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On pseudospherical congruencies in E^4

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Geometric Bäcklund transformations of pseudospherical surfaces in four-dimensional Euclidean space are studied. An analog of the classical theorems by Bäcklund, Tenenblat and Terng is proved.

Let $F^2, \tilde{F}^2$ be regular two-dimensional surfaces in four-dimensional Euclidean space E^4. *A line congruence* $\psi : F^2 \to \tilde{F}^2$ is a diffeomorphism which possesses the following bitangency property: for each point $P \in F^2$ the straight line joining P with $\psi(P) = \tilde{P} \in \tilde{F}^2$ is a common tangent line for F^2 and $\tilde{F}^2$.

The line congruence $\psi : F^2 \to \tilde{F}^2$ is said to be *pseudospherical* if it satisfies two additional conditions:

B1) the distance between corresponding points $P \in F^2$ and $\tilde{P} \in \tilde{F}^2$ is equal to a non-zero constant independent of P, $|P\tilde{P}| \equiv l_0 \neq 0$;

B2) the angle between planes tangent to F^2 and $\tilde{F}^2$ at corresponding points is equal to a non-zero constant independent of P, $\angle(T_P F^2, T_{\tilde{P}} \tilde{F}^2) \equiv \omega_0 \neq 0$.

The constants l_0 and ω_0 are called *the parameters* of the pseudospherical congruence ψ.

This definiton corresponds to the classical definition of pseudospherical congruencies of n-dimensional submanifolds in $(2n-1)$-dimensional Euclidean space [2]. Generalising classical results by L. Bianchi and A.V. Bäcklund, K. Tenenblat and C.-L. Terng proved that if two n-dimensional submanifolds M, M^* in E^{2n-1} are connected by a pseudospherical congruence, then both M and M^* are of constant negative Gauss curvature $K = -\dfrac{\sin^2 \omega_0}{l_0^2}$ [1]. Moreover, an arbitrary submanifold $M^n \subset E^{2n-1}$ with constant negative Gauss curvature (usually reffered to as a pseudospherical submanifold) admits a large family of different pseudospherical congruencies. Using the classical terminology, M^* is called a *Bäcklund*

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transformation of M. This geometric construction is of greate importance for the soliton theory, where it initiated the development of some fundamental ideas and principles. Nowdays the pseudospherical submanifolds M^n in E^{2n-1} represent one of the most illustrative classical examples of integrable systems.

For two-dimensional surfaces in E^4 the situation with pseudospherical congruencies seems to be somewhat different from the classical one, although a number of results remain valid. In order to exclude the three-dimensional case, we will always assume that $F^2 \subset E^4$ is *in the general position*, i.e., the dimension of its first normal space is equal 2, so F^2 does not belong (even locally) to any affine hyperplane $E^3 \subset E^4$.

First of all, if F^2 in E^4 admits a line congruence $\psi : F^2 \to \tilde{F}^2$, then it necessarily has some specific properties. Namely, in the generic case F^2 carries a regular net of conjugate curves and the straight lines of the congruence ψ are tangent to one family of conjugate curves. The surface $\tilde{F}^2$ has the same property, so ψ may be viewed as a congruence of conjugate nets.

On the other hand, at each point P of $F^2 \subset E^4$ there exist at most two different conjugate directions in the tangent plane $T_P F^2$. Moreover, there is the following classification: $P \in F^2$ is said to be hyperbolic (parabolic, elliptic) if there are 2 (1 and 0 respectively) different conjugate directions in $T_P F^2$ (cf. [2, § 8.6]). A surface which consists of hyperbolic points carries a unique net of conjugate curves, it is usually referred to as a *Cartan surface* in E^4. Evidently it admits at most two different line congruencies [3]. Every surface in E^4 which consists of elliptic points, a E-surface, doesn't carry conjugate nets, so it doesn't admit line congruencies. Every surface in E^4 which consists of parabolic points is foliated by asymptotic lines, so it admits at most one line congruence. Recall that an arbitrary surface in E^3 admits infinitely many different line congruencies.

Our main result is the proof of the following statement.

Theorem. *Let F^2, $\tilde{F}^2$ be regular Cartan surfaces in E^4. Let $\psi : F^2 \to \tilde{F}^2$ be a pseudospherical congruence with parameters $l_0 > 0$, $\omega_0 \in (0, \pi/2]$. Then F^2 and $\tilde{F}^2$ are pseudospherical surfaces of Gauss curvature* $K = -\dfrac{\sin^2 \omega_0}{l_0^2}$.

The assumption that F^2 and $\tilde{F}^2$ are Cartan surfaces is not restrictive. It's not difficult to demonstrate that if a pseudospherical surface in E^4 admits a pseudospherical congruence, then, in the general case, it is a Cartan surface. For example, exactly for this reason K. Teneblat in [3] considered only Cartan surfaces in E^4.

Note that a pseudospherical surface in E^4 admits at most two different pseudospherical congruencies, contrary to the classical three-dimensional case. Besides, there is no reason to assume that a pseudospherical surface in E^4 has to be a Cartan surface. Therefore, contrary to the classical case, it seems to be true that some pseudospherical surfaces in E^4 do not admit pseudospherical congruencies, whereas another ones admits either one or two different pseudospherical congruencies, but this is an open question. In order to solve it we have to construct a pseudospherical surface in E^4 which consists of elliptic and parabolic points. Some other open problems will be formulated at the end of the article.

The studying of Bäcklund transformations of pseudospherical surfaces in E^4 was initiated by Yu. Aminov and A. Sym in [4].

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P r o o f. The Cartan surface in question, F^2, carries a net of conjugate lines. It is easy to see that F^2 can be locally parameterised in such a way that the coordinate curves are conjugate. If $r = r(u,v)$ is the corresponding position-vector, then the conjugacy of coordinate lines just means that $\partial^2_{uv} r$ is a linear combination of the tangent vectors $\partial_u r$ and $\partial_v r$.

Let n_1, n_2 denote vector fields normal to F^2 which at every point $P \in F^2$ form an orthonormal frame in the normal plane $N_P F^2$. Let g_{ij} stand for the coefficients of the first fundamental form of F^2 and let L^1_{ij} and L^2_{ij} be the coefficients of the second fundamental forms of F^2 with respect to n_1 and n_2. Finally, let $\mu_1 =< \partial_u n_1, n_2 >$, $\mu_2 =< \partial_v n_1, n_2 >$.

The conjugacy of coordinate lines means that the second fundamental forms are diagonal, $L^1_{12} \equiv 0$, $L^2_{12} \equiv 0$.

Introduce new functions $A(u,v)$, $B(u,v)$, $a(u,v)$, $b(u,v)$ to rewrite the matrices of coefficients L^σ_{ij} as follows [2]:

$$L^1 = \begin{pmatrix} A\cos a & 0 \\ 0 & B\cos b \end{pmatrix}, \quad L^2 = \begin{pmatrix} A\sin a & 0 \\ 0 & B\sin b \end{pmatrix}. \tag{1}$$

The dimension of the first normal space of F^2 is equal to the rank of the matrix

$$\begin{pmatrix} L^1_{11} & L^1_{12} & L^1_{22} \\ L^2_{11} & L^2_{12} & L^2_{22} \end{pmatrix} = \begin{pmatrix} A\cos a & 0 & B\cos b \\ A\sin a & 0 & B\sin b \end{pmatrix}.$$

Hence

$$AB\sin(b-a) \neq 0$$

since F^2 is assumed to be in the general position.

The fundamental Gauss–Codazzi–Ricci equations for F^2 have the following form [2, § 6.3]:

$$AB\cos(a-b)$$
$$= R_{1212} = \frac{\det g}{g_{11}}\left(\partial_v \Gamma^2_{11} - \partial_u \Gamma^2_{12} + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^2_{11} - \Gamma^2_{12}\Gamma^2_{12}\right), \tag{2.1}$$

$$\partial_v A - A\Gamma^1_{12} + B\Gamma^2_{11}\cos(a-b) = 0, \tag{2.2}$$

$$\partial_u B - B\Gamma^2_{12} + A\Gamma^1_{22}\cos(a-b) = 0, \tag{2.3}$$

$$A(\partial_v a + \mu_2) + \Gamma^2_{11} B\sin(b-a) = 0, \tag{2.4}$$

$$B(\partial_u b + \mu_1) + \Gamma^1_{22} A\sin(a-b) = 0, \tag{2.5}$$

$$\partial_u \mu_2 - \partial_v \mu_1 + \frac{AB\sin(a-b)}{\det g} g_{12} = 0. \tag{2.6}$$

Here Γ^i_{jk} denote the Christoffel symbols of the Levi–Civita connection on F^2, R_{1212} is the Riemannian curvature of F^2. The system of equations (2) admits some simplifications. For example, one can solve (2.4)–(2.5) with respect to μ_1 and μ_2 and substitute the result into (2.6). As well, one can solve (2.1) with respect to $\cos(a-b)$, etc. In any case, the system of equations (2) provides a simple and useful description for the Cartan surface F^2 in E^4.

For future purposes, multiply (2.2) by A and replace $AB\cos(a-b)$ by the right side of the Gauss equation (2.1). The resulting equation reads:

$$\frac{1}{2}\partial_v A^2 - A^2\Gamma^1_{12} + \Gamma^2_{11} R_{1212} = 0. \tag{2.2a}$$

Now consider the line congruence $\psi : F^2 \to \tilde{F}^2$. The position vector of $\tilde{F}^2$ can be represented as

$$r = \tilde{r}(u,v) = r - \frac{1}{\Gamma^2_{12}}\partial_u r \tag{3}$$

or as

$$r = \tilde{r}(u,v) = r - \frac{1}{\Gamma^1_{12}}\partial_v r; \tag{4}$$

see, for example, [3]. The surface $\tilde{F}^2$ is usually called *a Laplace transformation* of F^2. In the general situation, it is a Cartan surface, and the line congruence ψ may be viewed as a line congruence between two conjugate nets (cf. [1]). Without loss of generality, we suppose that $\tilde{F}^2$ is presented by (3). Note that (3) is invariant with respect to scale transformations $\hat{u} = \hat{u}(u)$, $\hat{v} = \hat{v}(v)$.

Let us find the distance between corresponding points of F^2 and $\tilde{F}^2$. It easy follows from (3) that

$$l(u,v) = |r(u,v) - \tilde{r}(u,v)| = \frac{\sqrt{g_{11}}}{\Gamma^2_{12}}. \tag{5}$$

Next, for each pair of points $P \in F^2$, $\tilde{P} \in \tilde{F}^2$ connected by ψ, the angle between the tangent planes $T_P F^2$ and $T_{\tilde{P}}\tilde{F}^2$ is expressed as follows:

$$\cos\omega(u,v) = \frac{\Gamma^2_{11}}{\sqrt{(\Gamma^2_{11})^2 + \dfrac{g_{11}}{\det g}\sum_\sigma (L^\sigma_{11})^2}}, \tag{6}$$

this formula can be obtained from (3) by trivial calculations.

Note that for two two-dimensional subspaces E_1^2, E_2^2 in E^4 there are two well-defined angles that determine how E_1^2 is placed with respect to E_2^2. If the intersection $\zeta = E_1^2 \cup E_2^2$ is a straight line, then one of two angles is zero. So, the relative position of E_1^2 and E_2^2 is determined by one angle, just like in the three-dimensional case. This angle is calculated as the angle between straight lines in E_1^2, E_2^2 orthogonal to the line of intersection ζ.

Let us return to the proof. The line congruence ψ is assumed to be pseudo-spherical, i.e., the conditions B1) and B2) are satisfied, $l(u,v) \equiv l_0$, $\omega(u,v) \equiv \omega_0$. Applying (5)–(6) and taking into account (1), we write

$$\Gamma_{12}^2 = \frac{1}{l_0}\sqrt{g_{11}}\,, \tag{7}$$

$$\cos\omega_0 = \frac{\Gamma_{11}^2}{\sqrt{(\Gamma_{11}^2)^2 + \dfrac{g_{11}}{\det g}A^2}}\,. \tag{8}$$

The last equality can be rewritten in the following form:

$$A^2 = \frac{\det g}{g_{11}}\,(\Gamma_{11}^2)^2\,\mathrm{tg}^2\omega_0. \tag{9}$$

Now we shall analyse the equations (2.1) and (2.2a) together with (7), (9). Replace A^2 in (2.2a) by the expression from (9):

$$\frac{1}{2}\partial_v\left(\frac{\det g}{g_{11}}\,(\Gamma_{11}^2)^2\,\mathrm{tg}^2\omega_0\right) - \Gamma_{12}^1\,\frac{\det g}{g_{11}}\,(\Gamma_{11}^2)^2\,\mathrm{tg}^2\omega_0 + \Gamma_{11}^2 R_{1212} = 0. \tag{10}$$

There are simple formulae for the derivatives involved in (10):

$$\partial_v g_{11} = 2\left(\Gamma_{12}^1 g_{11} + \Gamma_{12}^2 g_{12}\right), \quad \partial_v g_{22} = 2\left(\Gamma_{22}^1 g_{12} + \Gamma_{22}^2 g_{22}\right),$$

$$\partial_v g_{12} = \Gamma_{12}^1 g_{12} + \Gamma_{12}^2 g_{22} + \Gamma_{22}^1 g_{11} + \Gamma_{22}^2 g_{12},$$

$$\partial_v(\det g) = 2\left(\Gamma_{12}^1 + \Gamma_{22}^2\right)\,\det g.$$

Applying these formulae, find R_{1212} from (10):

$$R_{1212} = -\mathrm{tg}^2\omega_0\frac{\det g}{g_{11}}\left(\partial_v\Gamma_{11}^2 + \Gamma_{22}^2\Gamma_{11}^2 - \Gamma_{12}^1\Gamma_{11}^2 - \Gamma_{12}^2\Gamma_{11}^2\frac{g_{12}}{g_{11}}\right). \tag{11}$$

Thus, if B2) is assumed, then (2.2a) is equivalent to (11).

Write (11) in another form. By the definition of the Riemann curvature tensor,

$$R_{1212} = \frac{\det g}{g_{11}}\left(\partial_v\Gamma_{11}^2 + \Gamma_{22}^2\Gamma_{11}^2 - \Gamma_{12}^1\Gamma_{11}^2 - \partial_u\Gamma_{12}^2 + \Gamma_{11}^1\Gamma_{12}^2 - \Gamma_{12}^2\Gamma_{12}^2\right).$$

Therefore

$$\partial_v \Gamma^2_{11} + \Gamma^2_{22}\Gamma^2_{11} - \Gamma^1_{12}\Gamma^2_{11} = R_{1212}\frac{g_{11}}{\det g} + \partial_u \Gamma^2_{12} - \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{12}\Gamma^2_{12}. \tag{12}$$

Substitute (12) to (11)

$$R_{1212} = -\mathrm{tg}^2\omega_0 \, \frac{\det g}{g_{11}} \left(R_{1212}\frac{g_{11}}{\det g} + \partial_u \Gamma^2_{12} - \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^2_{12}\Gamma^2_{11}\frac{g_{12}}{g_{11}} \right)$$

and find R_{1212}:

$$\begin{aligned} R_{1212} &= -\sin^2\omega_0 \frac{\det g}{g_{11}} \left(\partial_u \Gamma^2_{12} - \Gamma^1_{11}\Gamma^2_{12} - \Gamma^2_{12}\Gamma^2_{11}\frac{g_{12}}{g_{11}} + \Gamma^2_{12}\Gamma^2_{12} \right) \\ &= -\sin^2\omega_0 \frac{\det g}{g_{11}} \left(\partial_u \Gamma^2_{12} - \frac{1}{g_{11}}\Gamma^2_{12}\Gamma_{11,1} + \Gamma^2_{12}\Gamma^2_{12} \right) \\ &= -\sin^2\omega_0 \frac{\det g}{g_{11}} \left(\partial_u \Gamma^2_{12} - \frac{1}{2}\Gamma^2_{12}\frac{\partial_u g_{11}}{g_{11}} + \Gamma^2_{12}\Gamma^2_{12} \right). \end{aligned} \tag{13}$$

Apply (5) and replace Γ^2_{12} in (13) by $\sqrt{g_{11}}/l$. Then (13) is rewritten as follows:

$$R_{1212} = -\frac{\sin^2\omega_0}{l^2}\left(1 - \frac{\partial_u l}{\sqrt{g_{11}}}\right)\det g.$$

Therefore, the formula for the Gauss curvature of F^2 reads

$$K = \frac{R_{1212}}{\det g} = -\frac{\sin^2\omega_0}{l^2}\left(1 - \frac{\partial_u l}{\sqrt{g_{11}}}\right).$$

As consequence, if B1) holds, $l \equiv l_0$, then

$$K = -\frac{\sin^2\omega_0}{l_0^2}$$

and F^2 is a pseudospherical surface.

Since the pseudospherical congruence ψ is a symmetric construction, $\tilde{F}^2$ is also a pseudospherical surface of Gauss curvature $K = -\dfrac{\sin^2\omega_0}{l_0^2}$. ■

Let us formulate some open problems connected to the proven theorem.

1. *Describe a generic pseudospherical Cartan surface in E^4 that admits a pseudospherical congruence.* The supposed answer would be a system of differential equations, SDE, similar to the classical sine–Gordon equation or to its generalisations constructed by Yu. Aminov, K. Teneblat and C.-L. Terng [1–3]. Another

problem is *what are the transformations of solutions of SDE that correspond to the pseudospherical congruencies viewed as transformations of surfaces?* These two problems are already solved in the particular case when $\omega_0 = \frac{\pi}{2}$, i.e., for the Bianchi transformations of pseudospherical surfaces in E^4. Namely, it was demonstrated by the author that the pseudospherical surfaces in E^4 that admit Bianchi transformations are well-described by the solutions $\varphi(u,v)$, $P(u,v)$, $Q(u,v)$ of the following system of partial differential equations:

$$\partial^2_{uu} \cdot \varphi e^{2\varphi} + 2(\partial_u \varphi)^2 e^{2\varphi} - \partial^2_{vv} \cdot \varphi e^{-2\varphi} + 2(\partial_v \varphi)^2 e^{-2\varphi} + PQ + 1 = 0,$$

$$\partial_u P - \partial_u \varphi Q e^{2\varphi} = 0, \quad \partial_v Q + \partial_v \varphi P e^{-2\varphi} = 0,$$

whereas the transformation is described as

$$\{\varphi(u,v), P(u,v), Q(u,v)\} \quad \to \quad \{-\varphi(v,u), Q(v,u), P(v,u)\}.$$

2. *Describe a generic pseudospherical Cartan surfaces in E^4 which admit two pseudospherical congruencies. Do such surfaces exist?* The supposed answer is negative. So it would be correct to apply the notion "a pair of pseudospherical Cartan surfaces in E^4 connected by a pseudospherical congruence". (Such situation is not unusual, recall the classical notion "a pair of isothermic surfaces connected by a Christoffel transformation".)

One of the ways of solving these problems is to analyze the system of six partial differential equations (2.1)–(2.6) together with an additional equation $R_{1212} = -k_0^2 \det g$ for nine functions g_{11}, g_{12}, g_{22}, A, B, a, b, μ_1, μ_2.

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