

$$\sigma_{\mathrm{e}}(H) = \widetilde{\sigma}_{\mathrm{e}}(H) \subseteq \overline{\Theta(H)}. \tag{4.8}$$

This follows from the fact that $\dim(\ker(H - zI)) \leq 2$, $\dim(\ker(H^* - \overline{z}I)) \leq 2$, equations (4.4)–(4.7), and [23, p. 269].

Theorem 4.4. *Assume Hypothesis 3.2. Then the point spectrum and residual spectrum of H are empty and hence the spectrum of H is purely continuous,*

$$\sigma_{\mathrm{p}}(H) = \sigma_{\mathrm{r}}(H) = \emptyset, \tag{4.9}$$

$$\sigma(H) = \sigma_{\mathrm{c}}(H) = \sigma_{\mathrm{e}}(H) = \sigma_{\mathrm{ap}}(H). \tag{4.10}$$

The first part of (4.9) follows from (2.55) and some Wronski determinant considerations, and the part on $\sigma_r(H)$ follows from the fact that the point spectrum of H^* is also empty by the same arguments. This proves that the spectrum of H is purely continuous. The remaining equalities in (4.10) then are clear.

The following result is a fundamental one:

Theorem 4.5. *Assume Hypothesis 3.2. Then the spectrum of H coincides with Σ and hence equals the conditional stability set of H,*

$$\sigma(H) = \big\{\lambda \in \mathbb{C} \,\big|\, \mathrm{Re}\big(\langle g(\lambda, \cdot)^{-1}\rangle\big) = 0\big\} \tag{4.11}$$

$$= \{\lambda \in \mathbb{C} \,|\, \textit{there exists at least one bounded distributional solution } 0 \neq \psi \in L^\infty(\mathbb{R}; dx) \textit{ of } H\psi = \lambda\psi\}. \tag{4.12}$$

In particular,

$$\{E_m\}_{m=0}^{2n} \subset \sigma(H), \tag{4.13}$$

and $\sigma(H)$ contains no isolated points.

The proof of

$$\sigma(H) \subseteq \Sigma \tag{4.14}$$

is obtained by adapting a method due to Chisholm and Everitt [8]. The proof of

$$\sigma(H) \supseteq \Sigma \tag{4.15}$$

is obtained by adapting a strategy of proof applied by Eastham in the case of (real-valued) periodic potentials [11] (reproduced in the proof of Theorem 5.3.2 of [12]) to the (complex-valued) quasi-periodic case at hand.

In the special self-adjoint case where V is real-valued, the result (4.11) is equivalent to the vanishing of the Lyapunov exponent of H which characterizes the (purely absolutely continous) spectrum of H as discussed by Kotani [24–27] (see also [7, p. 372]). In the case where V is periodic and complex-valued, this has also been studied by Kotani [27].

The explicit formula for Σ in (4.2) permits a qualitative description of the spectrum of H as follows. We recall (3.15) and write

$$\frac{d}{dz}\langle g(z,\cdot)^{-1}\rangle = -2\langle g(z,\cdot)\rangle = -i\frac{\prod_{j=1}^{n}\big(z-\widetilde{\lambda}_j\big)}{\big(\prod_{m=0}^{2n}(z-E_m)\big)^{1/2}}, \quad z\in\Pi, \tag{4.16}$$

for some constants

$$\{\widetilde{\lambda}_j\}_{j=1}^{n}\subset\mathbb{C}. \tag{4.17}$$

As in similar situations before, (4.16) extends to either side of the cuts in $\mathcal{C}$ by continuity with respect to z.

Theorem 4.6. *Assume Hypothesis 3.2. Then the spectrum $\sigma(H)$ of H has the following properties:*
(i) $\sigma(H)$ is contained in the semi-strip

$$\sigma(H)\subset\{z\in\mathbb{C}\,|\,\mathrm{Im}(z)\in[M_1,M_2],\ \mathrm{Re}(z)\geq M_3\}, \tag{4.18}$$

where

$$M_1=\inf_{x\in\mathbb{R}}[\mathrm{Im}(V(x))],\quad M_2=\sup_{x\in\mathbb{R}}[\mathrm{Im}(V(x))],\quad M_3=\inf_{x\in\mathbb{R}}[\mathrm{Re}(V(x))]. \tag{4.19}$$

(ii) $\sigma(H)$ consists of finitely many simple analytic arcs and one simple semi-infinite arc. These analytic arcs may only end at the points $\widetilde{\lambda}_1,\dots,\widetilde{\lambda}_n$, $E_0,\dots,E_{2n}$, and at infinity. The semi-infinite arc, σ_∞, asymptotically approaches the half-line $L_{\langle V\rangle}=\{z\in\mathbb{C}\,|\,z=\langle V\rangle+x,\ x\geq 0\}$ in the following sense: asymptotically, σ_∞ can be parameterized by

$$\sigma_\infty=\big\{z\in\mathbb{C}\,\big|\,z=R+i\,\mathrm{Im}(\langle V\rangle)+O\big(R^{-1/2}\big)\ \text{as}\ R\uparrow\infty\big\}. \tag{4.20}$$

(iii) Each E_m, $m=0,\dots,2n$, is met by at least one of these arcs. More precisely, a particular E_{m_0} is hit by precisely $2N_0+1$ analytic arcs, where $N_0\in\{0,\dots,n\}$ denotes the number of $\widetilde{\lambda}_j$ that coincide with E_{m_0}. Adjacent arcs meet at an angle $2\pi/(2N_0+1)$ at E_{m_0}. (Thus, generically, $N_0=0$ and precisely one arc hits E_{m_0}.)
(iv) Crossings of spectral arcs are permitted. This phenomenon takes place precisely when for a particular $j_0\in\{1,\dots,n\}$, $\widetilde{\lambda}_{j_0}\in\sigma(H)$ such that

$$\mathrm{Re}\big(\langle g(\widetilde{\lambda}_{j_0},\cdot)^{-1}\rangle\big)=0\ \text{ for some } j_0\in\{1,\dots,n\}\ \text{with}\ \widetilde{\lambda}_{j_0}\notin\{E_m\}_{m=0}^{2n}. \tag{4.21}$$

In this case $2M_0+2$ analytic arcs are converging toward $\widetilde{\lambda}_{j_0}$, where $M_0 \in \{1,\dots,n\}$ denotes the number of $\widetilde{\lambda}_j$ that coincide with $\widetilde{\lambda}_{j_0}$. Adjacent arcs meet at an angle $\pi/(M_0+1)$ at $\widetilde{\lambda}_{j_0}$. (Thus, generically, $M_0 = 1$ and two arcs cross at a right angle.)
(v) The resolvent set $\mathbb{C}\backslash\sigma(H)$ of H is path-connected.

Item (i) is clear from a numerical range consideration. To prove (ii) one introduces the meromorphic differential of the second kind

$$\Omega^{(2)} = \langle g(P,\cdot)\rangle dz = \frac{i\langle F_n(z,\cdot)\rangle dz}{2y} = \frac{i}{2}\frac{\prod_{j=1}^{n}\big(z-\widetilde{\lambda}_j\big)dz}{R_{2n+1}(z)^{1/2}},\ P=(z,y)\in\mathcal{K}_n\backslash\{P_\infty\}. \tag{4.22}$$

Then, by Lemma 3.4,

$$\big\langle g(P,\cdot)^{-1}\big\rangle = -2\int_{Q_0}^{P}\Omega^{(2)} + \big\langle g(Q_0,\cdot)^{-1}\big\rangle, \quad P\in\mathcal{K}_n\backslash\{P_\infty\} \tag{4.23}$$

for some fixed $Q_0 \in \mathcal{K}_n\backslash\{P_\infty\}$, is holomorphic on $\mathcal{K}_n\backslash\{P_\infty\}$. By (4.16), (4.17), the characterization (4.11) of the spectrum,

$$\sigma(H) = \big\{\lambda\in\mathbb{C}\,\big|\,\mathrm{Re}\big(\big\langle g(\lambda,\cdot)^{-1}\big\rangle\big)=0\big\}, \tag{4.24}$$

and the fact that $\mathrm{Re}\big(\big\langle g(z,\cdot)^{-1}\big\rangle\big)$ is a harmonic function on the cut plane Π, the spectrum $\sigma(H)$ of H consists of analytic arcs which may only end at the points $\widetilde{\lambda}_1,\dots,\widetilde{\lambda}_n$, $E_0,\dots,E_{2n}$, and possibly tend to infinity. To study the behavior of spectral arcs near infinity one computes

$$\big\langle g(z,\cdot)^{-1}\big\rangle \underset{|z|\to\infty}{=} -2iz^{1/2} + \frac{i}{z^{1/2}}\langle V\rangle + O\big(|z|^{-3/2}\big), \tag{4.25}$$

which readily yields (4.20). To prove (iii) one first recalls that by Theorem 4.5 the spectrum of H contains no isolated points. On the other hand, since $\{E_m\}_{m=0}^{2n}\subset\sigma(H)$ by (4.13), one concludes that at least one spectral arc meets each E_m, $m=0,\dots,2n$. Choosing $Q_0=(E_{m_0},0)$ in (4.23) one obtains

$$\begin{aligned}\big\langle g(z,\cdot)^{-1}\big\rangle &= -2\int_{E_{m_0}}^{z} dz'\,\langle g(z',\cdot)\rangle + \big\langle g(E_{m_0},\cdot)^{-1}\big\rangle\\ &\underset{z\to E_{m_0}}{=} -i[N_0+(1/2)]^{-1}(z-E_{m_0})^{N_0+(1/2)}[C+O(z-E_{m_0})] + \big\langle g(E_{m_0},\cdot)^{-1}\big\rangle,\\ &\qquad z\in\Pi\end{aligned} \tag{4.26}$$

for some $C = |C|e^{i\varphi_0} \in \mathbb{C}\backslash\{0\}$. Using

$$\mathrm{Re}\big(\langle g(E_m,\cdot)^{-1}\rangle\big) = 0, \quad m = 0,\dots,2n, \tag{4.27}$$

then readily implies the assertions made in item (iii). The proof of (iv) directly follows from (4.16). Finally assume that the spectrum of H contains a simple closed loop γ, $\gamma \subset \sigma(H)$. Then

$$\mathrm{Re}\big(\langle g(P,\cdot)^{-1}\rangle\big) = 0, \quad P \in \Gamma, \tag{4.28}$$

where the closed simple curve $\Gamma \subset \mathcal{K}_n$ denotes the lift of γ to $\mathcal{K}_n$, yields the contradiction

$$\mathrm{Re}\big(\langle g(P,\cdot)^{-1}\rangle\big) = 0 \text{ for all } P \text{ in the interior of } \Gamma \tag{4.29}$$

by Corollary 8.2.5 in [2]. Therefore, since there are no closed loops in $\sigma(H)$ and precisely one semi-infinite arc tends to infinity, the resolvent set of H is connected and hence path-connected, proving (v).

R e m a r k 4.7. Here $\sigma \subset \mathbb{C}$ is called an *arc* if there exists a parameterization $\gamma \in C([0,1])$ such that $\sigma = \{\gamma(t) \,|\, t \in [0,1]\}$. The arc σ is called *simple* if there exists a parameterization γ such that $\gamma\colon [0,1] \to \mathbb{C}$ is injective. The arc σ is called *analytic* if there is a parameterization γ that is analytic at each $t \in [0,1]$. Finally, σ_∞ is called a *semi-infinite* arc if there exists a parameterization $\gamma \in C([0,\infty))$ such that $\sigma_\infty = \{\gamma(t) \,|\, t \in [0,\infty)\}$ and σ_∞ is an unbounded subset of $\mathbb{C}$. Analytic semi-infinite arcs are defined analogously and by a simple semi-infinite arc we mean one that is without self-intersection (i.e., corresponds to a injective parameterization) with the additional restriction that the unbounded part of σ_∞ consists of precisely one branch tending to infinity.

R e m a r k 4.8. For simplicity we focused on $L^2(\mathbb{R}; dx)$-spectra thus far. However, since $V \in L^\infty(\mathbb{R}; dx)$, H in $L^2(\mathbb{R}; dx)$ is the generator of a C_0-semigroup $T(t)$ in $L^2(\mathbb{R}; dx)$, $t > 0$, whose integral kernel $T(t,x,x')$ satisfies a Gaussian upper bound. Thus, $T(t)$ in $L^2(\mathbb{R}; dx)$ defines, for $p \in [1,\infty)$, consistent C_0-semigroups $T_p(t)$ in $L^p(\mathbb{R}; dx)$ with generators denoted by H_p (i.e., $H = H_2$, $T(t) = T_2(t)$, etc.). One then infers the p-independence of the spectrum,

$$\sigma(H_p) = \sigma(H), \quad p \in [1,\infty). \tag{4.30}$$

We refer to [1] for more details.

Of course, these results apply to the special case of algebro-geometric complex-valued periodic potentials (see [5, 6, 41, 42]) and we briefly pointed out the corresponding connections between the algebro-geometric approach and standard Floquet theory in Appendix C of [1]. But even in this special case, items (iii) and

(iv) of Theorem 4.6 provide additional new details on the nature of the spectrum of H.

The methods of paper [1] extend to the case of algebro-geometric non-self-adjoint second order finite difference (Jacobi) operators associated with the Toda lattice hierarchy and to the case of Dirac-type operators related to the focusing nonlinear Schrödinger hierarchy. Moreover, they extend to the infinite genus limit $n \to \infty$ using the approach in [16]. This will be studied elsewhere.

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References

[1] *V. Batchenko and F. Gesztesy*, On the spectrum of Schrödinger operators with quasi-periodic algebro-geometric KdV potentials. Preprint (2003).

[2] *A.F. Beardon*, A primer on Riemann surfaces. London Math. Soc. Lecture Notes, v. 78. Cambridge University Press, Cambridge (1986).

[3] *E.F. Beckenbach and R. Bellman*, Inequalities, 4th printing. Springer, Berlin (1983).

[4] *E.D. Belokolos, A.I. Bobenko, V.Z. Enol'skii, A.R. Its, and V.B. Matveev*, Algebro-geometric approach to nonlinear integrable equations. Springer, Berlin (1994).

[5] *B. Birnir*, Complex Hill's equation and the complex periodic Korteweg-de Vries equations. — *Commun. Pure Appl. Math.* (1986), v. 39, p. 1–49.

[6] *B. Birnir*, Singularities of the complex Korteweg-de Vries flows. — *Commun. Pure Appl. Math.* (1986), v. 39, p. 283–305.

[7] *R. Carmona and J. Lacroix*, Spectral theory of random Schrödinger operators. — Birkhäuser, Boston (1990).

[8] *R.S. Chisholm and W.N. Everitt*, On bounded integral operators in the space of integrable-square functions. — *Proc. Roy. Soc. Edinburgh Sect.* (1970/71), v. A69, p. 199–204.

[9] *B.A. Dubrovin*, Periodic problems for the Korteweg-de Vries equation in the class of finite-gap potentials. — *Funct. Anal. Appl.* (1975), v. 9, p. 215–223.

[10] *B.A. Dubrovin, V.B. Matveev, and S.P. Novikov*, Non-linear equations of Korteweg-de Vries type, finite-zone linear operators, and Abelian varieties. — *Russian Math. Surv.* (1976), v. 31:1, p. 59–146.

[11] *M.S.P. Eastham*, Gaps in the essential spectrum associated with singular differential operators. — *Quart. J. Math.* (1967), v. 18, p. 155–168.

[12] *M.S.P. Eastham*, The spectral theory of periodic differential equations. — Scottish Acad. Press, Edinburgh and London (1973).

[13] *D.E. Edmunds and W.D. Evans*, Spectral theory and differential operators. Clarendon Press, Oxford (1989).

[14] *H. Flaschka*, On the inverse problem for Hill's operator. — *Arch. Rat. Mech. Anal.* (1975), v. 59, p. 293–309.

[15] *I.M. Gel'fand and L.A. Dikii*, Asymptotic behaviour of the resolvent of Sturm-Liouville equations and the algebra of the Korteweg-de Vries equations. — *Russian Math. Surv.* (1975), v. 30:5, p. 77–113.

[16] *F. Gesztesy*, Integrable systems in the infinite genus limit. — *Diff. Integr. Eqs.* (2001), v. 14, p. 671–700.

[17] *F. Gesztesy and H. Holden*, Soliton equations and their algebro-geometric solutions. V. I: (1+1)-dimensional continuous models. Cambridge Studies in Adv. Math., v. 79. Cambridge Univ. Press (2003).

[18] *F. Gesztesy, R. Ratnaseelan, and G. Teschl*, The KdV hierarchy and associated trace formulas. In: Recent Developments in Operator Theory and Its Applications, I. Gohberg, P. Lancaster, and P.N. Shivakumar (Eds.). Operator Theory: Adv. and Appl., Birkhäuser, Basel (1996), v. 87, p. 125–163.

[19] *F. Gesztesy and R. Weikard*, Floquet theory revisited. In: Diff. Eq. and Math. Phys. I. Knowles (Ed.). Int. Press, Boston (1995), p. 67–84.

[20] *F. Gesztesy and R. Weikard*, Picard potentials and Hill's equation on a torus. — *Acta Math.* (1996), v. 176, p. 73–107.

[21] *I.M. Glazman*, Direct methods of qualitative spectral analysis of singular differential operators. Moscow (1963). Engl. transl. by Israel Program for Sci. Transl. (1965).

[22] *A.R. Its and V.B. Matveev*, Schrödinger operators with finite-gap spectrum and N-soliton solutions of the Korteweg-de Vries equation. — *Theoret. Math. Phys.* (1975), v. 23, p. 343–355.

[23] *T. Kato*, Perturbation theory for linear operators. — Corr. printing of the 2nd ed. Springer, Berlin (1980).

[24] *S. Kotani*, Ljapunov indices determine absolutely continuous spectra of stationary random one-dimensional Schrödinger operators. Stochastic Analysis, K. Itǒ (Ed.), North-Holland, Amsterdam (1984), p. 225–247.

[25] *S. Kotani*, On an inverse problem for random Schrödinger operators. — *Contemporary Math.* (1985), v. 41, p. 267–281.

[26] *S. Kotani*, One-dimensional random Schrödinger operators and Herglotz functions. In: Probabilistic Meth. Math. Phys., K. Itǒ and N. Ikeda (Eds.). Acad. Press, New York (1987), p. 219–250.

[27] *S. Kotani*, Generalized Floquet theory for stationary Schrödinger operators in one dimension. — *Chaos, Solitons & Fractals* (1997), v. 8, p. 1817–1854.

[28] *P.D. Lax*, Periodic solutions of the Korteweg–deVries equation. — *Commun. Pure Appl. Math.* (1975), v. 28, p. 141–188.

[29] *V.A. Marchenko*, A periodic Korteweg–deVries problem. — *Soviet Math. Dokl.* (1974), v. 15, p. 1052–1056.

[30] *V.A. Marchenko*, The periodic Korteweg–deVries problem. — *Math. USSR Sb.* (1974), v. 24, p. 319–344.

[31] *V.A. Marchenko*, Sturm-Liouville operators and applications. — Birkhäuser, Basel (1986).

[32] *A.I. Markushevich*, Introduction to the classical theory of Abelian functions. Amer. Math. Soc, Providence (1992).

[33] *H.P. McKean and P. van Moerbeke*, The spectrum of Hill's equation. — *Invent. Math.* (1975), v. 30, p. 217–274.

[34] *S.P. Novikov*, The periodic problem for the Korteweg-de Vries equation. — *Funct. Anal. Appl.* (1974), v. 8, p. 236–246.

[35] *S.P. Novikov, S.V. Manakov, L.P. Pitaevskii, and V.E. Zakharov*, Theory of Solitons. Consultants Bureau, New York (1984).

[36] *L.A. Pastur and V.A. Tkachenko*, Geometry of the spectrum of the one-dimensional Schrödinger equation with a periodic complex-valued potential. — *Math. Notes* (1991), v. 50, p. 1045–1050.

[37] *M. Reed and B. Simon*, Methods of modern mathematical physics. IV: Analysis of Operators. Acad. Press, New York (1978).

[38] *F.S. Rofe-Beketov*, The spectrum of non-selfadjoint differential operators with periodic coefficients. — *Soviet Math. Dokl.* (1963), v. 4, p. 1563–1566.

[39] *M.I. Serov*, Certain properties of the spectrum of a non-selfadjoint differential operator of the second kind. — *Soviet Math. Dokl.* (1960), v. 1, p. 190–192.

[40] *V.A. Tkachenko*, Spectral analysis of the one-dimensional Schrödinger operator with a periodic complex-valued potential. — *Soviet Math. Dokl.* (1964), v. 5, p. 413–415.

[41] *R. Weikard*, Picard operators. — *Math. Nachr.* (1998), v. 195, p. 251–266.

[42] *R. Weikard*, On Hill's equation with a singular complex-valued potential. — *Proc. London Math. Soc.* (1998), v. (3)76, p. 603–633.