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Strong asymptotic stability and constructing of stabilizing controls*

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We show the role which plays a recent theorem on the strong asymptotic stability in the analysis of the strong stabilizability problem in Hilbert spaces. We consider a control system with skew-adjoint operator and one-dimensional control. We examine in details the property for a linear feedback control to stabilize such a system. A robustness analysis of stabilizing controls is also given.

Dedicated to the memory of V.Ya. Shirman

0. Introduction

The initial point of our work is the assertion:

Theorem 0.1. *Let A be the generator of a C_0-semigroup $\{e^{At}\}_{t\geq 0}$ on a Banach space X and*

i) the set $\sigma(A) \cap (iR)$ is at most countable,

ii) the adjoint operator A^* has no pure imaginary eigenvalues.

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Then the semigroup $\{e^{At}\}_{t\geq 0}$ is strongly asymptotically stable if and only if it is uniformly bounded.

This fact was first proved by Sklyar and Shirman in 1982 [1] for the case of bounded operator A. The method of treating of this problem given in [1] was picked up by Lyubich and Vu Phong [2] who extended in 1988 the result to the general case. Independently, in 1988 Theorem 0.1 was obtained by Arendt and Batty [3] who used some different approach.

In the present work we will use the following equivalent formulation (see [1]) of this theorem

Theorem 0.2. *If A is as in Theorem 0.1, the strong asymptotic stability of $\{e^{At}\}_{t\geq 0}$ occurs if and only if there is a norm $||\cdot||_1$ equivalent to the initial norm $||\cdot||$ of X such that A is dissipative in $||\cdot||_1$, i.e. (see [4]) for any $x \in X$: $||e^{At}x||_1 \leq ||x||_1$, $t \geq 0$.*

In particular, Theorem 0.2 suggests that, even in the case when $X = H$ is a Hilbert space, the study of the strong stability of semigroups in terms of dissipativity of their generators may require taking into account equivalent but non-Hilbert norms in X.

One of the most natural areas for application of the mentioned above results is the problem of strong stabilizability of contractive systems.

We consider a system

$$\dot{x} = Ax + Bu, \tag{0.1}$$

where A generates a contractive semigroup $\{e^{At}\}_{t\geq 0}$ in a Hilbert space H; B is a bounded operator from a Hilbert space U to H. One needs to find a feedback control $u = Px$, $P \in [H, U]$ such that the closed-loop system $\dot{x} = (A + BP)x$ is strongly asymptotically stable.

This problem has been studied in [5–13], see also the bibliography in [14, 15]. Majority of these works uses stability property of the semigroup generated by $A - BB^*$. In particular, it is known [13] that if

i) $\sigma(A) \cap (iR)$ is at most countable,

ii) there is no eigenvector v of A corresponding to a pure imaginary eigenvalue and such that $B^*v = 0$,

then the system (0.1) is strong stabilizable and a solution of this problem is given by $u = -B^*x$.

This fact is a consequence of Theorem 0.1. However, the problem becomes much more complicated if one needs to determine some other strong stabilizing controls or, especially, to describe the set of such controls.

This question appears to be insufficiently explored (one can mention only the works [8–11], where a Riccati equation approach was used) but, at the same time, important, for example, when examining robustness of stabilizing controls.

We suggest Theorem 0.2 as a tool for determination of stabilizing controls. The present work contains an analysis of this problem in a case which is often met with in applications. We consider equation (0.1) under the assumptions:

i) A is a skew-adjoint unbounded operator with discrete spectrum consisting of simple eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$,

ii) there exists a constant $C_\sigma \equiv \frac{1}{2}\min\limits_{i\neq j}|\lambda_i - \lambda_j| > 0$,

iii) the space U is one dimensional, so we associate B with a vector $b \in H$; besides, if $\{\phi_n\}_{n=1}^{\infty}$ is an orthonormal eigenbasis $A\phi_n = \lambda_n\phi_n$, then $b_n = \langle b, \phi_n\rangle \neq 0$, $n \in N$.

Our goal is to examine for feedback control $u = \langle x, q\rangle$, $q \in H$ the property to be stabilizing.

In Section 1 we show that operator $A + bq^*$ preserves the Riesz basis property of its eigenelements if $||b|| \cdot ||q|| < C_\sigma/2$ (Theorem 1). This fact, in particular, implies (compare Theorem 0.2) that for the strong stability of the semigroup $\{e^{(A+bq^*)t}\}_{t\geq 0}$ occurs if and only if there exists an equivalent Hilbert norm in which the operator $A + bq^*$ is dissipative. A complete description of all such norms is obtained in Section 2 (Theorem 4). The final result of Sections 1, 2 is a development of Theorem 0.2 to the case of semigroup $\{e^{(A+bq^*)t}\}_{t\geq 0}$. In Section 3 we give a robustness analysis for a stabilizing control $u = q^*x$. Note that the results of this section essentially generalize the robustness analysis given in [16] for the control $u = -b^*x$. Finally, in Section 4 we obtain conditions under which a feedback control is stabilizing. These results rest on the application of the Rouché theorem as well as of the Routh–Hurwitz method [17].

The results of this paper were announced in [18].

1. Basis and spectral properties of $A + bq^*$

Theorem 1. *Let $||b|| \cdot ||q|| < C_\sigma/2$, where $C_\sigma \equiv \frac{1}{2}\min|\lambda_i - \lambda_j| > 0$. Then the eigenvectors ψ_k of the operator $\widetilde{A} \equiv A + bq^*$ constitute a Riesz basis in H.*

P r o o f. Let us consider the equation for the eigenvectors ψ_k: $(A - \widetilde{\lambda}_k I + bq^*)\psi_n = 0$ and apply the resolvent $R_{\widetilde{\lambda}_k}(A)$. We get $\psi_n = \alpha_n \cdot R_{\widetilde{\lambda}_k}(A)b$, where $\alpha_n = -\langle\psi_n, q\rangle$. It gives

$$1 + \langle R_{\widetilde{\lambda}_k}(A)b, q\rangle = 0.$$

This together with the property $|\lambda_n - \widetilde{\lambda}_k| > C_\sigma$ (for all $n \neq k$) and the implicit form of the resolvent $R_{\widetilde{\lambda}_k}(A)b = \sum_{k=1}^{\infty} b_k\phi_k/(\lambda_k - \widetilde{\lambda}_n)$ lead to

$$\frac{\langle b, \phi_n\rangle\langle\phi_n, q\rangle}{\lambda_n - \widetilde{\lambda}_n} = -1 - \sum_{k\neq n}\frac{\langle b, \phi_k\rangle\langle\phi_k, q\rangle}{\lambda_k - \widetilde{\lambda}_n}. \tag{1.1}$$

Now we need the following

Lemma 1. *Let* $||b|| \cdot ||q|| < C_\sigma/2$. *Then the following equation*

$$1 + \langle R_\lambda(A)b, q\rangle = 0 \tag{1.2}$$

has a unique root in each ring

$$K_n = \left\{\lambda \in C \quad : \quad \frac{2}{3}|\langle b, \phi_n\rangle\langle q, \phi_n\rangle| \le |\lambda - \lambda_n| \le 2|\langle b, \phi_n\rangle\langle q, \phi_n\rangle|\right\}. \tag{1.3}$$

P r o o f o f L e m m a 1. Let us rewrite the equation (see also (1.1)) in the form $g(\lambda) = g_1^n(\lambda) + g_2^n(\lambda) = 0$, where

$$g_1^n(\lambda) \equiv \frac{\langle b, \phi_n\rangle\langle \phi_n, q\rangle}{\lambda_n - \lambda} + 1, \quad g_2^n(\lambda) \equiv \sum_{k \ne n} \frac{\langle b, \phi_k\rangle\langle \phi_k, q\rangle}{\lambda_k - \lambda}. \tag{1.4}$$

Since for any $\lambda \in K_n$ we have $|g_2^n(\lambda)| \le ||b||\,||q|| C_\sigma^{-1} < 1/2$, then on the boundary ∂K_n of the ring K_n one has

$$|g_1^n(\lambda)| \ge \left|\frac{\langle b, \phi_n\rangle\langle \phi_n, q\rangle}{\lambda_n - \lambda}\right| - 1 = 3/2 - 1 = 1/2 > |g_2^n(\lambda)|$$

for $|\lambda - \lambda_n| = \frac{2}{3}|\langle b, \phi_n\rangle\langle q, \phi_n\rangle|$ and

$$|g_1^n(\lambda)| \ge 1 - \left|\frac{\langle b, \phi_n\rangle\langle \phi_n, q\rangle}{\lambda_n - \lambda}\right| = 1 - 1/2 = 1/2 > |g_2^n(\lambda)|$$

for $|\lambda - \lambda_n| = 2|\langle b, \phi_n\rangle\langle q, \phi_n\rangle|$. Now we apply Rouché theorem to deduce that the function $g(\lambda)$ has the same number of roots as $g_1^n(\lambda)$, namely the only one root in K_n. The proof of Lemma 1 is complete.

From (1.1) and the assumptions on $\|b\|$ and $\|q\|$ we easily get $|\lambda_n - \widetilde{\lambda}_n| \le 2|\langle b, \phi_n\rangle\langle \phi_n, q\rangle|$ (c.f. the definition of K_n). Consider

$$f_n \equiv \frac{\lambda_n - \widetilde{\lambda}_n}{b_n} \sum_{k=1}^{\infty} \frac{b_k}{\lambda_k - \widetilde{\lambda}_n}\, \phi_k = \phi_n + \frac{\lambda_n - \widetilde{\lambda}_n}{b_n} \sum_{k \ne n}^{\infty} \frac{b_k}{\lambda_k - \widetilde{\lambda}_n}\, \phi_k\,. \tag{1.5}$$

It is evidently that $f_n = \psi_n \cdot c_n$ with a constant c_n. We have then

$$\| \phi_n - f_n \|^2 \le \left\| \frac{\lambda_n - \tilde{\lambda}_n}{b_n} \sum_{k \ne n}^{\infty} \frac{b_k}{\lambda_k - \widetilde{\lambda}_n}\, \phi_k \right\|^2 \le 4|q_n|^2 C_\sigma^{-2} \| b \|^2 .$$

So we get $\sum_{n=1}^{\infty} \| \phi_n - f_n \|^2 \le 4C_\sigma^{-2} \| b \|^2 \| q \|^2$. The assumptions on $\|b\|$ and $\|q\|$ imply

$$\sum_{n=1}^{\infty} \| \phi_n - f_n \|^2 < 1.$$

The last estimate means that the sequences ϕ_n and f_n are quadratically close. Taking into account Lemma 1 it remains to apply [19, Theorem V.2.20] that completes the proof of Theorem 1.

Remark 1. *One can notice that under the condition* $||b||\cdot||q|| < C_\sigma/2$ *we have proved that*

$$\sum_{n=1}^{\infty}|\lambda_n-\widetilde{\lambda}_n| < 2\sum_{n=1}^{\infty}|\langle b,\phi_n\rangle\langle\phi_n,q\rangle| \le 2||b||||q|| < 2C_\sigma/2 = C_\sigma\,.$$

So the strengthening of this assumptions by $||b||\cdot||q|| < C_\sigma/4$ *yields*

$$\sum_{n=1}^{\infty}|\lambda_n-\widetilde{\lambda}_n| < C_\sigma/2, \qquad n\in N. \tag{1.6}$$

In turns that implies

$$|\widetilde{\lambda}_i+\overline{\widetilde{\lambda}_j}| \ge |\lambda_i+\overline{\lambda_j}| - |\widetilde{\lambda}_i-\lambda_i| - |\overline{\widetilde{\lambda}_i}-\overline{\lambda_j}| = |\lambda_i+\lambda_j| - |\widetilde{\lambda}_i-\lambda_i| - |\widetilde{\lambda}_i-\lambda_j|$$
$$> 2C_\sigma - C_\sigma/2 - C_\sigma/2 = C_\sigma, \qquad i\neq j, \quad i,j\in N. \tag{1.7}$$

Theorem 2. *Let* $\{\widetilde{\lambda}_n\}_{n=1}^{\infty}$ *be any set of complex numbers such that:*
i) $|\lambda_n-\widetilde{\lambda}_n| < C_\sigma, \quad n\in N$;
ii) $\sum_{n=1}^{\infty}\frac{|\lambda_n-\widetilde{\lambda}_n|^2}{|b_n|^2} < \frac{C_\sigma}{||b||^2}$,
where $C_\sigma, b_n \equiv \langle b,\phi\rangle$ *and* λ_n *are as in Theorem 1. Then there exists a unique control* $u(x) = q^*x$ *such that the spectrum* $\sigma(\widetilde{A})$ *of the operator* $\widetilde{A} = A + bq^*$ *is* $\{\widetilde{\lambda}_n\}_{n=1}^{\infty}$ *and, moreover, the corresponding eigenvectors* $\widetilde{A}\psi_n = \widetilde{\lambda}_n\psi_n$, *constitute a Riesz basis.*

P r o o f. Consider the vectors f_n defined by (1.5). From i) we get

$$\|\phi_n - f_n\|^2 \le \left\|\frac{\lambda_n-\widetilde{\lambda}_n}{b_n}\sum_{k\neq n}^{\infty}\frac{b_k}{\lambda_k-\widetilde{\lambda}_n}\phi_k\right\|^2 \le \left|\frac{\lambda_n-\widetilde{\lambda}_n}{b_n}\right|^2 C_\sigma^{-2}\,\|b\|^2\,.$$

Hence ii) brings $\sum_{n=1}^{\infty}\|\phi_n-f_n\|^2 < 1$ and then $\{f_n\}_{n=1}^{\infty}$ is a Riesz basis by [19, Theorem V.2.20]. That gives that the equations $\langle f_n,q\rangle = c_n$, $n\in N$ have a unique solution $q\in H$ iff $\sum_{n=1}^{\infty}|c_n|^2<\infty$. In our case, since $\sum_{n=1}^{\infty}|\lambda_n-\widetilde{\lambda}_n|^2\cdot|b_n|^{-2}<\infty$, then the infinite system of equations

$$\langle f_n,q\rangle = -\frac{\lambda_n-\widetilde{\lambda}_n}{b_n}, \qquad n\in N$$

has a unique solution $q\in H$. Then $\langle R_{\widetilde{\lambda}_k}(A)b,q\rangle = -1$, $n\in N$. Since, in addition, $\{f_n\}_{n=1}^{\infty}$ is a Riesz basis, then the proof is complete.

2. Strong stability of a closed-loop system

Theorem 3. *Let the system* $\dot{x} = (A + bq^*)x$ *be strongly asymptotically stable and* $\|b\| \cdot \|q\| < C_\sigma/2$. *Then there exists a Hilbert norm* $\| \cdot \|_F = \langle F\cdot, \cdot \rangle^{1/2}$ *with positive definite* F *such that the operator* $A + bq^*$ *is dissipative in this norm. Moreover, if* $\langle F\cdot, \cdot\rangle^{1/2}$ *is such a norm, then*
a) *for any solution* $x(t)$ *of* $\dot{x} = (A + bq^*)x = \widetilde{A}x$ *one has*

$$\frac{d}{dt}\|x(t)\|_F^2 = \frac{d}{dt}\langle Fx(t), x(t)\rangle = -\langle W_1 x(t), x(t)\rangle,$$

where W_1 *is a nonnegative compact operator:* $W_1 = \sum_{i=1}^{\infty} \mu_i \omega_i \omega_i^*$, $\{\omega_i\}_{i=1}^{\infty}$ *is an orthonormal basis of eigenvectors corresponding to eigenvalues* $\mu_i \geq 0, i \in N$, *such that*

(i) $\sum_{i=1}^{\infty} \mu_i < \infty$,

(ii) $\exists C_1, C_2 > 0$ *such that for any normed eigenelement* ψ_i *of* $\widetilde{A}$, $\widetilde{A}\psi_i = \widetilde{\lambda}_i \psi_i$, $i \in N$, *the following estimate holds:*

$$C_1 \leq \sum_{k=1}^{\infty} \frac{\mu_k |\langle \omega_k, \psi_i\rangle|^2}{|Re\,\widetilde{\lambda}_i|} \leq C_2;$$

b) *the operator* F *can be found as*

$$F = \int_0^\infty \sum_{k=1}^{\infty} \mu_k e^{\widetilde{A}^* t} \omega_k \omega_k^* e^{\widetilde{A}t}\, dt, \tag{2.1}$$

where the integral is convergent in the weak sense:

$$\langle Fx, y\rangle = \lim_{T\to\infty} \int_0^T \sum_{k=1}^{\infty} \mu_k \langle e^{\widetilde{A}t}x, \omega_k\rangle \langle \omega_k, e^{\widetilde{A}t}y\rangle\, dt, \quad x, y \in H.$$

P r o o f. Let us prove item **a**). By Theorem 1 we have that the eigenvectors ψ_k of the operator $\widetilde{A} \equiv A + bq^*$ constitute a Riesz basis. Hence, there exists [20] a bounded invertible operator F_1 which maps the basis ψ_k into an orthonormal basis, i.e., $\langle F_1\psi_i, F_1\psi_k\rangle = \delta_{ik}$. It is easy to deduce from the stability of the system that the norm $\langle Fx(t), x(t)\rangle$ is not increasing for any solution $x(t)$ of the equation $\dot{x} = \widetilde{A}x$, where $F \equiv F_1^* F_1 > 0$. So we have $d/dt\langle Fx(t), x(t)\rangle \leq 0$. More precisely,

$$\frac{d}{dt}\langle Fx, x\rangle = \langle (FA + Fbq^* - AF + qb^*F)x, x\rangle = -\langle W_1 x, x\rangle,$$

here we denote $W_1 \equiv -FA - Fbq^* + AF - qb^*F \geq 0$.

Let us prove that W_1 is compact. By the definition of operator W_1 we have $W \equiv AF - FA = W_1 + Fbq^* + qb^*F$. Since $Fbq^* + qb^*F$ is a self-adjoint two-dimensional operator and $W_1 \geq 0$, we can rewrite W in the form $W = W_2 - \lambda_- x_- x_-^*$ where $W_2 \geq 0$ and x_- is the eigenvector of $Fbq^* + qb^*F$ corresponding to the negative eigenvalue $-\lambda_- < 0$. We note that for any eigenvector ϕ_i of A one has $\langle (FA - AF)\phi_i, \phi_i\rangle = 0$. Hence $\langle W\phi_i, \phi_i\rangle = \langle W_2\phi_i, \phi_i\rangle - \lambda_-|\langle x_-, \phi_i\rangle|^2 = 0$, and we can write $\langle W_2\phi_i, \phi_i\rangle = \lambda_-|\langle x_-, \phi_i\rangle|^2 \equiv \alpha_i^2, \alpha_i \geq 0$. Further we use the nonnegativeness of W_2 to define $W_2^{1/2}$ and get $\|W_2^{1/2}\phi_i\| = \alpha_i \geq 0$. Since $\{\phi_i\}_{i=1}^{\infty}$ is an orthonormal basis in H we have $\{\alpha_i\}_{i=1}^{\infty} \in \ell^2$. Using this property of the operator $W_2^{1/2}$ and the standard proof of the complete boundedness of "Hilbert cube" (see, e.g., [21]), we get that $W_2^{1/2}$ is a compact operator. Hence, $W_2 = W_2^{1/2}W_2^{1/2}$ is also compact together with W_1 which is a finite-dimensional perturbation of W_2.

Since W_1 is compact and nonnegative definite it can be represented as $W_1 = \sum_{k=1}^{\infty} \mu_k\omega_k\omega_k^*$. This yields for the eigenelements ψ_i of $\widetilde{A}$:

$$\langle (F\widetilde{A} + \widetilde{A}F)\psi_i, \psi_i\rangle = \langle F\widetilde{\lambda}_i\psi_i, \psi_i\rangle + \langle F\psi_i, \widetilde{\lambda}_i\psi_i\rangle$$

$$= 2Re\widetilde{\lambda}_i\langle F\psi_i, \psi_i\rangle = -\langle W_1\psi_i, \psi_i\rangle = -\sum_{k=1}^{\infty}\mu_k|\langle\omega_k, \psi_i\rangle|^2.$$

Then taking into account the existence of such constants $C_1, C_2 > 0$ that $C_1/2 \leq \langle F\psi_i, \psi_i\rangle \leq C_2/2$, $i \in N$, and since $Re\,\widetilde{\lambda}_i = -|Re\,\widetilde{\lambda}_i|$ we obtain property (ii). To prove (i) one can observe

$$\sum_{k=1}^{\infty}\mu_k = \sum_{k=1}^{\infty}\mu_k\sum_{i=1}^{\infty}|\langle\omega_k, \psi_i\rangle|^2 = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\mu_k|\langle\omega_k, \psi_i\rangle|^2$$

$$\leq C_2\sum_{i=1}^{\infty}|Re\,\widetilde{\lambda}_i| \leq C_2\sum_{i=1}^{\infty}|\widetilde{\lambda}_i - \lambda_i| \leq C_2 \cdot C_\sigma < \infty.$$

Remark 2. *The property (i) together with $\mu_i \geq 0$ mean that W_1 is a kernal operator (for the definition and properties see, e.g., [20]).*

Let us prove item **b)**. First note that since $Re\,\widetilde{\lambda}_i < 0$, $i \in N$, then the operator F given by (b.3) is convergent in the weak sense. Next observe that this operator satisfies

$$\langle (F\widetilde{A} + \widetilde{A}^*F)x, y\rangle = -\langle W_1x, y\rangle, \qquad x, y \in H. \tag{2.2}$$

It remains to prove that F is the unique operator satisfying (2.2). Conversely, if F_1 is another solution of (2.2), then $\langle ((F - F_1)\widetilde{A} + \widetilde{A}^*(F - F_1))x, y\rangle = 0$, $x, y \in H$. For $x = \psi_i$, $y = \psi_j$, $i, j \in N$, this yields $\langle ((F - F_1)\psi_i, \psi_j\rangle = 0$, $i, j \in N$, that immediately implies $F = F_1$. The proof of Theorem 3 is complete.

Theorem 4. *Let the system $\dot{x} = (A + bq^*)x$ be strongly asymptotically stable and $\|b\| \cdot \|q\| < C_\sigma/4$. Let W_1 be any nonnegative compact operator satisfying properties (i), (ii) from Theorem 3 and F be defined by (2.1).*

Then F is positive definite and bounded and, therefore, $\| \cdot \|_F = \langle F\cdot, \cdot\rangle^{1/2}$ is an equivalent norm in which $\widetilde{A} = A + bq^*$ is dissipative.

P r o o f. Let $\{\psi'_n\}_{n=1}^{\infty}$ be the biorthogonal normed basis to the Riesz basis $\{\psi_n\}_{n=1}^{\infty}$ of normed eigenvectors of $\widetilde{A}$, $\widetilde{A}\psi_n = \widetilde{\lambda}_n\psi_n$, $n \in N$ (see Theorem 1). Then for the operator F given by (2.1) we obtain

$$\langle Fx, x\rangle = \sum_{k=1}^{\infty} \mu_k \int_0^{\infty} \langle \sum_{i=1}^{\infty} e^{\widetilde{\lambda}_i t} \langle x, \psi'_i\rangle \psi_i, \omega_k\rangle \langle \omega_k, \sum_{j=1}^{\infty} e^{\overline{\widetilde{\lambda}_j} t} \langle x, \psi'_j\rangle \psi_j\rangle dt$$

$$= - \sum_{i,j=1}^{\infty} \frac{1}{\widetilde{\lambda}_i + \overline{\widetilde{\lambda}_j}} \langle x, \psi'_i\rangle \langle \psi'_j, x\rangle \sum_{k=1}^{\infty} \mu_k \langle \psi_i, \omega_k\rangle \langle \omega_k, \psi_j\rangle$$

$$= \sum_{i=1}^{\infty} \frac{1}{2|Re\widetilde{\lambda}_i|} |\langle x, \psi'_i\rangle|^2 \sum_{k=1}^{\infty} \mu_k |\langle \psi_i, \omega_k\rangle|^2$$

$$- \sum_{\substack{i,j=1 \\ i\neq j}}^{\infty} \frac{\sqrt{|Re\widetilde{\lambda}_i|}\sqrt{|Re\widetilde{\lambda}_j|}}{\widetilde{\lambda}_i + \overline{\widetilde{\lambda}_j}} \frac{\langle x, \psi'_i\rangle \langle \psi'_j, x\rangle}{\sqrt{|Re\widetilde{\lambda}_i|}\sqrt{|Re\widetilde{\lambda}_j|}} \sum_{k=1}^{\infty} \mu_k \langle \psi_i, \omega_k\rangle \langle \omega_k, \psi_j\rangle.$$

This yields the estimate

$$|\langle Fx, x\rangle - \frac{1}{2} \sum_{i=1}^{\infty} y_i^2| \leq$$

$$\sum_{\substack{i,j=1 \\ i\neq j}}^{\infty} \frac{\sqrt{|Re\widetilde{\lambda}_i|}\sqrt{|Re\widetilde{\lambda}_j|}}{|\widetilde{\lambda}_i + \overline{\widetilde{\lambda}_j}|} \frac{|\langle x, \psi'_i\rangle||\langle \psi'_j, x\rangle|}{\sqrt{|Re\widetilde{\lambda}_i|}\sqrt{|Re\widetilde{\lambda}_j|}} \sqrt{\sum_{k=1}^{\infty} \mu_k |\langle \psi_i, \omega_k\rangle|^2} \sqrt{\sum_{k=1}^{\infty} \mu_k |\langle \psi_j, \omega_k\rangle|^2}$$

$$= \sum_{\substack{i,j=1 \\ i\neq j}}^{\infty} \frac{\sqrt{|Re\widetilde{\lambda}_i|}\sqrt{|Re\widetilde{\lambda}_j|}}{|\widetilde{\lambda}_i + \overline{\widetilde{\lambda}_j}|} y_i y_j, \tag{2.3}$$

where $y_i = |\langle x, \psi'_i\rangle| \left(\sqrt{|Re\widetilde{\lambda}_i|}\right)^{-1} \sqrt{\sum_{k=1}^{\infty} \mu_k |\langle \psi_i, \omega_k\rangle|^2}$.

Taking into account property (ii) of the operator W_1 and the basis property of $\{\psi_i\}_{i=1}^{\infty}$, we get

$$C'_1 \, ||x||^2 \leq ||y||^2 = \sum_{i=1}^{\infty} y_i^2 \leq C'_2 \, ||x||^2, \qquad C'_1, C'_2 > 0. \tag{2.4}$$

On the other hand, (1.6) and (1.7) imply

$$\sum_{\substack{i,j=1\\ i\neq j}}^{\infty} \frac{\sqrt{|Re\widetilde{\lambda}_i|}\sqrt{|Re\widetilde{\lambda}_j|}}{|\widetilde{\lambda}_i+\overline{\widetilde{\lambda}_j}|} y_i y_j < C_\sigma^{-1} \sum_{i,j=1}^{\infty} \left(\sqrt{|Re\widetilde{\lambda}_i|}\, y_i\right)\left(\sqrt{|Re\widetilde{\lambda}_j|}\, y_j\right)$$

$$= C_\sigma^{-1}\left(\sum_{i=1}^{\infty}\sqrt{|Re\widetilde{\lambda}_i|}\, y_i\right)^2 \le C_\sigma^{-1}\left(\sum_{i=1}^{\infty}|Re\widetilde{\lambda}_i|\right)||y||^2$$

$$= C_\sigma^{-1}\left(\sum_{i=1}^{\infty}|\lambda_i-\widetilde{\lambda}_i|\right)||y||^2 = \gamma||y||^2, \qquad (2.5)$$

where $\gamma < 1/2$. Finally, from (2.3)–(2.5) we conclude

$$(\frac{1}{2}-\gamma)||x||^2 \le (\frac{1}{2}-\gamma)||y||^2 \le \langle Fx,x\rangle \le (\frac{1}{2}+\gamma)||y||^2 \le (\frac{1}{2}+\gamma)||x||^2$$

which yields boundedness and positive definiteness of F. From (2.2) for any solution $x(t)$ of $\dot{x} = (A+bq^*)x$ we get the inequality

$$\frac{d}{dt}\langle Fx(t),x(t)\rangle = \langle (F\widetilde{A}+\widetilde{A}^*F)x(t),x(t)\rangle = -\langle W_1x(t),x(t)\rangle \le 0.$$

Thus the norm $||\cdot||_F = \langle F\cdot,\cdot\rangle^{1/2}$ brings the dissipativity for $\widetilde{A}$. The proof is complete.

Summarizing Theorems 1, 3, 4, we obtain the following development of Theorem 0.1 in the examined case:

Theorem 5. *Let $||b||\cdot||q|| < C_\sigma/2$. Then the following assertions are equivalent:*
i) *the equation $\dot{x} = (A+bq^*)x$ is strongly asymptotically stable;*
ii) *all the eigenvalues $\widetilde{\lambda}_n$ of $\widetilde{A} = A+bq^*$ are of negative real part: $Re\,\widetilde{\lambda}_n < 0$, $n\in N$;*
iii) *operator $\widetilde{A}$ is dissipative in some Hilbert norm and has no pure imaginary eigenvalues.*

Under the assumption $||b||\cdot||q|| < C_\sigma/4$ the set of norms from iii) is completely described by Theorem 4.

P r o o f. i) → ii) is obvious if we take into account Theorem 1.

ii) → iii). Let $Re\,\widetilde{\lambda}_n < 0$, $n \in N$. Since eigenelements $\{\psi_n\}_{n=1}^{\infty}$ of $\widetilde{A}$ constitute a Riesz basis (Theorem 1) then the operator F_1 given by $F_1\psi_n = \phi_n$, $n\in N$, where $\{\phi_n\}_{n=1}^{\infty}$ is an orthonormal basis in H, is bounded and invertible. Therefore, if $x(t)$ is a solution of $\dot{x} = \widetilde{A}x$, $x(0) = x^0$ then

$$\langle F_1^*F_1x(t),x(t)\rangle = \sum_{i=1}^{\infty} e^{2Re\,\widetilde{\lambda}_i t}|\langle x^0,\psi_i'\rangle|^2$$

is a decreasing function of t. Thus $\widetilde{A}$ is dissipative in the norm $||\cdot||_F = \langle F\cdot,\cdot\rangle^{1/2}$, $F = F_1^*F_1$ and, as a consequence, the equation $\dot{x} = \widetilde{A}x$ is strongly asymptotically stable by Theorem 0.2.

iii) $\to$ i) follows immediately from Theorem 0.2. That completes the proof.

3. Robustness of a stabilizing control

Consider an asymptotically stable system $\dot{x} = (A + bq^*)x$ and a Hilbert norm $\langle F\cdot,\cdot\rangle^{1/2}$ from Theorem 3. With a feedback control $u(x) = q^*x + p^*x$, i.e. $u(x) = \langle x, q+p\rangle$ the system $\dot{x} = Ax + bu$ takes the following form:

$$\dot{x}(t) = (A + bq^* + bp^*)x \equiv (\widetilde{A} + bp^*)x, \quad b,q,p \in H, \quad t \geq 0. \tag{3.1}$$

Consider an arbitrary finite or infinite orthonormal system $\{\omega_i\}_{i=1}^N \subset H$ and $\{\mu_i\}_{i=1}^N \subset \ell_1$; $\mu_i \geq 0$, $N \leq \infty$. Define a compact operator $W_0 = \sum_{i=1}^N \mu_i\omega_i\omega_i^* \geq 0$. For any positive δ and stable operator $\widetilde{A} \equiv A + bq^*$ we can consider Lyapunov equation with the right hand side $-W_0$:

$$F[D + (1-\delta)I\,]\widetilde{A} + \widetilde{A}^*F[D + (1-\delta)I\,] = -W_0. \tag{3.2}$$

and its unique operator solution $F[D + (1-\delta)I] \geq 0$. This means that for any $x \in H$ one has

$$\langle F(D+I)x, x\rangle \geq \delta\langle Fx, x\rangle \geq C_1\|x\|^2. \tag{3.3}$$

We denote by $\lambda_\pm$ and $x_\pm$ the eigenvalues and eigenvectors of the two-dimensional self-adjoint operator $R_2 \equiv F(I+D)bp^* + pb^*F(I+D)$. They are given by $\lambda_\pm = \langle F(I+D)b, p\rangle \pm \|F(I+D)b\| \cdot \|p\|$, $\lambda_+ \geq 0$, $\lambda_- \leq 0$,

$$x_\pm = F(I+D)b\|p\| \pm p\|F(I+D)b\|, \quad \langle x_+, x_-\rangle = 0. \tag{3.4}$$

Theorem 6. *Let the system $\dot{x} = (A + bq^*)x$ be strongly asymptotically stable and $\|b\| \cdot \|q\| < C_\sigma/2$. Then for any vector p for which there exist a finite or infinite orthonormal system $\{\omega_i\}_{i=1}^{N(p)} \subset H$ and $\{\mu_i\}_{i=1}^{N(p)} \subset \ell_1$; $\mu_i \geq 0$ such that*

$$p, F(I+D)b \in Span\{\omega_i\}_{i=1}^{N(p)} \quad \text{and} \quad \lambda_+|\langle x_+, \omega_i\rangle| < \mu_i||x_+||, \quad i = 1,\ldots,N(p),$$

the system $\dot{x} = Ax + bu$ is asymptotically stabilizable with the aid of the control $u(x) = \langle x, q+p\rangle$. Here the vector x_+ is defined by (3.4).

Remark 3. *Let us note that under the conditions of Theorem 6 on vectors b, q and p we in fact show (c.f. Theorem 3), that for any solution of the system $d/dt\langle\tilde{F}x(t), x(t)\rangle = -\langle\tilde{W}x(t), x(t)\rangle$, where $\tilde{W}$ is a nonnegative kernal operator.*

P r o o f o f T h e o r e m 6. Our goal is to find a new norm $\|\cdot\|_1$ which satisfies $\|\cdot\|_1 \geq C\|\cdot\|$ and $\|x(t)\|_1 \leq \|x(0)\|_1$, $t \geq 0$ for all solutions of (3.1). We wish the new norm to be a perturbation of the stable one: $\|x\|_1^2 = \langle F(D+I)x, x\rangle$, with a bounded self-adjoint operator D (see (3.2)). An easy calculation shows that on a solution of (3.1) one has

$$\frac{d}{dt}\|x(t)\|_1^2 = \langle (F(D+I)\widetilde{A}+\widetilde{A}^*F(D+I)+F(D+I)bp^*+pb^*F(D+I))x, x\rangle. \quad (3.5)$$

Using the definition of the operator R_2, we deduce from (3.5) for any positive δ that

$$\frac{d}{dt}\|x(t)\|_1^2 = \langle (F[D+(1-\delta)I\,]\widetilde{A}+\widetilde{A}^*F[D+(1-\delta)I\,]+\delta(F\widetilde{A}+\widetilde{A}^*F)+R_2)x, x\rangle. \quad (3.6)$$

Consider the compact nonnegative operator $W_\delta = \sum_{i=1}^{N(p)} \mu_i\omega_i\omega_i^* \geq 0$, constructed by $\{\omega_i\}_{i=1}^{N(p)}$, $\{\mu_i\}_{i=1}^{N(p)}$ and Lyapunov equation (3.2) with the right hand side $-W_\delta$. Theorem 3 implies $F\widetilde{A} + \widetilde{A}^*F = -W_1 \geq 0$. Now, using (3.2) and the definitions of operators $-W_\delta$ and R_2, we get from (3.6)

$$\frac{d}{dt}\|x(t)\|_1^2 = \langle [-W_\delta - \delta\cdot W_1 + R_2]x, x\rangle. \quad (3.7)$$

It is easy to see that the condition $\lambda_+|\langle x_+, \omega_i\rangle| < \mu_i||x_+||$, $i = 1,\ldots,N(p)$, is sufficient for the right hand side of (3.7) to be nonpositive.

To apply Theorem 0.1 it is sufficient to prove that $A + bq^* + bp^*$ has no imaginary eigenvalues. Assume the opposite i.e., let there exists $\lambda \in i\mathbf{R}$ such that $(A + bq^* + bp^*)\hat{x} = \lambda\hat{x}$. It is easy to deduce from (3.7) that $\langle[-W_\delta - \delta\cdot W_1 + R_2]\hat{x}, \hat{x}\rangle = 0$. If $p, F(I+D)b \in Span\{\omega_i\}_{i=1}^{N(p)}$ then $\hat{x}$ is orthogonal to $Span\{\omega_i\}_{i=1}^{N(p)}$. Hence $\langle\hat{x}, p\rangle = 0$ which implies $(A + bq^*)\hat{x} = \lambda\hat{x}$. This contradicts the asymptotic stability of $A + bq^*$ since λ is a pure imaginary eigenvalue.

It completes the proof of Theorem 6.

We can find conditions which guarantee the equivalence of the norms $\|\cdot\|_1$ and $\|\cdot\|$. For fixed vectors b and q we consider the normal basis of eigenfunctions $\{\psi_i\}_{i=1}^\infty$ and the sequence of eigenvalues $\{\widetilde{\lambda}_i\}_{i=1}^\infty$ of the operator $\widetilde{A} \equiv A + bq^*$ (see Theorem 1) and define the set

$$W \equiv \{w \in H : \frac{|\langle w, \psi_i\rangle|}{|\mathrm{Re}\ \widetilde{\lambda}_i|} \leq \frac{C}{i^\alpha},\ \text{ for some } \alpha > 1, C > 0;\ i = 1, 2, \ldots\}. \quad (3.8)$$

Remark 4. *The condition $w \in W$ means that the coefficients $\langle w, \psi_i\rangle$ decay in some sense more quickly than the coefficients $\langle b, \phi_i\rangle$, as $i \to \infty$ (see Theorem 1).*

Since $F[D+(1-\delta)I]$ is the solution of Lyapunov equation (3.2), then

$$F[D+(1-\delta)I] = \int\limits_0^\infty e^{\widetilde{A}^* t} W_\delta e^{\widetilde{A}t} x\, dt = \sum_{k=1}^\infty \mu_k \int\limits_0^\infty e^{\widetilde{A}^* t} w_k w_k^* e^{\widetilde{A}t}\, dt. \tag{3.9}$$

It is easy to see that if we assume $w_k \in W$, then (3.9) gives

$$\langle (F[D+(1-\delta)I])x, x\rangle \le \|x\|^2 \sum_{k=1}^\infty \mu_k \cdot \left[\sum_{i,j=1}^\infty \frac{|\langle w_k, \psi_i\rangle \langle w_k, \psi_j\rangle|}{|\operatorname{Re} \widetilde{\lambda}_i| |\operatorname{Re} \widetilde{\lambda}_j|} \right] \le C\|x\|^2 \sum_{k=1}^\infty \mu_k.$$

Assumptions $\{\mu_i\}_{i=1}^{N(p)} \subset \ell_1$ and $\mu_i \ge 0$ imply that $\|x\|_1 \le C\|x\|$. This together with (3.3) gives the equivalence of the norms $\|\cdot\|_1$ and $\|\cdot\|$.

4. Determination of strong stabilizing controls

Let us denote by K_n^- the intersection of the open left half plane and the ring K_n defined by (1.3). Then, using Lemma 1, we deduce that the system $\dot{x} = (A + bq^*)x$ is strongly asymptotically stable if and only if equation (1.2) has exactly N roots in each set $\cup_{n=1}^N K_n^-$, $N = 1, 2, \ldots$.

We rewrite (1.2) in the form (see also (1.1)): $g(\lambda) = \widetilde{g}_1^N(\lambda) + \widetilde{g}_2^N(\lambda) = 0$, where

$$\widetilde{g}_1^N(\lambda) \equiv \sum_{n=1}^N \frac{\langle b, \phi_n\rangle \langle \phi_n, q\rangle}{\lambda_n - \lambda} + 1, \quad \widetilde{g}_1^N(\lambda) \equiv \sum_{n=N+1} \frac{\langle b, \phi_n\rangle \langle \phi_n, q\rangle}{\lambda_n - \lambda}, \quad n \in \mathbf{N}.$$

Theorem 7. *Assume that there exists $\gamma_1 \in (0,1)$ such that for any $n \in \mathbf{N}$ one has*

$$|Re\,(\langle b, \phi_n\rangle \langle \phi_n, q\rangle)| > \gamma_1 |\langle b, \phi_n\rangle \langle \phi_n, q\rangle|. \tag{4.1}$$

Let also $||b|| \cdot ||q|| < \gamma_1 C_\sigma/4$. Then the control $u(x) = \langle x, q\rangle$ brings the strong asymptotic stability if and only if for any $n \in \mathbf{N}$ the function $\widetilde{g}_1^N(\lambda)$ has no roots in the closed right half plane.

Remark 5. *For any fixed N the question on the lack of the roots in the closed right half plane for $\widetilde{g}_1^N(\lambda)$ can be solved by the standard Routh–Hurvitz method (see, e.g., [17]).*

Remark 6. *The property* (4.1) *is satisfied, for example, for elements $q \in H$ which are, in some sense, close to $\pm b$. More precisely, consider vector $q = \pm b + \alpha$. We denote for short $b_n \equiv \langle b, \phi_n\rangle$, $q_n \equiv \langle q, \phi_n\rangle$, $\alpha_n \equiv \langle \alpha, \phi_n\rangle$. It is easy to deduce for any $z \in \mathbf{C}$ that $|Re\, z| > \gamma_1|z|$ holds iff $|Re\, z| > \gamma_1(1-\gamma_1^2)^{-1/2}|Im\, z|$. Let us fix n, assume $|\alpha_n| \le |b_n|$ and denote $z \equiv b_n\overline{q_n}$. Using $|Re\, z| \ge |b_n|^2 - |b_n\overline{\alpha_n}| =$*

$|b_n|(|b_n|-|\alpha_n|)$ *and* $|Im\, z| = |Im\,(b_n\overline{\alpha_n})| \le |b_n||\alpha_n|$ *we easily get that* $|b_n|-|\alpha_n| > \gamma_1(1-\gamma_1^2)^{-1/2}|\alpha_n|$ *implies (4.1). So we obtain*

$$|\alpha_n| < C_{\gamma_1}|b_n|, \quad C_{\gamma_1} \equiv \frac{\sqrt{1-\gamma_1^2}}{\gamma_1+\sqrt{1-\gamma_1^2}}, \quad n = 1, 2, ...$$

as a sufficient condition for $q = \pm b + \alpha$ *to satisfy (4.1).*

P r o o f o f T h e o r e m 7. Our goal is to prove that for any $n \in \mathbf{N}$ and $\lambda \in \partial(\cup_{n=1}^N K_n^-)$ one has

$$|\widetilde{g}_1^N(\lambda)| \ge |\widetilde{g}_2^N(\lambda)|. \tag{4.2}$$

The assumption $||b|| \cdot ||q|| < \gamma_1 C_\sigma/4$ implies $|\widetilde{g}_2^N(\lambda)| \le \gamma_1/4$. The boundary $\partial(\cup_{n=1}^N K_n^-)$ consists of $2N$ semicircles and $2N$ intervals on the imaginary axis. Estimate (4.2) for λ belonging to the semicircles can be obtained exactly in the same way as in Lemma 1.

For $\lambda \in K_n^- \cap (i\mathbf{R})$ we first estimate $g_1^n(\lambda)$ defined by (1.4):

$$|g_1^n(\lambda)| \ge \left| Im\,\frac{\langle b,\phi_n\rangle\langle\phi_n,q\rangle}{\lambda_n-\lambda}\right| = \left| Im\,\frac{i\langle b,\phi_n\rangle\langle\phi_n,q\rangle}{Im\,(\lambda_n-\lambda)}\right| = \left|\frac{Re\,\langle b,\phi_n\rangle\langle\phi_n,q\rangle}{Im\,(\lambda_n-\lambda)}\right| \ge \frac{\gamma_1}{2}.$$

Using this, we get

$$|\widetilde{g}_1^N(\lambda)| = \left| g_1^n(\lambda) + \sum_{\substack{k=1\\k\ne n}}^N \frac{\langle b,\phi_k\rangle\langle\phi_k,q\rangle}{\lambda_k-\lambda}\right| \ge |g_1^n(\lambda)| - \frac{\gamma_1}{4} \ge \frac{\gamma_1}{2} - \frac{\gamma_1}{4} \ge \frac{\gamma_1}{4} \ge |\widetilde{g}_2^N(\lambda)|.$$

Now we apply Rouché theorem for the boundary $\partial(\cup_{n=1}^N K_n^-)$, to deduce that the function $g(\lambda)$ has the same number of roots in $\cup_{n=1}^N K_n^-$ as $\widetilde{g}_1^N(\lambda)$. The proof of Theorem 7 is complete.

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