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The solution of one time-optimal problem on the basis of the Markov moment min-problem with even gaps

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The time-optimal problem for a linear system which matrix has the spectrum $\sigma(A) = \{(2k-1)\lambda\}_{k=1}^{n}$ is considered. This problem is reduced to the power Markov moment *min*-problem with even gaps. The new generating function is suggested for finding the optimal time. The explicit form of the polynomial for which the set of nonnegative roots coincides with the set of switchings of the time-optimal control is given. The analytical form of the 0-controllability set at time Θ is obtained.

1. Introduction

Let us consider the linear time-optimal problem

$$\dot{x} = Ax + bu, \quad |u| \le 1, \quad x \in E_n,$$

$$x(0) = x^0, \quad x(\Theta) = 0, \quad \Theta \to \min, \tag{1.1}$$

$$\operatorname{rank}(b, Ab, A^2b, \ldots, A^{n-1}b) = n,$$

where A is a matrix of dimension $n \times n$, b is a n-dimensional vector, Θ is a time of motion from the point x^0 to the origin.

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It follows from the maximum principle [1] that the time-optimal control $u(t)$ is a piecewise constant function on the interval $[0, \Theta]$ taking values ± 1 and having a finite number of points of discontinuity. Moreover, if all eigenvalues of A are real then the control $u(t)$ has no more than $n-1$ switchings (points of discontinuity).

Thus, the solution of the time-optimal problem is reduced to finding the optimal time Θ, the switchings $T_1, T_2, \dots, T_k$, $k \le n-1$, and the sign of the control $u(t)$ at the interval $[T_k, \Theta]$ for all points x^0 from the 0-controllability set. Simultaneously the problem arises to describe the 0-controllability set of system (1.1), i.e., the set of all initial points x^0 from which it is possible to reach the origin at the time Θ.

The solution of the linear time-optimal problem by reducing of this problem to the abstract moment problem, namely to the moment L-problem, was proposed by N.N. Krasovskii [2]. The abstract moment L-problem is to determine the linear functional taking the given values at the given n elements and having the minimal norm. In terms of the abstract moment problem a number of numerical methods for calculation of the time-optimal control are developed.

If the restrictions on control have the form $(\int\limits_{t_0}^{\infty} |u(\tau)|^2\, d\tau)^{1/2} \le L$, it is possible to obtain the algebraic equation to determine the optimal time [2]. However, when the restrictions are of the form $|u(t)| \le 1$, in [2] it is suggested to use the numerical methods which are based on minimization of some function with certain restrictions on variables.

Another approach to the problems with restrictions $|u(t)| \le 1$ based on the reduction of the time-optimal problem to the Markov moment *min*-problem [3] was proposed in [4, 5]. This problem is completely solved for the power and the trigonometric cases. Namely, the polynomials for determining of the optimal time and the switchings [4–7] are found. One of the ways to obtain such polynomials is as follows. A certain rational function is associated to the moment min-problem and the fact that there exist certain relations between the coefficients of expansion of the rational function in the series is used. In [4] the solution of the time-optimal synthesis problem is also given.

The present paper develops the work [5]. We give the solution of the time-optimal problem (1.1) for the case when the spectrum of the matrix A has the form $\sigma(A) = \{(2k-1)\lambda\}_{k=1}^{n}$ reducing this problem to the Markov moment *min*-problem with even gaps.

In the case of such matrix the time-optimal problem (1.1) is reduced to the following form:

$$\begin{gathered} \dot{x}_k = (2k-1)\lambda x_k + u, \quad k = 1, \dots, n, \quad |u| \le 1, \\ x \in E_n, \quad x(0) = x^0, \quad x(\Theta) = 0, \quad \Theta \to \min . \end{gathered} \tag{1.2}$$

Let us reduce this problem to the Markov moment min-problem. Suppose the control $u(t)$ transfers the point x^0 to the origin in the time Θ. Then the following equalities hold

$$x_k^0 = -\int\limits_0^{\Theta} e^{-(2k-1)\lambda\tau}\, u(\tau)\, d\tau, \quad k = 1, \ldots, n. \tag{1.3}$$

Consider the case $\lambda < 0$. Then as it is well known for any $x^0 \in R^n$ there exist such a time Θ and a control $u(t)$, $|u(t)| \leq 1$, that (1.3) holds.

Having changed the variable $t = e^{-\lambda\tau}$ we rewrite equalities (1.3) in the form

$$x_k^0 = \frac{1}{\lambda} \int\limits_1^{e^{-\lambda\Theta}} t^{2k-2}\, u(t)\, dt, \quad k = 1, \ldots, n. \tag{1.4}$$

Thus, the solution of the time-optimal problem (1.2) coincides with the solution of the following Markov moment *min*-problem

$$s_k = \int\limits_1^{\tilde{\Theta}} t^{2k-2}\, u(t)\, dt, \quad k = 1, \ldots, n, \quad |u(t)| \leq 1, \quad t \in [1, \tilde{\Theta}], \quad \tilde{\Theta} \to \min, \tag{1.5}$$

where $s = \lambda x^0$, $\tilde{\Theta} = e^{-\lambda\Theta}$.

Further we denote by $\tilde{T}_i$ the points of discontinuity of the function $u(t)$ solving the moment min-problem (1.5). Note that $\tilde{T}_i = e^{-\lambda T_i}$ where T_i are switchings of the time-optimal control of problem (1.2). In what follows we also call $\tilde{\Theta}$ the optimal time and $\tilde{T}_i$ the switchings.

In [5] such moment problem is called the Markov moment *min*-problem with even gaps. As it is noted earlier, the method of solving of problem (1.5) given in [5] is based on the using of properties of some rational function. In the present paper we suggest to introduce into consideration the new rational function, namely the hyperbolic area-tangent. This allows to obtain the polynomial for finding the optimal time $\tilde{\Theta}$ the degree of which is almost two times less than the degree of the polynomial considered in [5]. In the Markov moment problem such function was not considered earlier (see [8]). Moreover, we give the explicit form of the polynomial for which the set of nonnegative roots coincides with the set of all switchings $\tilde{T}_i$. The analytical description of the 0-controllability set of system (1.2) is also given.

2. The equations for the optimal time

Consider problem (1.2) which is equivalent to problem (1.4). Further we consider only such points x^0 for which the optimal control has $n-1$ switchings.

Then the solving of system of the moment equalities (1.4) leads to the nonlinear system of equations

$$(-1)^n \sum_{i=1}^{n-1} (-1)^{i+1} \tilde{T}_i^{2k-1} = \frac{\tilde{\Theta}^{2k-1} + (-1)^n - (2k-1)\tilde{u}\lambda x_k^0}{2}, \quad k = 1, \dots, n, \quad (2.1)$$

where $\tilde{u}$ is a control on the last interval $[\tilde{T}_{n-1}, \tilde{\Theta}]$ ($\tilde{u} = \pm 1$). Let us call the control $u(t)$ to be the control of the first kind (the second kind respectively), if $\tilde{u} = -1$ ($\tilde{u} = +1$ respectively).

Let us denote the right-hand sides of the equations of system (2.1) by C_{2k-1} ($k = 1, \dots, n$) i.e.

$$C_{2k-1} = \frac{\tilde{\Theta}^{2k-1} + (-1)^n - (2k-1)\tilde{u}\lambda x_k^0}{2}, \qquad k = 1, \dots, n. \qquad (2.2)$$

T h e c a s e o f t h e e v e n n. Consider system (2.1) for the even n ($n = 2p$). Let us assume that the switchings $\tilde{T}_1, \tilde{T}_2, \dots, \tilde{T}_{n-1}$ are known. Let us supplement system (2.1) by analogous equations for $k \geq n+1$ and in what follows let us consider the infinite system of the equations

$$\sum_{i=1}^{n-1} (-1)^{i+1} \tilde{T}_i^{2k-1} = C_{2k-1}, \quad k = 1, 2, \dots . \qquad (2.3)$$

Consider the equality

$$\sum_{i=1}^{n-1} (-1)^{i+1} \operatorname{ar\,th} \frac{\tilde{T}_i}{z} = \sum_{k=1}^{\infty} \frac{C_{2k-1}}{(2k-1) z^{2k-1}}, \quad z \in C. \qquad (2.4)$$

Here and further the value $|z|$ is sufficiently large such that considered series are converged. Relations (2.3) are obtained from equality (2.4) if we expand the functions $\operatorname{ar\,th} \frac{\tilde{T}_i}{z}$ ($i = 1, \dots, n-1$) into series and equate the coefficients at the same powers of z. Taking into account that $\operatorname{ar\,th} x + \operatorname{ar\,th} y = \operatorname{ar\,th} \frac{x+y}{1+xy}$, equality (2.4) will be written in the form

$$\operatorname{ar\,th} \frac{b_{n-2} z^{n-2} + b_{n-4} z^{n-4} + \dots + b_2 z^2 + b_0}{z^{n-1} + a_{n-3} z^{n-3} + \dots + a_3 z^3 + a_1 z} = \sum_{k=1}^{\infty} \frac{C_{2k-1}}{(2k-1) z^{2k-1}}. \qquad (2.5)$$

Represent the rational function

$$R(z) = \frac{P(z)}{Q(z)}, \qquad (2.6)$$

where $P(z) = b_{n-2}z^{n-2} + b_{n-4}z^{n-4} + \ldots + b_2z^2 + b_0$, $Q(z) = z^{n-1} + a_{n-3}z^{n-3} + \ldots + a_3z^3 + a_1z$, in the form of the series

$$R(z) = \sum_{k=1}^{\infty} \frac{\gamma_{2k-1}}{z^{2k-1}}. \tag{2.7}$$

Then it follows from equalities (2.5), (2.7) that

$$\gamma_1 = C_1,\ \gamma_{2k-1} = \frac{1}{2k-1}\left(C_{2k-1} - \sum_{i=1}^{k-1} C_{2k-2i-1} \sum_{j=1}^{i} \gamma_{2j-1}\gamma_{2i-2j+1}\right),\ k = 2, 3, \ldots. \tag{2.8}$$

From relations (2.8) we obtain the equalities

$$C_{2k-1} = \begin{vmatrix} \gamma_1 & 3\gamma_3 & 5\gamma_5 & \ldots & (2k-1)\gamma_{2k-1} \\ -1 & \gamma_1^2 & 2\gamma_1\gamma_3 & \ldots & \sum\limits_{i=1}^{k-1} \gamma_{2i-1}\gamma_{2k-2i-1} \\ 0 & -1 & \gamma_1^2 & \ldots & \sum\limits_{i=1}^{k-2} \gamma_{2i-1}\gamma_{2k-2i-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 2\gamma_1\gamma_3 \\ 0 & 0 & 0 & \ldots & \gamma_1^2 \end{vmatrix}, \quad k = 1, \ldots, n. \tag{2.9}$$

Let us obtain the equation for finding the optimal time $\tilde{\Theta}$ in the case of the even n. From the equality

$$\frac{b_{n-2}z^{n-2} + b_{n-4}z^{n-4} + \ldots + b_2z^2 + b_0}{z^{n-1} + a_{n-3}z^{n-3} + \ldots + a_3z^3 + a_1z} = \sum_{k=1}^{\infty} \frac{\gamma_{2k-1}}{z^{2k-1}} \tag{2.10}$$

it follows that

$$\begin{aligned} &a_1\gamma_3 + a_3\gamma_5 + \ldots + a_{n-3}\gamma_{n-1} + \gamma_{n+1} = 0, \\ &a_1\gamma_5 + a_3\gamma_7 + \ldots + a_{n-3}\gamma_{n+1} + \gamma_{n+3} = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &a_1\gamma_{n+1} + a_3\gamma_{n+3} + \ldots + a_{n-3}\gamma_{2n-3} + \gamma_{2n-1} = 0, \end{aligned}$$

whence

$$\begin{vmatrix} \gamma_3 & \gamma_5 & \cdots & \gamma_{n+1} \\ \gamma_5 & \gamma_7 & \cdots & \gamma_{n+3} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n+1} & \gamma_{n+3} & \cdots & \gamma_{2n-1} \end{vmatrix} = 0. \tag{2.11}$$

The left-hand side of equality (2.11) is a polynomial of $\tilde{\Theta}, x^0, \tilde{u}$ by virtue of equalities (2.2), (2.9). Then the optimal time $\tilde{\Theta}$ is some root of equation (2.11) for $\tilde{u} = -1$ or $\tilde{u} = +1$. Further we explain how to select this root.

T h e c a s e o f t h e o d d n. Consider system (2.1) for the odd n ($n = 2p+1$). Just as in previous case we have

$$\operatorname{ar\,th} R(z) = \operatorname{ar\,th} \sum_{k=1}^{\infty} \frac{\gamma_{2k-1}}{z^{2k-1}} = \sum_{k=1}^{\infty} \frac{C_{2k-1}}{(2k-1)z^{2k-1}},$$

where

$$R(z) = \frac{b_{n-2}z^{n-2} + b_{n-4}z^{n-4} + \ldots + b_3 z^3 + b_1 z}{z^{n-1} + a_{n-3}z^{n-3} + \ldots + a_2 z^2 + a_0}. \tag{2.12}$$

The polynomials $\gamma_{2k-1}(x^0, \tilde{\Theta}, \tilde{u})$ $(k = 1, \ldots, n)$ are defined by formulas (2.8) or (2.9) as before.

From the equality

$$\frac{b_{n-2}z^{n-2} + b_{n-4}z^{n-4} + \ldots + b_3 z^3 + b_1 z}{z^{n-1} + a_{n-3}z^{n-3} + \ldots + a_2 z^2 + a_0} = \sum_{k=1}^{\infty} \frac{\gamma_{2k-1}}{z^{2k-1}} \tag{2.13}$$

it follows that

$$\begin{vmatrix} \gamma_1 & \gamma_3 & \cdots & \gamma_n \\ \gamma_3 & \gamma_5 & \cdots & \gamma_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_n & \gamma_{n+2} & \cdots & \gamma_{2n-1} \end{vmatrix} = 0. \tag{2.14}$$

The optimal time $\tilde{\Theta}$ is a root of the equation (2.14) for $\tilde{u} = -1$ or $\tilde{u} = +1$.

Note that since the elements γ_{2k-1} $(k = 1, \ldots, n)$ are the polynomials of the $(2k-1)$th degree from the optimal time $\tilde{\Theta}$ then in the case of the even n $(n = 2p)$ we have the equation of degree $p(2p+1) = \frac{n^2}{2} + \frac{n}{2}$ for finding the optimal time $\tilde{\Theta}$. And in the case of the odd n we have the equation of degree $\frac{n^2}{2} + \frac{n}{2}$ as well. The equation for the determination of the optimal time obtained in [5] has the degree n^2 both for an even and an odd n. Thus, the introduction of the hyperbolic area-tangent allows to decrease the degree of the polynomial in the equation for the optimal time $\tilde{\Theta}$ almost in two times for sufficiently large n since $\dfrac{\frac{n^2}{2} + \frac{n}{2}}{n^2} \to \frac{1}{2}$ for $n \to \infty$.

Let us show how to select the root of equations (2.11) and (2.14) which is the optimal time. The following Lemmas 1–3 are used to prove the theorem on the selection of the optimal time and to describe the 0-controllability set.

Lemma 1. *Let the rational function $R(z) = \frac{P(z)}{Q(z)}$ has form (2.6) in the case of an even n $(n = 2p)$, $n > 2$ or form (2.12) in the case of an odd n $(n = 2p+1)$,*

$n > 1$. *Then the roots* $p_1, p_2, \ldots, p_{n-2}$ *and the poles* $q_1, q_2, \ldots, q_{n-1}$ *of the function* $R(z)$ *are real and alternate i.e.* $q_1 < p_1 < \ldots < q_{n-2} < p_{n-2} < q_{n-1}$.

P r o o f. Let us prove the lemma for the even n (in the case of the odd n the proof is the same).

By virtue of equalities (2.4), (2.5), (2.6) we have

$$\operatorname{ar\,th}\frac{P(z)}{Q(z)} = \sum_{i=1}^{n-1}(-1)^{i+1}\operatorname{ar\,th}\frac{\tilde{T}_i}{z}. \tag{2.15}$$

Let us denote $P_1(z) = Q(z) + P(z)$, $Q_1(z) = Q(z) - P(z)$. Using the relation $\operatorname{ar\,th} x = \frac{1}{2}\ln\frac{1+x}{1-x}$ in the both sides of equality (2.15), we obtain

$$\operatorname{ar\,th}\frac{P(z)}{Q(z)} = \frac{1}{2}\ln\frac{P_1(z)}{Q_1(z)} = \frac{1}{2}\ln\frac{(z+\tilde{T}_1)(z-\tilde{T}_2)\ldots(z+\tilde{T}_{n-1})}{(z-\tilde{T}_1)(z+\tilde{T}_2)\ldots(z-\tilde{T}_{n-1})},$$

and, consequently,

$$P_1(z) = (z+\tilde{T}_1)(z-\tilde{T}_2)\ldots(z+\tilde{T}_{n-1}), \quad Q_1(z) = (z-\tilde{T}_1)(z+\tilde{T}_2)\ldots(z-\tilde{T}_{n-1}).$$

Let us show that $P_1(-\tilde{T}_{n-2}) > 0$. Really, we have

$$P_1(-\tilde{T}_{n-2}) = (-\tilde{T}_{n-2}+\tilde{T}_1)(-\tilde{T}_{n-2}-\tilde{T}_2)\ldots(-\tilde{T}_{n-2}-\tilde{T}_{n-2})(-\tilde{T}_{n-2}+\tilde{T}_{n-1}) > 0$$

since the number n is even and $\tilde{T}_{n-2} > \tilde{T}_k$ for $k = 1, \ldots, n-3$ and $\tilde{T}_{n-2} < \tilde{T}_{n-1}$.

Arguing by the same way, we obtain the relations

$$\begin{gathered}
P_1(-\tilde{T}_{n-1}) = 0,\ P_1(-\tilde{T}_{n-2}) > 0,\ P_1(-\tilde{T}_{n-3}) = 0,\ P_1(-\tilde{T}_{n-4}) < 0, \\
P_1(-\tilde{T}_{n-5}) = 0, \ldots, \operatorname{sign} P_1(-\tilde{T}_2) = (-1)^{\frac{n}{2}},\ P_1(-\tilde{T}_1) = 0, \\
\operatorname{sign} P_1(\tilde{T}_1) = (-1)^{\frac{n}{2}+1},\ P_1(\tilde{T}_2) = 0, \ldots, P_1(\tilde{T}_{n-5}) > 0, \\
P_1(\tilde{T}_{n-4}) = 0,\ P_1(\tilde{T}_{n-3}) < 0,\ P_1(\tilde{T}_{n-2}) = 0,\ P_1(\tilde{T}_{n-1}) > 0, \\
Q_1(-\tilde{T}_{n-1}) < 0,\ Q_1(-\tilde{T}_{n-2}) = 0,\ Q_1(-\tilde{T}_{n-3}) > 0,\ Q_1(-\tilde{T}_{n-4}) = 0, \\
Q_1(-\tilde{T}_{n-5}) < 0, \ldots, Q_1(-\tilde{T}_2) = 0,\ \operatorname{sign} Q_1(-\tilde{T}_1) = (-1)^{\frac{n}{2}}, \\
Q_1(\tilde{T}_1) = 0,\ \operatorname{sign} Q_1(\tilde{T}_2) = (-1)^{\frac{n}{2}+1}, \ldots, Q_1(\tilde{T}_{n-5}) = 0, \\
Q_1(\tilde{T}_{n-4}) > 0,\ Q_1(\tilde{T}_{n-3}) = 0,\ Q_1(\tilde{T}_{n-2}) < 0,\ Q_1(\tilde{T}_{n-1}) = 0.
\end{gathered}$$

Hence

$$\begin{gathered}
Q(-\tilde{T}_{n-1}) = \frac{1}{2}(P_1(-\tilde{T}_{n-1}) + Q_1(-\tilde{T}_{n-1})) < 0, \\
Q(-\tilde{T}_{n-2}) = \frac{1}{2}(P_1(-\tilde{T}_{n-2}) + Q_1(-\tilde{T}_{n-2})) > 0.
\end{gathered}$$

Then the point $z = q_1$ such that $Q(q_1) = 0$ will be found in the interval $(-\tilde{T}_{n-1}, -\tilde{T}_{n-2})$. Analogously it is shown that in each interval $(-\tilde{T}_{n-3}, -\tilde{T}_{n-4})$, $\ldots$, $(\tilde{T}_{n-4}, \tilde{T}_{n-3})$, $(\tilde{T}_{n-2}, \tilde{T}_{n-1})$ there exists the point of vanishing function $Q(z)$. There is no other poles besides $q_1, q_2, \ldots, q_{n-1}$ of the rational function $R(z)$ since the polynomial $Q(z)$ has degree $n-1$.

For the function $P(z)$ in each interval $(-\tilde{T}_{n-2}, -\tilde{T}_{n-3})$, $(-\tilde{T}_{n-4}, -\tilde{T}_{n-5}), \ldots$, $(\tilde{T}_{n-5}, \tilde{T}_{n-4})$, $(\tilde{T}_{n-3}, \tilde{T}_{n-2})$ there exists the point of vanishing of the function $P(z)$. Hence the function $R(z)$ has $n-2$ real roots, and there is no other points besides $p_1, p_2, \ldots, p_{n-2}$ such that $P(z) = 0$.

Thus, the rational function $R(z) = \dfrac{P(z)}{Q(z)}$ has $n-2$ roots $p_1, p_2, \ldots, p_{n-2}$ and $n-1$ poles $q_1, q_2, \ldots, q_{n-1}$, and as it is obvious from the above disposition of p_i $(i = 1, \ldots, n-2)$ and q_j $(j = 1, \ldots, n-1)$ inside of the corresponding intervals the roots and the poles of the rational function $R(z)$ alternate. The lemma is proved.

R e m a r k 1. For the case of $n = 1$ we have $\gamma_1 = C_1 = \dfrac{\tilde{\Theta} - 1 - \tilde{u}\lambda x_1^0}{2} = 0$ and whence the optimal time $\tilde{\Theta}$ is determined. For the case of $n = 2$ the rational function $R(z)$ has one pole and no roots. For $n = 2$ we have $\gamma_3 = 0$ and $\gamma_1 = C_1 = T_1 > 0$.

R e m a r k 2. Evidently the roots and the poles of rational function (2.6) or (2.12) are situated symmetric with respect to the point $z = 0$. In the case of the even n $(n = 2m)$ rational function (2.6) has the roots $\pm p'_1, \pm p'_2, \ldots, \pm p'_{m-1}$ and the poles $0, \pm q'_1, \pm q'_2, \ldots, \pm q'_{m-1}$ and $0 < p'_1 < q'_1 < p'_2 < q'_2 < \ldots < p'_{m-1} < q'_{m-1}$. In the case of the odd n $(n = 2m+1)$ rational function (2.12) has the roots $0, \pm p'_1, \pm p'_2, \ldots, \pm p'_{m-1}$ and the poles $\pm q'_1, \pm q'_2, \ldots, \pm q'_m$ and $0 < q'_1 < p'_1 < q'_2 < p'_2 < \ldots < q'_{m-1} < p'_{m-1} < q'_m$.

Consider the case of the even n $(n = 2m)$. Having multiplied the both sides of equality (2.10) by z and denoted z^2 by ω, we have

$$\frac{b_{2m-2}\omega^{m-1} + b_{2m-4}\omega^{m-2} + \ldots + b_2\omega + b_0}{\omega^{m-1} + a_{2m-3}\omega^{m-2} + \ldots + a_3\omega + a_1} = \sum_{k=1}^{\infty} \frac{\gamma_{2k-1}}{\omega^{k-1}}, \tag{2.16}$$

whence we obtain

$$1 + \gamma_1 - \frac{b_{2m-2}\omega^{m-1} + b_{2m-4}\omega^{m-2} + \ldots + b_2\omega + b_0}{\omega^{m-1} + a_{2m-3}\omega^{m-2} + \ldots + a_3\omega + a_1} = 1 - \sum_{k=1}^{\infty} \frac{\gamma_{2k+1}}{\omega^{k}}. \tag{2.17}$$

Having denoted

$$\tilde{P}(\omega) = b_{2m-2}\omega^{m-1} + b_{2m-4}\omega^{m-2} + \ldots + b_2\omega + b_0, \tag{2.18}$$

$$\tilde{Q}(\omega) = \omega^{m-1} + a_{2m-3}\omega^{m-2} + \ldots + a_3\omega + a_1, \tag{2.19}$$

we rewrite equality (2.17) in the form

$$\frac{\tilde{P}_1(\omega)}{\tilde{Q}(\omega)} = 1 - \sum_{k=1}^{\infty} \frac{\gamma_{2k+1}}{\omega^k}, \tag{2.20}$$

where $\tilde{P}_1(\omega) = (1+\gamma_1)\tilde{Q}(\omega) - \tilde{P}(\omega)$ or

$$\tilde{P}_1(\omega) = (1+\gamma_1 - b_{2m-2})\omega^{m-1} + (a_{2m-3} - b_{2m-4})\omega^{m-2} + \ldots + (a_3 - b_2)\omega + a_1 - b_0. \tag{2.21}$$

The following lemma holds.

Lemma 2. *The rational function*

$$\tilde{R}_1(\omega) = \frac{\tilde{P}_1(\omega)}{\tilde{Q}(\omega)}, \tag{2.22}$$

where the polynomials $\tilde{P}_1(\omega)$ and $\tilde{Q}(\omega)$ are defined by equalities (2.21) and (2.19) respectively has the real roots $\tilde{p}'_1, \tilde{p}'_2, \ldots, \tilde{p}'_{m-1}$ and the poles $\tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_{m-1}$ which alternate, i.e., $\tilde{q}_1 < \tilde{p}'_1 < \tilde{q}_2 < \tilde{p}'_2 < \ldots < \tilde{q}_{m-1} < \tilde{p}'_{m-1}$.

P r o o f. By virtue of Remark 2 to Lemma 1 the polynomials $\tilde{P}(\omega)$ and $\tilde{Q}(\omega)$ defined by equalities (2.18) and (2.19) respectively can be written in the form $\tilde{P}(\omega) = b_{2m-2}(\omega-\tilde{p}_1)(\omega-\tilde{p}_2)\ldots(\omega-\tilde{p}_{m-1})$, $\tilde{Q}(\omega) = (\omega-\tilde{q}_1)(\omega-\tilde{q}_2)\ldots(\omega-\tilde{q}_{m-1})$, where $\tilde{p}_i = p'^2_i$, $\tilde{q}_i = q'^2_i$ $(i = 1, \ldots, m-1)$ and the inequalities

$$\tilde{p}_1 < \tilde{q}_1 < \tilde{p}_2 < \tilde{q}_2 < \ldots < \tilde{p}_{m-1} < \tilde{q}_{m-1} \tag{2.23}$$

hold.

Let us show that at the interval $(\tilde{q}_1, \tilde{q}_2)$ there exists the point $\tilde{p}'_1$ such that $\tilde{P}_1(\tilde{p}'_1) = 0$. We have

$$\tilde{P}_1(\tilde{q}_1) = (1+\gamma_1)\tilde{Q}(\tilde{q}_1) - \tilde{P}(\tilde{q}_1) = -\tilde{P}(\tilde{q}_1) = -b_{2m-2}(\tilde{q}_1-\tilde{p}_1)(\tilde{q}_1-\tilde{p}_2)\ldots(\tilde{q}_1-\tilde{p}_{m-1}).$$

Here $b_{2m-2} > 0$ since from (2.16) we have $b_{2m-2} = \gamma_1 = C_1 > 0$. Taking into account inequality (2.23) in the case of the even m, we have the inequality $\tilde{P}_1(\tilde{q}_1) < 0$, and in the case of the odd m we obtain the inequality $\tilde{P}_1(\tilde{q}_1) > 0$. Further, we have

$$\tilde{P}_1(\tilde{q}_2) = (1+\gamma_1)\tilde{Q}(\tilde{q}_2) - \tilde{P}(\tilde{q}_2) = -\tilde{P}(\tilde{q}_2) = -b_{2m-2}(\tilde{q}_2-\tilde{p}_1)(\tilde{q}_2-\tilde{p}_2)\ldots(\tilde{q}_2-\tilde{p}_{m-1}).$$

By virtue of inequality (2.23) in the case of the even m the inequality $\tilde{P}_1(\tilde{q}_2) > 0$ holds, and in the case of the odd m we have the inequality $\tilde{P}_1(\tilde{q}_2) < 0$. Thus, in the interval $(\tilde{q}_1, \tilde{q}_2)$ there exists the point $\omega = \tilde{p}'_1$ such that $\tilde{P}_1(\tilde{p}'_1) = 0$.

Analogously it is shown that in each interval $(\tilde{q}_i, \tilde{q}_{i+1})$ $(i = 2, \ldots, m-2)$ there exists the point $\tilde{p}'_i$ $(i = 2, \ldots, m-2)$ such that $\tilde{P}_1(\tilde{p}'_i) = 0$ $(i = 2, \ldots, m-2)$.

Let us show that there exists the point $\omega = \tilde{p}'_{m-1}$ $(\tilde{p}'_{m-1} \in (\tilde{q}_{m-1}, +\infty))$ such that $\tilde{P}_1(\tilde{p}'_{m-1}) = 0$. Since $\lim\limits_{\omega\to+\infty} \tilde{P}(\omega) > 0$, $\lim\limits_{\omega\to+\infty} \tilde{Q}(\omega) > 0$, the leading coefficient of the polynomial $\tilde{P}(\omega)$ is equal to $b_{2m-2} = \gamma_1$ and the leading coefficient of the polynomial $(1+\gamma_1)\tilde{Q}(\omega)$ is equal to $1+\gamma_1$ then in the interval $(\tilde{q}_{m-1}, +\infty)$ there exists the point $\tilde{p}'_{m-1}$ such that $\tilde{P}(\tilde{p}'_{m-1}) = (1+\gamma_1)\tilde{Q}(\tilde{p}'_{m-1})$, hence $\tilde{P}_1(\tilde{p}'_{m-1}) = 0$.

Thus, the polynomial $\tilde{P}_1(\omega)$ has $m-1$ real roots $\tilde{p}'_1, \tilde{p}'_2, \ldots, \tilde{p}'_{m-1}$, moreover, $\tilde{q}_1 < \tilde{p}'_1 < \tilde{q}_2 < \tilde{p}'_2 < \ldots < \tilde{q}_{m-1} < \tilde{p}'_{m-1}$, i.e. the roots and the poles of the rational function $\tilde{R}_1(\omega) = \dfrac{\tilde{P}_1(\omega)}{\tilde{Q}(\omega)}$ alternate. The lemma is proved.

Consider the case of $n = 2m+1$. Having divided the both sides of equality (2.13) by z and denoted z^2 by ω, we have

$$\frac{b_{2m-1}\omega^{m-1} + b_{2m-3}\omega^{m-2} + \ldots + b_3\omega + b_1}{\omega^m + a_{2m-2}\omega^{m-1} + \ldots + a_2\omega + a_0} = \sum_{k=1}^{\infty} \frac{\gamma_{2k-1}}{\omega^k},$$

whence we obtain

$$\frac{\omega^m + a_{2m-2}\omega^{m-1} + \ldots + a_2\omega + a_0 - \left(b_{2m-1}\omega^{m-1} + b_{2m-3}\omega^{m-2} + \ldots + b_3\omega + b_1\right)}{\omega^m + a_{2m-2}\omega^{m-1} + \ldots + a_2\omega + a_0}$$
$$= 1 - \sum_{k=1}^{\infty} \frac{\gamma_{2k-1}}{\omega^k}. \tag{2.24}$$

Having denoted

$$\overline{P}(\omega) = b_{2m-1}\omega^{m-1} + b_{2m-3}\omega^{m-2} + \ldots + b_3\omega + b_1,$$
$$\overline{Q}(\omega) = \omega^m + a_{2m-2}\omega^{m-1} + \ldots + a_2\omega + a_0, \tag{2.25}$$

we rewrite equality (2.24) in the form

$$\frac{\overline{P}_1(\omega)}{\overline{Q}(\omega)} = 1 - \sum_{k=1}^{\infty} \frac{\gamma_{2k-1}}{\omega^k}, \tag{2.26}$$

where $\overline{P}_1(\omega) = \overline{Q}(\omega) - \overline{P}(\omega)$, or

$$\overline{P}_1(\omega) = \omega^m + (a_{2m-2} - b_{2m-1})\omega^{m-1} + \ldots + (a_2 - b_3)\omega + a_0 - b_1. \tag{2.27}$$

The following lemma is proved by analogy with Lemma 2.

Lemma 3. *The rational function*

$$\overline{R}_1(\omega)=\frac{\overline{P}_1(\omega)}{\overline{Q}(\omega)}, \tag{2.28}$$

where the polynomials $\overline{P}_1(\omega)$ and $\overline{Q}(\omega)$ are defined by equalities (2.27) and (2.25) respectively has the real roots $\overline{p}'_1,\ \overline{p}'_2,\dots,\overline{p}'_m$ and the poles $\overline{q}_1,\ \overline{q}_2,\dots,\overline{q}_m$ which alternate, i.e., $\overline{q}_1<\overline{p}'_1<\overline{q}_2<\overline{p}'_2<\dots<\overline{q}_m<\overline{p}'_m$.

Let us denote

$$\Delta_1=\gamma_1,\quad \Delta_3=\gamma_3,\quad \Delta_5=\begin{vmatrix}\gamma_1 & \gamma_3\\ \gamma_3 & \gamma_5\end{vmatrix},\quad \Delta_7=\begin{vmatrix}\gamma_3 & \gamma_5\\ \gamma_5 & \gamma_7\end{vmatrix},\dots,$$

$$\Delta_{4p-1}=\begin{vmatrix}\gamma_3 & \gamma_5 & \cdots & \gamma_{2p+1}\\ \gamma_5 & \gamma_7 & \cdots & \gamma_{2p+3}\\ \dots & \dots & \ddots & \dots\\ \gamma_{2p+1} & \gamma_{2p+3} & \cdots & \gamma_{4p-1}\end{vmatrix},$$

$$\Delta_{4p+1}=\begin{vmatrix}\gamma_1 & \gamma_3 & \cdots & \gamma_{2p+1}\\ \gamma_3 & \gamma_5 & \cdots & \gamma_{2p+3}\\ \dots & \dots & \ddots & \dots\\ \gamma_{2p+1} & \gamma_{2p+3} & \cdots & \gamma_{4p+1}\end{vmatrix},\dots .$$

Since the roots and the poles of rational functions (2.22) and (2.28) having the expansions into series (2.20) and (2.26) respectively alternate then the conditions of Lemma 3 from [5] hold. Then the following corollary is valid.

Corollary 1. *The matrices $(\gamma_{2i+2j-1})_{i,j=1}^{n/2}$ and $(\gamma_{2i+2j-3})_{i,j=1}^{(n+1)/2}$ (in the case of even and odd n respectively) are positive semidefinite and, moreover, $\Delta_{2n-1}=0$ and $\Delta_{2k-1}\geq 0$ for $k=1,\dots,n-1$.*

Let us denote the polynomials γ_{2k-1} by α_{2k-1} $(k=1,\dots,n)$ in the case of the control of the first kind and by β_{2k-1} $(k=1,\dots,n)$ in the case of the control of the second kind. Then we have the recurrence formulas (2.8) for determining the polynomials α_{2k-1} and β_{2k-1}. Let us set $\gamma_{2k-1}=\alpha_{2k-1}$ $(k=1,\dots,n)$ and denote by Δ^-_{2n-1} the determinants

$$\begin{vmatrix}\alpha_3 & \alpha_5 & \cdots & \alpha_{n+1}\\ \alpha_5 & \alpha_7 & \cdots & \alpha_{n+3}\\ \dots & \dots & \ddots & \dots\\ \alpha_{n+1} & \alpha_{n+3} & \cdots & \alpha_{2n-1}\end{vmatrix},\qquad \begin{vmatrix}\alpha_1 & \alpha_3 & \cdots & \alpha_{n}\\ \alpha_3 & \alpha_5 & \cdots & \alpha_{n+2}\\ \dots & \dots & \ddots & \dots\\ \alpha_{n} & \alpha_{n+2} & \cdots & \alpha_{2n-1}\end{vmatrix}$$

in the case of even and odd n respectively. If we take β_{2k-1} $(k = 1, \ldots, n)$ instead of γ_{2k-1} $(k = 1, \ldots, n)$, then we denote by Δ^{+}_{2n-1} the determinants

$$\begin{vmatrix} \beta_3 & \beta_5 & \ldots & \beta_{n+1} \\ \beta_5 & \beta_7 & \ldots & \beta_{n+3} \\ \ldots & \ldots & \ddots & \ldots \\ \beta_{n+1} & \beta_{n+3} & \ldots & \beta_{2n-1} \end{vmatrix}, \qquad \begin{vmatrix} \beta_1 & \beta_3 & \ldots & \beta_n \\ \beta_3 & \beta_5 & \ldots & \beta_{n+2} \\ \ldots & \ldots & \ddots & \ldots \\ \beta_n & \beta_{n+2} & \ldots & \beta_{2n-1} \end{vmatrix}$$

in the case of even and odd n respectively. Finally we get the following theorem.

Theorem 1. *The optimal time $\tilde{\Theta}$ is a maximal real root of the equation*

$$\Delta^{-}_{2n-1}(\tilde{\Theta}, x^0)\, \Delta^{+}_{2n-1}(\tilde{\Theta}, x^0) = 0.$$

Moreover, if $\tilde{\Theta}$ is a maximal real root of equation

$$\Delta^{-}_{2n-1}(\tilde{\Theta}, x^0) = 0, \tag{2.29}$$

then the optimal control is of the first kind, if $\tilde{\Theta}$ is a maximal real root of equation

$$\Delta^{+}_{2n-1}(\tilde{\Theta}, x^0) = 0, \tag{2.30}$$

then the optimal control is of the second kind.

For problem (1.2) the optimal time is $\Theta = -\dfrac{1}{\lambda} \ln \tilde{\Theta}$.

3. The 0-controllability set of the system

The problem of description of the 0-controllability set of the system is closely connected with the time-optimal problem. Let us denote the 0-controllability set of system (1.2) by $S(0, \Theta)$. The following theorem gives the analytical description of the 0-controllability set $S(0, \Theta)$.

Theorem 2. *The set $S(0, \Theta)$ has a form*

$$S(0, \Theta) = \{x^0 : \Delta^{-}_{2k-1}(\tilde{\Theta}, x^0) \geq 0,\ \Delta^{+}_{2k-1}(\tilde{\Theta}, x^0) \geq 0\, (k = 1, \ldots, n)\},$$

where $\tilde{\Theta} = e^{-\lambda\Theta}$.

The proof is based on Theorem 1 and Corollary 1.

E x a m p l e. For $n = 2$ the 0-controllability set of system (1.2) at the time Θ is written by means of the following system of the inequalities

$$S(0, \Theta) = \{x^0 : \alpha_1(\tilde{\Theta}, x^0) \geq 0,\ \beta_1(\tilde{\Theta}, x^0) \geq 0,\ \alpha_3(\tilde{\Theta}, x^0) \geq 0,\ \beta_3(\tilde{\Theta}, x^0) \geq 0\},$$

where $\tilde{\Theta} = e^{-\lambda\Theta}$. Since

$$\alpha_1 = \frac{1}{2}\left(\tilde{\Theta} + 1 + \lambda x_1^0\right), \quad \beta_1 = \frac{1}{2}\left(\tilde{\Theta} + 1 - \lambda x_1^0\right),$$

$$\alpha_3 = \frac{1}{24}\left(3\tilde{\Theta}^3 - 3\tilde{\Theta}^2\left(1 + \lambda x_1^0\right) - 3\tilde{\Theta}\left(1 + \lambda x_1^0\right)^2 - \left(1 + \lambda x_1^0\right)^3 + 4\left(1 + 3\lambda x_2^0\right)\right),$$

$$\beta_3 = \frac{1}{24}\left(3\tilde{\Theta}^3 - 3\tilde{\Theta}^2\left(1 - \lambda x_1^0\right) - 3\tilde{\Theta}\left(1 - \lambda x_1^0\right)^2 - \left(1 - \lambda x_1^0\right)^3 + 4\left(1 - 3\lambda x_2^0\right)\right),$$

then the 0-controllability set has the form

$$\begin{cases} x_2^0 \le \frac{1}{12}\lambda^2\left(x_1^0\right)^3 + \frac{1}{4}\lambda\left(\tilde{\Theta}+1\right)\left(x_1^0\right)^2 + \frac{1}{4}\left(\tilde{\Theta}+1\right)^2 x_1^0 - \frac{1}{4\lambda}\left(\tilde{\Theta}+1\right)\left(\tilde{\Theta}-1\right)^2, \\ x_2^0 \ge \frac{1}{12}\lambda^2\left(x_1^0\right)^3 - \frac{1}{4}\lambda\left(\tilde{\Theta}+1\right)\left(x_1^0\right)^2 + \frac{1}{4}\left(\tilde{\Theta}+1\right)^2 x_1^0 + \frac{1}{4\lambda}\left(\tilde{\Theta}+1\right)\left(\tilde{\Theta}-1\right)^2, \end{cases}$$

where $\tilde{\Theta} = e^{-\lambda\Theta}$.

For example, for $\lambda = -1$ we obtain the following system of the inequalities describing the 0-controllability set at the time $\Theta = \ln 3$.

$$\begin{cases} x_2^0 \le \dfrac{1}{12}\left(x_1^0\right)^3 - \left(x_1^0\right)^2 + 4x_1^0 + 4, \\ x_2^0 \ge \dfrac{1}{12}\left(x_1^0\right)^3 + \left(x_1^0\right)^2 + 4x_1^0 - 4. \end{cases}$$

This set is represented on figure.

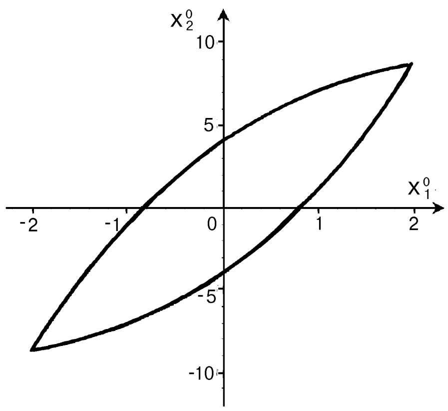

Figure.

4. The equations for the switchings

Having found the optimal time $\tilde\Theta$ and the kind of control it is necessary to find the switchings $\tilde T_1, \tilde T_2, \ldots, \tilde T_{n-1}$.

Theorem 3. *The switchings $\tilde T_1, \tilde T_2, \ldots, \tilde T_{n-1}$ are the positive roots of the equation*

$$\sum_{l=1}^{n} t^{2l-2} \sum_{k=l}^{n} \frac{\partial \Delta_{2n-1}(\gamma_1, \ldots, \gamma_{2n-1})}{\partial \gamma_{2k-1}} \left(\gamma_{-1}^2 + \sum_{m=1}^{k-l} \gamma_{2m-1}\gamma_{2k-2l-2m+1} \right) = 0, \quad (4.1)$$

with $\gamma_{2i-1} = \gamma_{2i-1}(x^0, \tilde\Theta, \tilde u), i = 1, \ldots, n$. Here x^0 is an initial point, $\tilde\Theta$ is an optimal time from x^0 to 0, $\tilde u$ is a control at the last interval $[\tilde T_{n-1}, \tilde\Theta]$. Here and further we set $\gamma_{-1}^2 = -1$ in the case of $k = l$ and $\gamma_{-1}^2 = 0$ in the case of $k > l$.

For the proof of the theorem the following auxiliary result is used.

Lemma 4. *The following equalities hold:*

$$\frac{\partial \gamma_{2k-1}}{\partial C_{2k-1}} = \frac{1}{2k-1}, \quad (4.2)$$

$$\frac{\partial \gamma_{2k-1}}{\partial C_{2j-1}} = -\frac{1}{2j-1} \sum_{m=1}^{k-j} \gamma_{2m-1}\gamma_{2k-2j-2m+1}, \qquad j = 1, \ldots, k-1. \quad (4.3)$$

For the proof of equalities (4.2) relations (2.8) are used. Equalities (4.3) are proved with the aid of formulas (2.9).

R e m a r k. Write relations (4.2) and (4.3) in the form

$$\frac{\partial \gamma_{2k-1}}{\partial C_{2l-1}} = -\frac{1}{2l-1} \left(\gamma_{-1}^2 + \sum_{m=1}^{k-l} \gamma_{2m-1}\gamma_{2k-2l-2m+1} \right), \qquad l = 1, \ldots, k. \quad (4.4)$$

P r o o f o f t h e t h e o r e m. Write equations (2.29) and (2.30) in the form

$$\Delta_{2n-1}(\gamma_1(x^0, \tilde\Theta, \tilde u), \ldots, \gamma_{2n-1}(x^0, \tilde\Theta, \tilde u)) = 0, \quad (4.5)$$

i.e., for $\tilde u = -1$ we have equation (2.29), for $\tilde u = +1$ we have equation (2.30).

For $\tilde\Theta = \tilde\Theta(x^0)$ and $\tilde u = \tilde u(x^0)$ equation (4.5) becomes identity. Since

$$\Delta_{2n-1} = \Delta_{2n-1}(\gamma_1, \ldots \gamma_{2n-1}),$$

$$\gamma_{2k-1} = \gamma_{2k-1}(C_1, \dots, C_{2k-1}), \qquad k = 1, \dots, n,$$
$$C_{2l-1} = C_{2l-1}(\tilde{T}_1, \dots, \tilde{T}_{n-1}), \qquad l = 1, \dots, n,$$

then we obtain that

$$\Delta_{2n-1} = \Delta_{2n-1}(\tilde{T}_1, \dots, \tilde{T}_{n-1}) \equiv 0.$$

Having differentiated this identity by $\tilde{T}_j$ $(j = 1, \dots, n-1)$, we obtain the system

$$\frac{\partial \Delta_{2n-1}}{\partial \tilde{T}_j} = 0, \qquad j = 1, \dots, n-1,$$

or

$$\begin{aligned}
&\sum_{k=1}^{n} \frac{\partial \Delta_{2n-1}}{\partial \gamma_{2k-1}} \frac{\partial \gamma_{2k-1}}{\partial C_1} \frac{\partial C_1}{\partial \tilde{T}_1} + \sum_{k=2}^{n} \frac{\partial \Delta_{2n-1}}{\partial \gamma_{2k-1}} \frac{\partial \gamma_{2k-1}}{\partial C_3} \frac{\partial C_3}{\partial \tilde{T}_1} + \dots \\
&\qquad + \frac{\partial \Delta_{2n-1}}{\partial \gamma_{2n-1}} \frac{\partial \gamma_{2n-1}}{\partial C_{2n-1}} \frac{\partial C_{2n-1}}{\partial \tilde{T}_1} = 0, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\sum_{k=1}^{n} \frac{\partial \Delta_{2n-1}}{\partial \gamma_{2k-1}} \frac{\partial \gamma_{2k-1}}{\partial C_1} \frac{\partial C_1}{\partial \tilde{T}_{n-1}} + \sum_{k=2}^{n} \frac{\partial \Delta_{2n-1}}{\partial \gamma_{2k-1}} \frac{\partial \gamma_{2k-1}}{\partial C_3} \frac{\partial C_3}{\partial \tilde{T}_{n-1}} + \dots \\
&\qquad + \frac{\partial \Delta_{2n-1}}{\partial \gamma_{2n-1}} \frac{\partial \gamma_{2n-1}}{\partial C_{2n-1}} \frac{\partial C_{2n-1}}{\partial \tilde{T}_{n-1}} = 0.
\end{aligned} \tag{4.6}$$

Since system (2.1) has the form $C_{2k-1} = (-1)^n \sum\limits_{j=1}^{n-1} (-1)^{j+1} \tilde{T}_j^{2k-1}$, $(k = 1, \dots, n)$, then

$$\frac{\partial C_{2k-1}}{\partial T_j} = (-1)^{n+j+1} (2k-1) T_j^{2k-2}. \tag{4.7}$$

Substituting equality (4.7) into system (4.6), we obtain

$$\begin{aligned}
&(-1)^{n+2} \left(\sum_{k=1}^{n} \frac{\partial \Delta_{2n-1}}{\partial \gamma_{2k-1}} \frac{\partial \gamma_{2k-1}}{\partial C_1} + 3\tilde{T}_1^2 \sum_{k=2}^{n} \frac{\partial \Delta_{2n-1}}{\partial \gamma_{2k-1}} \frac{\partial \gamma_{2k-1}}{\partial C_3} \right. \\
&\left. + 5\tilde{T}_1^4 \sum_{k=3}^{n} \frac{\partial \Delta_{2n-1}}{\partial \gamma_{2k-1}} \frac{\partial \gamma_{2k-1}}{\partial C_5} + \dots + (2n-1)\tilde{T}_1^{2n-2} \frac{\partial \Delta_{2n-1}}{\partial \gamma_{2n-1}} \frac{\partial \gamma_{2n-1}}{\partial C_{2n-1}} \right) = 0, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

$$(-1)^{2n}\left(\sum_{k=1}^{n}\frac{\partial\Delta_{2n-1}}{\partial\gamma_{2k-1}}\,\frac{\partial\gamma_{2k-1}}{\partial C_1}+3\tilde{T}_{n-1}^2\sum_{k=2}^{n}\frac{\partial\Delta_{2n-1}}{\partial\gamma_{2k-1}}\,\frac{\partial\gamma_{2k-1}}{\partial C_3}\right.$$

$$\left.+5\tilde{T}_{n-1}^4+\sum_{k=3}^{n}\frac{\partial\Delta_{2n-1}}{\partial\gamma_{2k-1}}\,\frac{\partial\gamma_{2k-1}}{\partial C_5}+\ldots+(2n-1)\tilde{T}_{n-1}^{2n-2}\frac{\partial\Delta_{2n-1}}{\partial\gamma_{2n-1}}\,\frac{\partial\gamma_{2n-1}}{\partial C_{2n-1}}\right)=0,$$

whence it follows that the switchings $\tilde{T}_1,\ \tilde{T}_2,\ldots,\tilde{T}_{n-1}$ and only they are the positive roots of the equation

$$\sum_{k=1}^{n}\frac{\partial\Delta_{2n-1}}{\partial\gamma_{2k-1}}\,\frac{\partial\gamma_{2k-1}}{\partial C_1}+3t^2\sum_{k=2}^{n}\frac{\partial\Delta_{2n-1}}{\partial\gamma_{2k-1}}\,\frac{\partial\gamma_{2k-1}}{\partial C_3}+\ldots$$

$$+(2n-1)t^{2n-2}\frac{\partial\Delta_{2n-1}}{\partial\gamma_{2n-1}}\,\frac{\partial\gamma_{2n-1}}{\partial C_{2n-1}}=0,$$

or

$$\sum_{l=1}^{n}(2l-1)t^{2l-2}\sum_{k=l}^{n}\frac{\partial\Delta_{2n-1}}{\partial\gamma_{2k-1}}\,\frac{\partial\gamma_{2k-1}}{\partial C_{2l-1}}=0. \tag{4.8}$$

For the given initial point x^0, $\tilde{\Theta}=\tilde{\Theta}(x^0)$ and $\tilde{u}=\tilde{u}(x^0)$ the left-hand side of equation (4.8) is the polynomial of degree $2l-2$ where all powers are even. Hence renaming t^2 by y we obtain the polynomial of the degree $n-1$ which has $n-1$ real roots. Then returning to the variable t, we have that the polynomial in equation (4.8) has $n-1$ real positive roots $\tilde{T}_1,\ \tilde{T}_2,\ldots,\tilde{T}_{n-1}$.

Substituting relation (4.4) into equation (4.8), we have equation (4.1) as to be proved.

Let us find the derivatives $\dfrac{\partial\Delta_{2n-1}}{\partial\gamma_{2k-1}}$. Denoting the determinant obtained from Δ_{2n-1} by crossing out the ith row and the jth column by $\Delta_{2n-1}^{(i,\,j)}$, we have for the odd n $(n=2p+1)$

$$\frac{\partial\Delta_{2n-1}}{\partial\gamma_{2k-1}}=(-1)^{k+1}\sum_{\substack{i+j=k+1\\1\le i,\ j\le p}}\Delta_{2n-1}^{(i,\,j)},\qquad k=1,\ldots,n,$$

and for the even n $(n=2p)$

$$\frac{\partial\Delta_{2n-1}}{\partial\gamma_{2k-1}}=(-1)^{k}\sum_{\substack{i+j=k\\1\le i,\ j\le p}}\Delta_{2n-1}^{(i,\,j)},\qquad k=1,\ldots,n.$$

Substituting the obtained expressions to equality (4.8) and using relations (4.4), we obtain the equations for finding all switchings $\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_{n-1}$ for $n = 2p+1$

$$\sum_{l=1}^{n} t^{2l-2} \sum_{k=l}^{n} (-1)^{k+1} \left(\gamma_{-1}^2 + \sum_{m=1}^{k-l} \gamma_{2m-1}\gamma_{2k-2l-2m+1} \right) \sum_{\substack{i+j=k+1 \\ 1 \le i,\, j \le p}} \Delta_{2n-1}^{(i,\,j)} = 0, \tag{4.9}$$

and for $n = 2p$

$$\sum_{l=1}^{n} t^{2l-2} \sum_{k=l}^{n} (-1)^{k} \left(\gamma_{-1}^2 + \sum_{m=1}^{k-l} \gamma_{2m-1}\gamma_{2k-2l-2m+1} \right) \sum_{\substack{i+j=k \\ 1 \le i,\, j \le p}} \Delta_{2n-1}^{(i,\,j)} = 0.$$

The theorem is proved.

Returning to the initial time-optimal problem (1.2), we obtain

$$T_i = -\frac{1}{\lambda} \ln \tilde{T}_i, \qquad i = 1, \ldots, n-1.$$

E x a m p l e. Let us consider the time-optimal problem for system (1.2) for $n = 3$, i.e., for the system

$$\dot{x}_1 = \lambda x_1 + u, \quad \dot{x}_2 = 3\lambda x_2 + u, \quad \dot{x}_3 = 5\lambda x_3 + u$$

from the initial point $(x_1^0,\, x_2^0,\, x_3^0)$ to 0.

In this case the system of the equations (2.1) has the form

$$\begin{cases} -\tilde{T}_1 + \tilde{T}_2 &= C_1, \\ -\tilde{T}_1^3 + \tilde{T}_2^3 &= C_3, \\ -\tilde{T}_1^5 + \tilde{T}_2^5 &= C_5, \end{cases}$$

where

$$C_1 = \frac{\tilde{\Theta} - 1 - \tilde{u}\lambda x_1^0}{2}, \quad C_3 = \frac{\tilde{\Theta}^3 - 1 - 3\tilde{u}\lambda x_2^0}{2}, \quad C_5 = \frac{\tilde{\Theta}^5 - 1 - 5\tilde{u}\lambda x_3^0}{2}.$$

Then we have

$$\gamma_1 = \frac{1}{2}\left(\tilde{\Theta} - 1 - \tilde{u}\lambda x_1^0\right),$$

$$\gamma_3 = \frac{1}{24}\Big[3\tilde{\Theta}^3 + 3\left(1 + \tilde{u}\lambda x_1^0\right)\tilde{\Theta}^2 - 3\left(1 + \tilde{u}\lambda x_1^0\right)^2\tilde{\Theta} + \left(1 + \tilde{u}\lambda x_1^0\right)^3 - 4\left(1 + 3\tilde{u}\lambda x_2^0\right)\Big],$$

$$\gamma_5 = \frac{1}{240}\Big[15\tilde\Theta^5 + 15\left(1+\tilde u\lambda x_1^0\right)\tilde\Theta^4 + 10\Big(1+3\tilde u\lambda x_2^0 - \left(1+\tilde u\lambda x_1^0\right)^3\Big)\tilde\Theta^2$$
$$+5\Big(\left(1+\tilde u\lambda x_1^0\right)^4 - 4\left(1+\tilde u\lambda x_1^0\right)\left(1+3\tilde u\lambda x_2^0\right)\Big)\tilde\Theta$$
$$+10\left(1+\tilde u\lambda x_1^0\right)^2\left(1+3\tilde u\lambda x_2^0\right) - 24\left(1+5\tilde u\lambda x_3^0\right) - \left(1+\tilde u\lambda x_3^0\right)^5\Big].$$

Equation (2.14) for finding the optimal time $\tilde\Theta$ for $n = 3$ has the form

$$\begin{vmatrix} \gamma_1 & \gamma_3 \\ \gamma_3 & \gamma_5 \end{vmatrix} = 0,$$

whence

$$45\tilde\Theta^6 - 90\left(1+\tilde u\lambda x_1^0\right)\tilde\Theta^5 - 45\left(1+\tilde u\lambda x_1^0\right)^2\tilde\Theta^4$$
$$+180\left(1+3\tilde u\lambda x_2^0\right)\tilde\Theta^3 + \Big(15\left(1+\tilde u\lambda x_1^0\right)^4 - 60\left(1+3\tilde u\lambda x_2^0\right)\Big)\tilde\Theta^2$$
$$+\Big(60\left(1+\tilde u\lambda x_1^0\right)^2\left(1+3\tilde u\lambda x_2^0\right) - 6\left(1+\tilde u\lambda x_1^0\right)^5 - 144\left(1+5\tilde u\lambda x_3^0\right)\Big)\tilde\Theta$$
$$+\left(1+\tilde u\lambda x_1^0\right)^6 - 20\left(1+\tilde u\lambda x_1^0\right)^2\left(1+3\tilde u\lambda x_2^0\right)$$
$$+144\left(1+\tilde u\lambda x_1^0\right)\left(1+5\tilde u\lambda x_3^0\right) - 80\left(1+3\tilde u\lambda x_2^0\right)^2 = 0.$$

The optimal time $\tilde\Theta$ is a maximal real root of this equation.

Equation (4.9) for the determination of the switchings $\tilde T_1$, $\tilde T_2$ will be written in the form

$$\gamma_1 t^4 - (\gamma_1^3 + 2\gamma_3)t^2 + \gamma_5 = 0,$$

where the functions γ_{2i-1} $(i = 1, 2, 3)$ depend on the optimal time $\tilde\Theta$ and the coordinates of the initial point $(x_1^0,\, x_2^0,\, x_3^0)$. Then we find the switchings $\tilde T_1$, $\tilde T_2$ from the last equation:

$$\tilde T_1 = \sqrt{\frac{\gamma_1^3 + 2\gamma_3 - \sqrt{(\gamma_1^3+2\gamma_3)^2 - 4\gamma_1\gamma_5}}{2\gamma_1}},$$

$$\tilde T_2 = \sqrt{\frac{\gamma_1^3 + 2\gamma_3 + \sqrt{(\gamma_1^3+2\gamma_3)^2 - 4\gamma_1\gamma_5}}{2\gamma_1}}.$$

Returning to the initial time-optimal problem we have

$$\Theta = -\frac{1}{\lambda}\ln\tilde\Theta, \quad T_1 = -\frac{1}{\lambda}\ln\tilde T_1, \quad T_2 = -\frac{1}{\lambda}\ln\tilde T_2.$$

So for the initial point (0, 2, 114) and $\lambda = -2$ we have the equations for the optimal time $\tilde{\Theta}$ determination

$$\tilde{\Theta}^6 - 2\tilde{\Theta}^5 - \tilde{\Theta}^4 + 52\tilde{\Theta}^3 - 17\tilde{\Theta}^2 - 3634\tilde{\Theta} + 3345 = 0,$$

$$\tilde{\Theta}^6 - 2\tilde{\Theta}^5 - \tilde{\Theta}^4 - 44\tilde{\Theta}^3 + 15\tilde{\Theta}^2 + 3630\tilde{\Theta} - 3855 = 0$$

in the case of the control of the first and the second kind respectively. Whence we find that the maximal real root $\tilde{\Theta} = 5$ and $\tilde{u} = -1$.

Define the switchings $\tilde{T}_1$, $\tilde{T}_2$ as the roots of a polynomial. For the found $\tilde{\Theta}$ and $\tilde{u}$ we have $\gamma_1 = 2, \gamma_3 = 16, \gamma_5 = 128$. Then we obtain the equation to determine the switchings

$$t^4 - 20t^2 + 64 = 0.$$

From the last equation we find $\tilde{T}_1 = 2$, $\tilde{T}_2 = 4$.

Finally, we obtain

$$\Theta = \frac{1}{2}\ln 5 \approx 0.805, \quad T_1 = \frac{1}{2}\ln 2 \approx 0.347, \quad T_2 = \frac{1}{2}\ln 4 \approx 0.693.$$

References

[1] *L.S. Pontryagin et al.*, The mathematical theory of optimal processes. Nauka, Moscow (1976).

[2] *N.N. Krasovskii*, Theory of movement control. Nauka, Moscow (1968).

[3] *V.I. Korobov and G.M. Sklyar*, The Markov moment problem on the minimally possible closed interval. — Dokl. AN USSR (1989), v. 308, No. 3, p. 525–528.

[4] *V.I. Korobov and G.M. Sklyar*, Time optimality and the power moment problem. — *Mat. Sb.* (1987), v. 134(176), No. 2(10), p. 186–206.

[5] *V.I. Korobov and G.M. Sklyar*, Markov power min-moment problem with periodic gaps. — *J. Math. Sci.* (1996), v. 80, No. 1, p. 1559–1581.

[6] *V.I. Korobov and G.M. Sklyar*, Time-optimal control and the trigonometric moment problem. — *Izv. AN USSR*, Ser. Mat. (1989), v. 53, No. 4, p. 868–885.

[7] *V.I. Korobov, G.M. Sklyar, and V.V. Florinsky*, The polynomial of the minimal degree for the determination of all switchings in the time- optimal problem. — *Mat. fiz., anal., geom.* (2000), v. 7, No. 3, p. 308–320.

[8] *M.G. Krein and A.A. Nudel'man*, The Markov moment problem and extremal problems. Nauka, Moscow (1973).