Matematicheskaya fizika, analiz, geometriya
2004, v. 11, No. 1, p. 25–44

On the zeros of entire absolutely monotonic functions

Olga M. Katkova and Anna M. Vishnyakova

Department of Mechanics and Mathematics V.N. Karazin Kharkov National University
4 Svobody Sq., Kharkov, 61103, Ukraine
E-mail:olga.m.katkova@univer.kharkov.ua
anna.m.vishnyakova@univer.kharkov.ua

Received November 10, 2003
Communicated by I.V. Ostrovskii

By the definition, an entire absolutely monotonic function f is an entire function representable in the form $f(z) = \int_0^\infty e^{zu} P(du)$, where P is a nonnegative finite Borel measure on $\boldsymbol{R}^+$ and the integral converges absolutely for each $z \in \boldsymbol{C}$. This paper is devoted to the problem of characterization of the sets which can serve as zero sets of entire absolutely monotonic functions. We give the solution to the problem for the sets that do not intersect some angle $\{z : |\arg z - \pi| < \alpha\}$ for $\alpha > 0$.

Introduction

This paper is devoted to the problem of characterization of the sets which can serve as zero sets of entire absolutely monotonic functions. This problem was posed in [1] and has been solved for finite sets there. Here we give the solution to the problem for the sets that do not intersect some angle $\{z : |\arg z - \pi| < \alpha\}$ for $\alpha > 0$.

By the definition, an entire absolutely monotonic function f is an entire function representable in the form

$$f(z) = \int_0^\infty e^{zu} P(du),$$

where P is a nonnegative finite Borel measure on $\boldsymbol{R}^+$ and the integral converges absolutely for each $z \in \boldsymbol{C}$. By the well-known S. Bernstein's theorem [2], the class of such functions can be defined as the class of entire functions f such that

$$f^{(k)}(x) > 0, \quad \forall k \in \mathbf{N} \bigcup \{0\}, \ \forall x \in \boldsymbol{R}.$$

Mathematics Subject Classification 2000: 30D10, 30D50, 44A10.

Entire absolutely monotonic functions form a proper subclass of the class of entire functions representable in the form

$$f(z) = \int_{-\infty}^{\infty} e^{zu} P(du), \tag{1}$$

where P is a finite nonnegative Borel measure on $\boldsymbol{R}$ and the integral converges absolutely on $\boldsymbol{C}$. The zero sets corresponding to the class described by (1) were completely characterized in [3]. This characterization is the following:

Theorem A ([3]). *A set $E \subset \boldsymbol{C}$ without finite accumulation points is the zero set of a function of the form (1) iff the following conditions are satisfied:*

(a)

$$E \bigcap \boldsymbol{R} = \emptyset, \quad a \in E \Leftrightarrow \bar{a} \in E \tag{2}$$

(multiplicities of a and $\bar{a}$ are equal);

(b) for every $H > 0$

$$\log n(r, H) = o(r), \quad r \to \infty, \tag{3}$$

holds, where

$$n(r, H) := \#\{z : z \in E, |Im\ z| \le r, |Re\ z| \le H\} \tag{4}$$

(points of E are counted with their multiplicities).

Sure, zero sets of entire absolutely monotonic functions form a subclass of sets described in the above theorem. On the other hand, it is evident that entire absolutely monotonic functions form a subclass of the class of entire functions bounded in each half-plane of the kind

$$\boldsymbol{C}_\omega := \{z : \mathrm{Re}\ z \le \omega\}, \quad \omega \in \boldsymbol{R}.$$

Therefore the characterization of the zero sets of entire functions bounded in each half-plane $\boldsymbol{C}_\omega$ is of interest. The following theorem from ([4]) gives the complete characterization of these sets:

Theorem B ([4]). *A set $E = \{a_k\}_{k=1}^{\infty} \subset \boldsymbol{C}$ without finite accumulation points is the zero set of an entire function bounded in $\boldsymbol{C}_\omega$, $\forall \omega \in \boldsymbol{R}$, iff*

$$\sum_{a_k \in E \bigcap \boldsymbol{C}_\omega} \frac{|Re\ a_k| + 1}{|a_k|^2 + 1} < \infty, \quad \forall \omega \in \boldsymbol{R}. \tag{5}$$

Note that the necessity of the condition (5) is an easy consequence of the well-known Blaschke condition for a half-plane. It can be easily shown that the condition (5) implies (b) in Theorem A.

It turned out that, if we add (5) to the conditions of Theorem A, we do not obtain the complete characterization of zero sets of entire absolutely monotonic functions. In [1] it was mentioned the following necessary condition not depending on all previous ones:

$$\operatorname{dist}(x, E) \to +\infty, \quad x \to -\infty. \tag{6}$$

I.V. Ostrovskii showed (oral communication) that the following independent condition is also necessary:

$$\exists \alpha \in (0, \pi/2) : \sum_{a_k \in E \bigcap \{z: |\arg z - \pi| < \alpha\}} \operatorname{Re} \frac{1}{x - a_k} \to 0, \quad x \to -\infty. \tag{7}$$

In [5] we obtained one more independent condition for a set E to be the zero set of entire absolutely monotonic function:

$$\exists \alpha \in (0, \pi/2) : \sum_{a_k \in E \bigcap \{z: |\arg z - \pi| < \alpha\}} \operatorname{Re} \frac{1}{(x - a_k)^2} \in L^1(-\infty, -1]. \tag{8}$$

At the moment we do not know whether or not the set of conditions: (2), (5)–(8) gives a complete characterization of zero sets of entire absolutely monotonic functions.

In [4] we obtained the complete characterization of zero sets of entire absolutely monotonic functions situated in the right half-plane. It was proved that for a set $E = \{a_k\}_{k=1}^{\infty} \subset \{z : \operatorname{Re} z \geq 0\}$ without finite accumulation points conditions (2) and (5) are necessary and sufficient to be the zero set of an entire absolutely monotonic function .

The main result of the paper is the following characterization of zero sets of entire absolutely monotonic functions that do not intersect some angle $\{z : |\arg z - \pi| < \alpha\}$ for $\alpha > 0$.

Theorem 1. *Let $E = \{a_k\}_{k=1}^{\infty}$ be a set without finite accumulation points. Suppose $\exists \alpha \in (0, \pi/2]$ such that $E \cap \{z : |\arg z - \pi| < \alpha\} = \emptyset$. The set E is the zero set of an entire absolutely monotonic function iff the conditions (2) and (5) are satisfied.*

The necessity of these conditions is obvious. Note that in our case the conditions (6), (7) and (8) are satisfied automatically.

In [4] we have proved that Theorem 1 is a consequence of the following fact.

Theorem 2. *Let $E = \{a_k\}_{k=1}^{\infty}$ be a set without finite accumulation points satisfying conditions (2) and (5). Suppose $\exists\alpha \in (0, \pi/2]$ such that $E \cap \{z : |\arg z - \pi| < \alpha\} = \emptyset$. There exists an entire function $\psi_1(z)$ with zero set E representable by the absolutely convergent in $\boldsymbol{C}$ integral*

$$\psi_1(z) = \int_0^{\infty} e^{zx} p_1(x) dx, \tag{9}$$

where p_1 is a real continuous on $\boldsymbol{R}^+$ function positive on an interval $(0, x_1)$, $x_1 > 0$.

In the proof of Theorem 2 we will use an entire function bounded in each half-plane C_ω with zero set E.

1. Construction of an entire function bounded in each half-plane C_ω with zero set E

The construction below is based on an idea of I.V. Ostrovskii.

Let E be a set satisfying the conditions of Theorem B. Note that $E = E_+ \cup E_-$, where

$$E_- := E \bigcap \{z : \text{Re } z < 0\}, \; E_+ := E \bigcap \{z : \text{Re } z \geq 0\}.$$

In [4] we have proved that there exists an entire absolutely monotonic function with zero set E_+. Since the class of entire absolutely monotonic functions is closed under multiplication it remains to prove that there exists an entire absolutely monotonic function with zero set E_-. Further we will suppose that

$$E = \{a_k\}_{k=1}^{\infty}, \; \text{Re} a_k < 0.$$

There exists a sequence of positive $\delta_k \uparrow +\infty, k \uparrow +\infty$, such that

$$\sum_{k=1}^{\infty} \frac{|\text{Re } a_k| \cdot \delta_k + \delta_k^2}{|a_k|^2 + 1} < \infty. \tag{10}$$

Note that (10) implies

$$\delta_k = o(|a_k|), \quad k \to \infty. \tag{11}$$

Set

$$B(z) := \prod_{k=1}^{\infty} \frac{1 - z/a_k}{1 - z/(\delta_k - \bar{a}_k)}. \tag{12}$$

By (10) the infinite product (12) converges and is a meromorphic function. We shall show that $B(z)$ is bounded in $\boldsymbol{C}_\omega \setminus K_\omega$, $\forall \omega \in \boldsymbol{R}$, where K_ω is a compact subset of $\{z : \text{Re } z \geq 0\}$. We have

$$|B(z)|^2 = \prod_{k=1}^{\infty} \left(1 + \frac{\delta_k^2 - 2\text{Re } a_k \cdot \delta_k}{|a_k|^2}\right) \prod_{k=1}^{\infty} \left|\frac{a_k - z}{\delta_k - \bar{a}_k - z}\right|^2 =: C \prod_{k=1}^{\infty} \left|\frac{a_k - z}{\delta_k - \bar{a}_k - z}\right|^2 .$$

Here and further we will denote by C not necessary equal positive constants.

For $\delta_k > 2\omega$

$$\left|\frac{a_k - z}{\delta_k - \bar{a}_k - z}\right| \leq 1, \quad \forall z \in \boldsymbol{C}_\omega. \tag{13}$$

In particular, for $\omega = 0$ (13) holds for any $k \in \mathbf{N}$, hence

$$|B(z)|^2 \leq C, \quad \forall z \in \{\text{Re } z \leq 0\}.$$

Since $\delta_k \uparrow +\infty$, (13) holds for all $\omega > 0$ and all sufficiently large $k \geq k_0(\omega)$. So, $B(z)$ is bounded in $\boldsymbol{C}_\omega \setminus K_\omega$, $\forall \omega > 0$, where $K_\omega \subset \{z : \text{Re } z \geq 0\}$ is a compact set including points $-\bar{a}_k + \delta_k$, $k = 1, \ldots k_0(\omega) - 1$.

Let $V(z)$ be an entire function with zero set coinciding with the set of all poles of $B(z)$. Let us consider the entire function

$$f_0(z) := B(z)V(z).$$

The zero set of $f_0(z)$ coincides with E. But $f_0(z)$ is not necessary bounded in $\boldsymbol{C}_\omega$, $\forall \omega \in \boldsymbol{R}$.

Further we shall need the following theorem being a simple particular case of the well-known theorem of M.V. Keldysh (see [6]).

Theorem C. *Let $\tau(x) > 0$ be a continuous nondecreasing function on $\boldsymbol{R}^+$ such that $\tau(x) \uparrow +\infty$, $x \uparrow +\infty$. Let $g(z)$ be a function analytic in the closed domain*

$$G = \boldsymbol{C} \setminus \{z : Re\ z > 0,\ |Im\ z| < \tau(Re\ z)\}.$$

Then there exists an entire function $\Phi(z)$ such that

$$|g(z) - \Phi(z)| \leq 1, \quad \forall z \in G.$$

Evidently, there exists a function $\tau(x)$ satisfying the conditions of Theorem C and such that the corresponding domain G is free of zeros of $V(z)$. Applying Theorem C to $g(z) = \log V(z)$, we get the entire function $\Phi(z)$ such that

$$|\log V(z) - \Phi(z)| \leq 1, \quad \forall z \in G. \tag{14}$$

So,

$$f(z) := f_0(z) \exp(-\Phi(z)) = B(z)V(z) \exp(-\Phi(z))$$

is an entire function bounded in $\boldsymbol{C}_\omega$, $\forall \omega$, with zero set E. ■

2. Proof of Theorem 2

Let E be a set satisfying conditions of Theorem 2. Let us construct an entire function $f(z)$ bounded in $\boldsymbol{C}_\omega$, $\forall\omega \in \boldsymbol{R}$ using the method of Sect. 1:

$$f(z) = B(z)V(z)\exp(-\Phi(z)), \tag{15}$$

where the zeros of $B(z)$ lie outside the angle $\{z : |\arg z - \pi| < \alpha\}$, $\alpha \in (0, \pi/2]$. Since E is symmetric with respect to $\boldsymbol{R}$, then by (12)

$$B(z) = \prod_{k=1}^{\infty}\left(1 + \frac{2|\mathrm{Re}\ \ a_k| \cdot \delta_k + \delta_k^2}{|a_k|^2}\right) \cdot \prod_{k=1}^{\infty} \frac{(a_k - z)(\bar{a}_k - z)}{(-a_k + \delta_k - z)(-\bar{a}_k + \delta_k - z)}. \tag{16}$$

In what follows we shall need estimations of $\log B(-r)$ and its derivatives. To write them, we introduce some notations.

Let us fix $0 < \beta < 1$ and set

$$q(r) := \frac{1}{r^{1+\beta}} + \sum_{k=1}^{\infty}\Big(\frac{2r(2|\mathrm{Re}\ a_k| + \delta_k)}{(|a_k|^2 + r^2)^2} + \frac{2|\mathrm{Re}\ a_k|\delta_k + \delta_k^2}{|a_k|} \cdot \frac{1}{|a_k|^2 + r^2}\Big). \tag{17}$$

By (10), (11) series in the right-hand side of (17) converge uniformly with respect to r on each compact subset of $\boldsymbol{R}^+$ and $q(r) \to 0$, as $r \to \infty$. Let

$$Q(r) := \int_r^{\infty} q(t)dt = \frac{1}{\beta} \cdot \frac{1}{r^\beta}$$
$$+ \sum_{k=1}^{\infty}\left(\frac{2|\mathrm{Re}\ a_k| + \delta_k}{|a_k|^2 + r^2} + \frac{2|\mathrm{Re}\ a_k|\delta_k + \delta_k^2}{|a_k|^2}\left(\frac{\pi}{2} - \arctan\frac{r}{|a_k|}\right)\right), \quad r \geq 1.$$

Note that $Q(r) \downarrow 0$, as $r \uparrow \infty$.

Lemma 1. *The following estimations hold:*

$$|(\log B(z))'|_{z=-r} \leq CQ(r), \quad r \geq 1; \tag{18}$$
$$\left|\left(\frac{d^j}{dz^j}\log B(z)\right)\right|_{z=-r} \leq C \cdot j!q(r)(Cr)^{2-j}, \quad j \geq 2,\ r \geq 1; \tag{19}$$
$$\log(|B(-r+iy)|/B(-r)) \leq Cq(r)y^2, \quad y \in \boldsymbol{R},\ r \geq 1; \tag{20}$$
$$\log(|B(-r+iy)|/B(-r)) \leq Cq(y)y^2, \quad y \in \boldsymbol{R}^+,\ 1 \leq r \leq y/2. \tag{21}$$

The proof of the lemma will be given in Sect. 3.

Theorem 2 is an immediate corollary of the following result.

Theorem. *Let $f(z)$ be the function defined by (15). There exists an entire function $\varphi(z)$ without zeros such that the product $\psi_1(z) := f(z)\varphi(z)$ is representable in the form*

$$\psi_1(z) = \int_0^\infty e^{zx} p_1(x)dx,$$

where p_1 is a real continuous function on $\boldsymbol{R}^+$ positive on some interval $(0, x_1)$, $x_1 > 0$.

Let

$$\Delta(t) := \left(-\frac{q(t)}{t}\right)' t^2 = \frac{2+\beta}{t^{1+\beta}}$$
$$+\sum_{k=1}^\infty \left(\frac{8(2|\mathrm{Re}\ a_k|+\delta_k)t^3}{(t^2+|a_k|^2)^3} + \frac{2|\mathrm{Re}\ a_k|+\delta_k}{|a_k|}\cdot\frac{1}{t^2+|a_k|^2} + \frac{2|\ \mathrm{Re}\ a_k|+\delta_k}{|a_k|}\cdot\frac{2t^2}{(t^2+|a_k|^2)^2}\right),$$
$$t \geq 1.$$

Lemma 2. *The function $\Delta(t)$ possess the following properties:*

(a) $\Delta(t) > 0, \quad t \geq 1;$

(b) $\int^\infty \Delta(t)dt < \infty;$

(c) $\Delta(t)t \to 0$, *as* $t \to +\infty;$

(d) $\Delta(t)t^3 \uparrow +\infty$, *as* $t \uparrow +\infty;$

(e) $\Delta(t) \leq 4q(t), \quad t \geq 1.$

Lemma 2 can be proved by easy estimations. Set

$$h(z) := \int_0^A (e^{tz}-1)\frac{\Delta(1/t)}{t^3}dt, \quad 0 < A \leq 1. \tag{22}$$

Since Lemma 2 (b), the integral in the right-hand side of (22) is absolutely convergent and $h(z)$ is an entire function. Since

$$h^{(k)}(x) > 0, \quad \forall k \in \mathbf{N}\bigcup\{0\},\ \forall x \in \boldsymbol{R},$$

the function $h(z)$ is entire absolutely monotonic. It is easy to see that

$$\mathrm{Re}\ h(x+iy) \leq h(x), \quad \forall x, y \in \boldsymbol{R}. \tag{23}$$

Let $\varphi_\alpha(z),\ \alpha \in (0,1)$, be the entire function defined by

$$\varphi_\alpha(z) := \exp\int_0^1 (e^{zt}-1)t^{-1-\alpha}dt.$$

Note that $\log\varphi_\alpha(z)$ is a particular case of (22) corresponding to $\Delta = \Delta_\alpha = t^{\alpha-2}$, $A = 1$.

We need the following Lemma from ([4]).

Lemma 3. *For a fixed ω, the following asymptotic equality holds in the half-plane $\boldsymbol{C}_\omega$:*

$$\log\varphi_\alpha(z) = -C_\alpha|z|^\alpha e^{i\alpha(\arg z-\pi)} + O(1), \tag{24}$$
$$\pi/2 < \arg z < 3\pi/2, \quad |z| \to \infty,$$

where $C_\alpha > 0$ does not depend on z.

Set

$$\begin{aligned}\psi_1(z) &:= f(z)\exp[M(h(z)-h(0))]\varphi_\alpha(z)\\ &= B(z)V(z)e^{-\Phi(z)}\exp[M(h(z)-h(0))]\varphi_\alpha(z),\end{aligned}$$

where the constant $M > 0$ will be chosen later. Let

$$\varphi(z) := \exp[M(h(z)-h(0))]\varphi_\alpha(z).$$

Evidently, $\varphi(z)$ is an entire absolutely monotonic function. Let

$$p_1(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\eta}\psi_1(i\eta)d\eta, \quad \eta \in \boldsymbol{R}. \tag{25}$$

Taking into account boundedness of $f(z)$ in $\boldsymbol{C}_\omega$, $\forall\omega \in \boldsymbol{R}$, and (23), (24), we see that the integral in the right-hand side of (25) converges absolutely and uniformly with respect to $x \in \boldsymbol{R}$. Using (23), (24), we can transfer the integration in (25) to the line $\{z : \mathrm{Im}\ z = \xi\}$, $\forall\xi \in \boldsymbol{R}$. Noting that

$$(\psi_1(x) \in \boldsymbol{R},\ \forall x \in \boldsymbol{R}) \Rightarrow (\psi_1(\xi+i\eta) = \overline{\psi_1(\xi-i\eta)},\ \xi,\eta \in \boldsymbol{R}),$$

we get

$$p_1(x) = \frac{e^{-x\xi}\psi_1(\xi)}{2\pi}\int_0^\infty \mathrm{Re}\left(e^{-ix\eta}\frac{\psi_1(\xi+i\eta)}{\psi_1(\xi)}\right)d\eta, \quad \forall\xi \in \boldsymbol{R}. \tag{26}$$

Hence $p_1(x) \in \boldsymbol{R}$ for $x \in \boldsymbol{R}$ and

$$\operatorname{sign} p_1(x) = \operatorname{sign} \int_0^\infty \operatorname{Re} \left(e^{-ix\eta} \frac{\psi_1(\xi + i\eta)}{\psi_1(\xi)} \right) d\eta, \quad \forall \xi \in \boldsymbol{R}. \tag{27}$$

Using (23) and (24) and putting $\xi \to -\infty$ in (26), we conclude that $p_1(x) = 0$ for $x < 0$. Taking sufficiently large positive ξ in (26), we get

$$p_1(x) = O(e^{-Cx}), \quad x \to +\infty, \ \forall C > 0.$$

Hence, by the Fourier inversion formula,

$$\psi_1(z) = \int_0^\infty e^{xz} p_1(x) dx, \quad \forall z \in \boldsymbol{C}.$$

We are going to show that $p_1(x) > 0$ on some interval $(0, x_1)$, $x_1 > 0$. For this, we represent the integral in the right-hand side of (27) in the form

$$\left(\int_0^{\varepsilon_1} + \int_{\varepsilon_1}^{\varepsilon_2} + \int_{\varepsilon_2}^{\infty} \right) \operatorname{Re} \left(e^{-ix\eta} \frac{\psi_1(\xi + i\eta)}{\psi_1(\xi)} \right) d\eta =: I_1 + I_2 + I_3, \tag{28}$$

where $\varepsilon_1 = \varepsilon_1(\xi)$, $\varepsilon_2 = \varepsilon_2(\xi)$ will be chosen later.

We shall estimate I_1 from below, $|I_2|$ and $|I_3|$ from above and are going to show that $I_1 > 0$ and $I_1 > |I_2| + |I_3|$ for $|\xi|$ being large enough.

Lemma 4. *Let*

$$\theta(r) := q(r) + \int_r^\infty \frac{q(u)}{u} du. \tag{29}$$

The following estimations hold:

$$h'(\xi) \geq \frac{2}{e} Q(|\xi|); \tag{30}$$

$$h''(\xi) \geq \frac{1}{e} q(|\xi|); \tag{31}$$

$$h'(\xi) \leq \int_{|\xi|}^\infty \Delta(u) du + C\Delta(|\xi|)|\xi|; \tag{32}$$

$$0 < h^{(j)}(\xi) \leq C \frac{j!}{|\xi|^{j-2}} \theta(|\xi|), \quad j = 2, 3, \ldots; \tag{33}$$

$$(\log \varphi_\alpha(\xi))' \geq C|\xi|^{\alpha-1}; \tag{34}$$

$$(\log \varphi_\alpha(\xi))'' \geq C|\xi|^{\alpha-2}; \tag{35}$$

$$0 < \frac{d^j}{d\xi^j} \log \varphi_\alpha(\xi) \leq C\, j!|\xi|^{\alpha-j}, \quad j = 1, 2, \ldots, \tag{36}$$

where $\xi \le \xi_0 < 0$.

Proof of Lemma 4 repeats the proof of Lemma 4 from [4].

Since the function $\log(V(\xi+z)e^{-\Phi(\xi+z)})$ is analytic in the disc $\{z : |z| < |\xi|/2\}$ for $\xi < 0$ and (14) holds, Caushy's inequality implies

$$\left|\frac{d^j}{d\xi^j}\log(V(\xi+z)e^{-\Phi(\xi+z)})\right| \le j!\frac{2^j}{|\xi|^j}\,. \tag{37}$$

Set

$$b(\xi) := \log\psi_1(\xi) = \log B(\xi) + \log\varphi_\alpha(\xi) + M(h(\xi)-h(0)) + \log(V(\xi)e^{-\Phi(\xi)}).$$

Since $f(\xi+z) \ne 0$ for $|z| < |\xi|$, $\xi < 0$, we have for $|\eta| < |\xi|/2$

$$\log\left\{e^{-ix\eta}\frac{\psi_1(\xi+i\eta)}{\psi_1(\xi)}\right\} = -ix\eta + \sum_{j=1}^{\infty}\frac{(i\eta)^j}{j!}b^{(j)}(\xi). \tag{38}$$

By (18),(30), (34) and (37),

$$\begin{aligned} b'(\xi) &= (\log B(\xi))' + (\log\varphi_\alpha(\xi))' + Mh'(\xi) + \left[\log\left(V(\xi)e^{-\Phi(\xi)}\right)\right]' \\ &\ge -CQ(|\xi|) + C|\xi|^{\alpha-1} + \frac{2M}{e}Q(|\xi|) - \frac{2}{|\xi|}\,. \end{aligned}$$

Further we shall assume that M is sufficiently large. Then, for $|\xi| > 1$

$$b'(\xi) \ge CQ(|\xi|) + C|\xi|^{\alpha-1} > 0. \tag{39}$$

On the other hand, (18), (32), (36) and (37) imply

$$b'(\xi) \le CQ(|\xi|) + C|\xi|^{\alpha-1} + \int_{|\xi|}^{\infty}\Delta(u)du + C\Delta(|\xi|)|\xi| + \frac{2}{|\xi|}.$$

Taking into account Lemma 2 (b) and (c), we conclude that

$$b'(\xi) \to 0, \quad \text{as} \quad \xi \to -\infty. \tag{40}$$

Moreover, (19), (31), (35) and (37) imply

$$\begin{aligned} b''(\xi) &= (\log B(\xi))'' + (\log\varphi_\alpha(\xi))'' + Mh''(\xi) + \left[\log\left(V(\xi)e^{-\Phi(\xi)}\right)\right]'' \\ &\ge -2Cq(|\xi|) + C|\xi|^{\alpha-2} + \frac{M}{e}q(|\xi|) - \frac{8}{|\xi|^2}\,. \end{aligned}$$

Assuming M being large enough, we shall have

$$b''(\xi) \geq 2Cq(|\xi|) + C|\xi|^{\alpha-2}, \quad |\xi| \geq 1. \tag{41}$$

From (39), (40), (41) we conclude that $b'(\xi) \downarrow 0$ as $\xi \downarrow -\infty$. Therefore the equation

$$b'(\xi) = x$$

has a unique solution $\xi(x)$ for every x, $0 \leq x \leq x_0$, such that

$$\xi(x) \downarrow -\infty, \quad \text{as} \quad x \downarrow 0. \tag{42}$$

Substituting $\xi = \xi(x)$ into (38), we get

$$\log\left\{e^{-ix\eta}\frac{\psi_1(\xi+i\eta)}{\psi_1(\xi)}\right\} = -\frac{b''(\xi)}{2}\eta^2 + \tau(\xi,\eta), \tag{43}$$

where

$$\tau(\xi,\eta) := \sum_{j=3}^{\infty}\frac{(i\eta)^j}{j!}b^{(j)}(\xi).$$

By (19), (36), (33) and (37),

$$\begin{aligned}|b^{(j)}(\xi)| \leq |(\log B(\xi))^{(j)}| + (\log\varphi_\alpha(\xi))^{(j)} \\ +Mh^{(j)}(\xi) + \left|\left[\log(V(\xi)e^{-\Phi(\xi)})\right]^{(j)}\right| \\ \leq Cj!q(|\xi|)(C|\xi|)^{2-j} + Cj!|\xi|^{\alpha-j} + Cj!|\xi|^{2-j}\theta(|\xi|) + j!\tfrac{2^j}{|\xi|^j},\end{aligned}$$

whence, using the definition of $\theta(r)$, we get

$$|b^{(j)}(\xi)| \leq Cj!(\theta(|\xi|)|\xi|^2 + |\xi|^\alpha)\left(\frac{C}{|\xi|}\right)^j, \quad j = 2,3,\dots . \tag{44}$$

For $|\eta| \leq |\xi|/4$, (44) implies

$$|\tau(\xi,\eta)| \leq C(\theta(|\xi|)|\xi|^2 + |\xi|^\alpha)\sum_{j=3}^{\infty}\left(\frac{C|\eta|}{|\xi|}\right)^j \leq K\frac{\theta(|\xi|) + |\xi|^{\alpha-2}}{|\xi|}|\eta|^3,\ K > 0.$$

We choose

$$\varepsilon_1 = \varepsilon_1(\xi) := \left(\frac{\pi}{3K}\cdot\frac{|\xi|}{\theta(|\xi|) + |\xi|^{\alpha-2}}\right)^{1/3}. \tag{45}$$

Then, for $|\eta| < \varepsilon_1$, the inequality holds

$$|\tau(\xi,\eta)| \leq \frac{\pi}{3}.$$

Using this and (43), we obtain

$$\begin{aligned} I_1 &= \int_0^{\varepsilon_1} \exp\left\{-\frac{b''(\xi)}{2}\eta^2 + \text{Re }\tau(\xi,\eta)\right\}\cos(\text{Im }\tau(\xi,\eta))d\eta \\ &\geq \frac{1}{2}e^{-\pi/3}\int_0^{\varepsilon_1} \exp\left(-\frac{b''(\xi)}{2}\eta^2\right) d\eta. \end{aligned}$$

Hence, by (44),

$$I_1 \geq \frac{1}{2}e^{-\pi/3}(\theta(|\xi|) + |\xi|^{\alpha-2})^{-1/2} \int_0^{(\theta(|\xi|)+|\xi|^{\alpha-2})^{1/2}\varepsilon_1} \exp(-Cu^2)du. \quad (46)$$

Note that (45) implies

$$(\theta(|\xi|)+|\xi|^{\alpha-2})^{1/2}\varepsilon_1 = C|\xi|^{1/3}(\theta(|\xi|)+|\xi|^{\alpha-2})^{1/6} \geq C|\xi|^{\alpha/6} \to \infty, \quad \text{as} \quad \xi \to -\infty.$$

Thus, (46) implies

$$I_1 \geq C(\theta(|\xi|) + |\xi|^{\alpha-2})^{-1/2} \to \infty, \quad \text{as} \quad \xi \to -\infty. \quad (47)$$

Set

$$\varepsilon_2 = \varepsilon_2(\xi) = 2|\xi|.$$

Evidently,

$$\varepsilon_1(\xi) = O(|\xi|^{1-\alpha/3}) < \varepsilon_2(\xi)$$

for sufficiently large $|\xi|$. We have

$$\begin{aligned} |I_2| &\leq \int_{\varepsilon_1}^{\varepsilon_2} \left|\frac{B(\xi+i\eta)}{B(\xi)}\right| \cdot \left|\frac{\varphi_\alpha(\xi+i\eta)}{\varphi_\alpha(\xi)}\right| \\ &\times \left|\frac{V(\xi+i\eta)e^{-\Phi(\xi+i\eta)}}{V(\xi)e^{-\Phi(\xi)}}\right| \exp\{M(\text{Re }h(\xi+i\eta) - h(\xi))\}d\eta. \end{aligned}$$

For sufficiently large $|\xi|$,

$$\begin{aligned} \text{Re }h(\xi+i\eta) - h(\xi) &= -2\int_0^A e^{-t|\xi|}\sin^2\frac{t\eta}{2}\Delta(\frac{1}{t})\frac{dt}{t^3} \\ &\leq -2\int_0^{1/|\xi|} e^{-t|\xi|}\sin^2\frac{t\eta}{2}\Delta(\frac{1}{t})\frac{dt}{t^3}. \end{aligned}$$

Since $t\eta/2 \le \eta/(2|\xi|) \le 1 < \pi/2$ for $\eta \le 2|\xi|$, we have

$$\mathrm{Re}\; h(\xi+i\eta) - h(\xi) \le -\frac{2}{e\pi^2}\eta^2 \int_0^{1/|\xi|} \Delta(\frac{1}{t})\frac{dt}{t}. \tag{48}$$

Integrating by parts, we get

$$\mathrm{Re}\; h(\xi+i\eta) - h(\xi) \le -\frac{2}{e\pi^2}\eta^2\theta(|\xi|). \tag{49}$$

Substituting $\Delta = \Delta_\alpha = u^{\alpha-2}$ into (48), we obtain

$$\log|\varphi_\alpha(\xi+i\eta)| - \log\varphi_\alpha(\xi) \le -\frac{2}{e\pi^2}\eta^2 \int_0^{1/|\xi|} t^{1-\alpha}dt = -\frac{2}{e\pi^2(2-\alpha)}\eta^2|\xi|^{\alpha-2}. \tag{50}$$

From (20), (49), (50), we derive

$$\begin{aligned}
&\log|\psi_1(\xi+i\eta)| - \log\psi_1(\xi)\\
&= (\log|B(\xi+i\eta)| - \log B(\xi)) + (\log|\varphi_\alpha(\xi+i\eta)|\\
&-\log\varphi_\alpha(\xi)) + M(\mathrm{Re}\; h(\xi+i\eta) - h(\xi)) + \Big[\log\Big|V(\xi+i\eta)e^{-\Phi(\xi+i\eta)}\Big|\\
&-\log\Big(V(\xi)e^{-\Phi(\xi)}\Big)\Big] \le Cq(|\xi|)\eta^2 - \frac{2}{e\pi^2(2-\alpha)}\eta^2|\xi|^{\alpha-2} - \frac{2}{e\pi^2}M\eta^2\theta(|\xi|) + C.
\end{aligned}$$

Assuming M being large enough, we get

$$\log|\psi_1(\xi+i\eta)| - \log\psi_1(\xi) \le -C(\theta(|\xi|) + |\xi|^{\alpha-2})\eta^2 + C.$$

Since $\eta \ge \varepsilon_1$, (45) implies

$$\begin{aligned}
&\log|\psi_1(\xi+i\eta)| - \log\psi_1(\xi) \le -C(\theta(|\xi|) + |\xi|^{\alpha-2})\varepsilon_1^2 + C\\
&\le -C|\xi|^{2/3}(\theta(|\xi|) + |\xi|^{\alpha-2})^{1/3} + C = -C(\theta(|\xi|)|\xi|^2 + |\xi|^{\alpha})^{1/3} + C.
\end{aligned}$$

Hence

$$|I_2| \le C|\xi|\exp[-C(\theta(|\xi|)|\xi|^2 + |\xi|^\alpha)^{1/3}] \to 0, \quad \text{as} \quad \xi \to -\infty. \tag{51}$$

We have

$$\begin{aligned}
|I_3| &\le \int_{\varepsilon_2}^{\infty} \left|\tfrac{B(\xi+i\eta)}{B(\xi)}\right| \cdot \left|\tfrac{\varphi_\alpha(\xi+i\eta)}{\varphi_\alpha(\xi)}\right| \qquad (52)\\
&\times \left|\tfrac{V(\xi+i\eta)e^{-\Phi(\xi+i\eta)}}{V(\xi)e^{-\Phi(\xi)}}\right| \exp\{M(\mathrm{Re}\; h(\xi+i\eta) - h(\xi))\}d\eta.
\end{aligned}$$

For $|\xi| > 1/(2A)$, we have $\eta \geq 2|\xi| \geq 1/A$, therefore

$$\text{Re } h(\xi+i\eta)-h(\xi) = -2\int_0^A e^{-t|\xi|}\sin^2\frac{t\eta}{2}\Delta\left(\frac{1}{t}\right)\frac{dt}{t} \leq -2\int_0^{1/\eta} e^{-t|\xi|}\sin^2\frac{t\eta}{2}\Delta\left(\frac{1}{t}\right)\frac{dt}{t^3}. \tag{53}$$

Since $t\eta/2 \leq 1/2 < \pi/2$ for $0 \leq t \leq 1/2$,

$$\text{Re } h(\xi+i\eta)-h(\xi) \leq -\frac{2\eta^2}{\pi^2}e^{-|\xi|/\eta}\int_0^{1/\eta}\Delta\left(\frac{1}{t}\right)\frac{dt}{t} \leq -\frac{2\eta^2}{\pi^2\sqrt{e}}\int_\eta^\infty \Delta(u)\frac{du}{u}.$$

Integrating by parts, we obtain

$$\text{Re } h(\xi+i\eta)-h(\xi) \leq -\frac{2}{\pi^2\sqrt{e}}\theta(\eta)\eta^2. \tag{54}$$

Since $|\xi| < \eta/2$, we derive from (21)

$$\log|B(\xi+i\eta)| - \log B(\xi) \leq Cq(\eta)\eta^2 \leq C\theta(\eta)\eta^2, \quad 1 \leq |\xi| < \frac{\eta}{2}.$$

Using 53), (54), (14) and the inequality

$$|\varphi_\alpha(\xi+i\eta)| \leq \varphi_\alpha(\xi)$$

(which is a particular case of (23)), we get

$$\begin{aligned}\left|\frac{\psi_1(\xi+i\eta)}{\psi_1(\xi)}\right| &= \left|\frac{B(\xi+i\eta)}{B(\xi)}\right|\cdot\left|\frac{\varphi_\alpha(\xi+i\eta)}{\varphi_\alpha(\xi)}\right|\cdot\left|\frac{V(\xi+i\eta)e^{-\Phi(\xi+i\eta)}}{V(\xi)e^{-\Phi(\xi)}}\right| \\ &\times \exp\{M(\text{Re } h(\xi+i\eta)-h(\xi))\} \\ &\leq C\exp\left\{C\theta(\eta)\eta^2 - \tfrac{2M}{\pi^2\sqrt{e}}\theta(\eta)\eta^2\right\}, \quad 2|\xi| < \eta < \infty.\end{aligned}$$

Assuming M being large enough and using (29), (17), we obtain

$$\left|\frac{\psi_1(\xi+i\eta)}{\psi_1(\xi)}\right| \leq C\exp(-C\theta(\eta)\eta^2) \leq C\exp(-Cq(\eta)\eta^2) \leq C\exp(-C\eta^{1-\beta}).$$

Thus,

$$|I_3| \leq C\int_{2|\xi|}^\infty \exp(-C\eta^{1-\beta})d\eta \to 0, \quad \text{as} \quad |\xi| \to \infty. \tag{55}$$

Substituting (47), (51), (55) into (28), we conclude that

$$\int_0^\infty \text{Re }\left(e^{-ix\eta}\frac{\psi_1(\xi+i\eta)}{\psi_1(\xi)}\right)d\eta > 0, \quad \text{for} \quad \xi \leq \xi_0 < 0.$$

Hence (27) and (42) imply that $p_1(x) > 0$ for $0 < x < x_0$.

3. Proof of Lemma 1

First we are going to show the validity of (18) and (19).

By (16)

$$\left(\tfrac{d^j}{dz^j}\log B(z)\right)_{z=-r} \tag{56}$$

$$= (j-1)!\sum_{k=1}^{\infty}\left\{-\tfrac{1}{(a_k+r)^j}-\tfrac{1}{(\bar a_k+r)^j}+\tfrac{1}{(-a_k+\delta_k+r)^j}+\tfrac{1}{(-\bar a_k+\delta_k+r)^j}\right\}$$

$$= (j-1)!\sum_{k=1}^{\infty}2\mathrm{Re}\,\left(\tfrac{1}{(-\bar a_k+r+\delta_k)^j}-\tfrac{1}{(a_k+r)^j}\right)$$

$$=: 2j!\sum_{k=1}^{\infty}\sigma_k,$$

where

$$\sigma_k := \int\limits_{-|\mathrm{Re}\ a_k|}^{|\mathrm{Re}\ a_k|+\delta_k}\mathrm{Re}\ \frac{1}{(r+u+i\mathrm{Im}\ a_k)^{j+1}}\,du. \tag{57}$$

Let us prove (18). We will estimate σ_k for $j=1$. Since points a_k lie outside the angle $\{z : |\pi-\arg z|<\alpha\}$ then for $u\in[-|\mathrm{Re}\ a_k|, |\mathrm{Re}\ a_k|+\delta_k]$ we have

$$|r+u+i\mathrm{Im}\ a_k|^2 \ge (1-\cos\alpha)\left(r^2+u^2+(\mathrm{Im}\ a_k)^2\right) \ge C_\alpha(r^2+|a_k|^2). \tag{58}$$

Here and further by C_α we will denote not necessary equal positive constants depending only on α. Hence by (57)

$$|\sigma_k| \le C_\alpha\frac{2|\mathrm{Re}\ a_k|+\delta_k}{|a_k|^2+r^2}\,. \tag{59}$$

Using (17), (56), (57) and (59), we get (18).

Let us prove (19). First we estimate σ_k for $j\ge 3$. By (57) and (58)

$$|\sigma_k| \le C_\alpha\tfrac{(2|\mathrm{Re}\ a_k|+\delta_k)r}{(|a_k|^2+r^2)^2}\cdot\tfrac{r^{j-3}}{(|a_k|^2+r^2)^{(j-3)/2}}\cdot\left(\tfrac{K_\alpha}{r}\right)^{j-2}$$

$$\le C_\alpha\tfrac{(2|\mathrm{Re}\ a_k|+\delta_k)r}{(|a_k|^2+r^2)^2}\cdot\left(\tfrac{K_\alpha}{r}\right)^{j-2},$$

where $K_\alpha>1$. Using (17), (10) and (56) we get (19) for $j\ge 3$.

Now we consider $j=2$. From (57) we obtain

$$\sigma_k = \mathrm{Re}\ \left(\tfrac{1}{(-\bar a_k+r+\delta_k)^2}-\tfrac{1}{(a_k+r)^2}\right) \tag{60}$$

$$= \left(\tfrac{1}{|-\bar a_k+r+\delta_k|^2}-\tfrac{1}{|a_k+r|^2}\right)-2(\mathrm{Im}\ a_k)^2\cdot\left(\tfrac{1}{|-\bar a_k+r+\delta_k|^4}-\tfrac{1}{|a_k+r|^4}\right)$$

Hence from (60) we obtain

$$|\sigma_k| \le \left|\frac{1}{|-\bar a_k+r+\delta_k|^2} - \frac{1}{|a_k+r|^2}\right| \left(1+2(\text{Im } a_k)^2\left(\frac{1}{|-\bar a_k+r+\delta_k|^2}+\frac{1}{|a_k+r|^2}\right)\right)$$
$$\le 3\left|\frac{1}{|-\bar a_k+r+\delta_k|^2} - \frac{1}{|a_k+r|^2}\right| \le 6\frac{(2|\text{Re } a_k|+\delta_k)r}{|-\bar a_k+r+\delta_k|^2\cdot|a_k+r|^2} + 3\frac{2|\text{Re } a_k|\delta_k+\delta_k^2}{|-\bar a_k+r+\delta_k|^2\cdot|a_k+r|^2}.$$

By (58)

$$|\sigma_k| \le C_\alpha\left(\frac{2r(2|\text{Re } a_k|+\delta_k)}{(|a_k|^2+r^2)^2} + \frac{2|\text{Re } a_k|\delta_k+\delta_k^2}{|a_k|^2r^2}\right). \tag{61}$$

Using (17), (10) and (61), we get (19) for $j=2$.

$$\left|(\log B(z))''\right|_{z=-r} \le C\Big(\frac{1}{r^2}+\sum_{k=1}^{\infty}\frac{2r(2|\text{Re } a_k|+\delta_k)}{(|a_k|^2+r^2)^2}\Big) \le C_\alpha q(r), \quad r\ge 1.$$

Let us prove (20) and (21). Substituting $z=-r+iy$, $a_k=-\alpha_k+i\beta_k$, $(r>0$, $\alpha_k>0$, $\beta_k>0$, $y>0)$ into (16), we get

$$2\log\frac{|B(z)|}{B(-r)} = \sum_{k=1}^\infty\left(\log\frac{(r-\alpha_k)^2+(y-\beta_k)^2}{(r+\alpha_k+\delta_k)^2+(y-\beta_k)^2} - \log\frac{(r-\alpha_k)^2+(\beta_k)^2}{(r+\alpha_k+\delta_k)^2+(\beta_k)^2}\right)$$
$$+\sum_{k=1}^\infty\left(\log\frac{(r-\alpha_k)^2+(y+\beta_k)^2}{(r+\alpha_k+\delta_k)^2+(y+\beta_k)^2} - \log\frac{(r-\alpha_k)^2+(\beta_k)^2}{(r+\alpha_k+\delta_k)^2+(\beta_k)^2}\right)$$
$$=\sum_{k=1}^\infty\left(\log\frac{(r-\alpha_k)^2+(y-\beta_k)^2}{(r-\alpha_k)^2+(\beta_k)^2} - \log\frac{(r+\alpha_k+\delta_k)^2+(y-\beta_k)^2}{(r+\alpha_k+\delta_k)^2+(\beta_k)^2}\right)$$
$$+\sum_{k=1}^\infty\left(\log\frac{(r-\alpha_k)^2+(y+\beta_k)^2}{(r-\alpha_k)^2+(\beta_k)^2} - \log\frac{(r+\alpha_k+\delta_k)^2+(y+\beta_k)^2}{(r+\alpha_k+\delta_k)^2+(\beta_k)^2}\right)$$
$$=: \sum_{k=1}^\infty\left(\gamma_k^{(1)}+\gamma_k^{(2)}\right).$$

We shall obtain the estimates for $\gamma_k^{(j)}, j=1,2$ with the help of the following elementary inequalities:

$$\log(1-u)-\log(1-pu) \le \frac{u}{pu-1}(1-p), \quad \text{for} \quad 0<u,\ p<1; \tag{62}$$
$$\log(1+u)-\log(1+pu) \le \frac{u}{pu+1}(1-p), \quad \text{for} \quad u>0,\ 0<p<1. \tag{63}$$

Let us consider $\gamma_k^{(1)}$. If $\beta>y/2$, then

$$0<\frac{2\beta y-y^2}{(r+\alpha)^2+\beta^2} = \frac{\beta^2-(y-\beta)^2}{(r+\alpha)^2+\beta^2} \le \frac{\beta^2}{(r+\alpha)^2+\beta^2}<1,$$

and we use (62) with

$$u_1=\frac{|y^2-2\beta_k y|}{(r-\alpha_k)^2+(\beta_k)^2}, \quad p_1=\frac{(r-\alpha_k)^2+(\beta_k)^2}{(r+\alpha_k+\delta_k)^2+(\beta_k)^2}. \tag{64}$$

If $\beta \le y/2$, then $y^2 - 2\beta y \ge 0$, and we use (63) with u_1 and p_1 defined by (64). In both cases we obtain

$$\gamma_k^{(1)} \le \frac{4\alpha_k r + 2\delta_k r + 2\delta_k\alpha_k + \delta_k^2}{(r-\alpha_k)^2 + (\beta_k)^2} \cdot \frac{y^2 - 2\beta_k y}{(r+\alpha_k+\delta_k)^2 + (y-\beta_k)^2}. \tag{65}$$

To estimate $\gamma_k^{(2)}$, we use (63) with

$$u_2 = \frac{2\beta_k y + y^2}{(r-\alpha_k)^2 + (\beta_k)^2}, \quad p_2 = p_1.$$

We obtain

$$\gamma_k^{(2)} \le \frac{4\alpha_k r + 2\delta_k r + 2\delta_k\alpha_k + \delta_k^2}{(r-\alpha_k)^2 + (\beta_k)^2} \cdot \frac{y^2 + 2\beta_k y}{(r+\alpha_k+\delta_k)^2 + (y+\beta_k)^2}. \tag{66}$$

Joining (66), (65) and (62), we get

$$\log\frac{|B(z)|}{B(-r)} \le 2y^2 \sum_{k=1}^{\infty} \frac{4\alpha_k r + 2\delta_k r + 2\delta_k\alpha_k + \delta_k^2}{(r-\alpha_k)^2 + (\beta_k)^2} D(\alpha_k, \beta_k, \delta_k), \tag{67}$$

where

$$D(\alpha,\beta,\nu) := \frac{(r+\alpha+\nu)^2 + y^2 - 3\beta^2}{[(r+\alpha+\nu)^2 + (y-\beta)^2][(r+\alpha+\nu)^2 + (y+\beta)^2]}.$$

Set

$$M := \{k \in \mathbf{N} : (r+\alpha_k+\delta_k)^2 + y^2 - 3(\beta_k)^2 > 0\}.$$

Then (67) can be rewritten in the form

$$\log\frac{|B(z)|}{B(-r)} \le 2y^2 \sum_{k\in M_1} \frac{4\alpha_k r + 2\delta_k r + 2\delta_k\alpha_k + \delta_k^2}{(r-\alpha_k)^2 + (\beta_k)^2} D(\alpha_k, \beta_k, \delta_k). \tag{68}$$

We need also the following lemma.

Lemma 5. *For each positive r, y, α, β, ν the following estimation holds*

$$D(\alpha,\beta,\nu) \le \frac{1}{r^2 + \alpha^2 + \beta^2}, \tag{69}$$

moreover, if $(r+\alpha+\nu)^2 + y^2 - 3\beta^2 > 0$, $1 \le r \le y/2$ and $\nu \le \frac{\sqrt{\alpha^2+\beta^2}}{4}$, then

$$D(\alpha,\beta,\nu) \le \frac{C}{y^2 + \alpha^2 + \beta^2}. \tag{70}$$

P r o o f o f L e m m a 5. Evidently,

$$(r^2+\alpha^2+\beta^2)D(\alpha,\beta,\nu)=\frac{(r^2+\alpha^2+\beta^2)[(r+\alpha+\nu)^2+y^2-3\beta^2]}{[(r+\alpha+\nu)^2+(y-\beta)^2][(r+\alpha+\nu)^2+(y+\beta)^2]}.$$

Denoting $x:=r+\alpha+\nu$, we get

$$(r^2+\alpha^2+\beta^2)D(\alpha,\beta,\nu)\le\frac{(x^2+\beta^2)(x^2+y^2-3\beta^2)}{[x^2+(y-\beta)^2][x^2+(y+\beta)^2]}.$$

To get (69), it suffices to show that

$$\frac{(x^2+\beta^2)(x^2+y^2-3\beta^2)}{[x^2+(y-\beta)^2][x^2+(y+\beta)^2]}\le 1.$$

This inequality is equivalent to

$$x^2(y^2+4\beta^2)+(y^2-\beta^2)^2-\beta^2(y^2-\beta^2)+2\beta^4\ge 0.$$

The last inequality follows from the calculation:

$$\begin{aligned}&x^2(y^2+4\beta^2)+(y^2-\beta^2)^2-\beta^2(y^2-\beta^2)+2\beta^4\\&\ge(y^2-\beta^2)^2-\beta^2(y^2-\beta^2)+2\beta^4\\&=\beta^4\left(\left(\tfrac{y^2-\beta^2}{\beta^2}\right)^2-\tfrac{y^2-\beta^2}{\beta^2}+2\right)>0\quad\text{for}\quad x>0,\ y>0,\ \beta>0.\end{aligned}$$

This completes the proof of (69).

Let us prove (70). If $\alpha>\beta/4,\beta\ge y/2$ then

$$\begin{aligned}D(\alpha,\beta,\nu)&\le\frac{3(r^2+\alpha^2+\nu^2)+y^2-\beta^2}{[(r+\alpha+\nu)^2+(y-\beta)^2][(r+\alpha+\nu)^2+(y+\beta)^2]}\\&\le\frac{3}{\alpha^2}\le\frac{C}{y^2+\alpha^2+\beta^2}.\end{aligned}$$

If $\alpha>\beta/4,\beta\le y/2$ then

$$\begin{aligned}D(\alpha,\beta,\nu)&\le\frac{3(r^2+\alpha^2+\nu^2)+y^2-\beta^2}{[(r+\alpha+\nu)^2+(y-\beta)^2][(r+\alpha+\nu)^2+(y+\beta)^2]}\\&\le\frac{3}{\alpha^2+y^2/4}\le\frac{C}{y^2+\alpha^2+\beta^2}.\end{aligned}$$

If $\alpha\le\beta/4$ then

$$(r+\alpha+\nu)^2+y^2-3\beta^2\le 3(r^2+\alpha^2+\nu^2)+y^2-3\beta^2\le\frac{7}{4}y^2-\frac{21}{8}\beta^2\le 0,\ \beta\ge\sqrt{2/3}y.$$

Hence if $(r+\alpha+\nu)^2+y^2-3\beta^2>0$, then $\beta<\sqrt{2/3}y$. Therefore

$$\frac{(y-\beta)^2}{y^2+\beta^2}=1-\frac{2(y/\beta)}{(y/\beta)^2+1}\geq 1-\frac{2\sqrt{3/2}}{5/2}=:K_1>0.$$

So

$$\begin{aligned}&\left[(r+\alpha+\nu)^2+(y+\beta)^2\right]\left[(r+\alpha+\nu)^2+(y-\beta)^2\right]\\ \geq&\left[y^2+\alpha^2+\beta^2\right]\left[\alpha^2+(y-\beta)^2\right]\geq K_1(y^2+\alpha^2+\beta^2)^2.\end{aligned}$$

We have

$$D(\alpha,\beta,\nu)\leq\frac{(7/4)y^2-(21/8)\beta^2}{K_1(y^2+\beta^2)^2}\leq C\frac{y^2}{(y^2+\beta^2)^2}\leq\frac{C}{y^2+\alpha^2+\beta^2}.$$

This completes the proof of (70) and Lemma 5. ■

To complete the proof of (20) we estimate (68) with the help of (69). Then we get (20) from the following calculation:

$$\begin{aligned}\log\frac{|B(z)|}{B(-r)}&\leq y^2\sum_{k\in M}\frac{2r(2\alpha_k+\delta_k)}{(r^2+|a_k|^2)((r-\alpha_k)^2+(\beta_k)^2)}\\&+y^2\sum_{k\in M}\frac{2\alpha_k\delta_k+\delta_k^2}{(r^2+|a_k|^2)((r-\alpha_k)^2+(\beta_k)^2)}\\&\leq C_\alpha y^2\left(\sum_{k=1}^{\infty}\frac{(2|\mathrm{Re}\ a_k|+\delta_k)r}{(r^2+|a_k|^2)^2}+\frac{1}{r^2}\right)\leq C_\alpha y^2 q(r).\end{aligned}$$

It remains to prove (21). Note that by (11) we can assume that $\delta_k/|a_k|\leq 1/4$. Hence we can use (70) to estimate (68). We get (21) from the following calculations:

$$\begin{aligned}\log\frac{|B(z)|}{B(-r)}&\leq C_\alpha y^2\sum_{k\in M}\frac{\delta_k^2+|\mathrm{Re}\ a_k|\delta_k}{r^2+|a_k|^2}\cdot\frac{1}{y^2+|a_k|^2}\\&+C_\alpha y^2\sum_{k\in M}\frac{r(2|\mathrm{Re}\ a_k|\delta_k+\delta_k^2)}{r^2+|a_k|^2}\cdot\frac{1}{y^2+|a_k|^2}\\&\leq C_\alpha y^2\left(\frac{1}{y^2}+\sum_{k=1}^{\infty}\frac{2|\mathrm{Re}\ a_k|\delta_k+\delta_k^2}{|a_k|}\cdot\frac{1}{y^2+|a_k|^2}\right)\leq C_\alpha y^2 q(y).\end{aligned}$$

Lemma 1 is proved.

Acknowledgement. The authors are deeply grateful to Prof. I.V. Ostrovskii for his attention to this work and valuable suggestions.

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