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On the growth of a subharmonic function with Riesz' measure on a ray

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We consider functions v subharmonic in $\mathbf{R}^n$, $n \geq 2$, which are natural counterparts of Weierstrass canonical products (so-called Weierstrass canonical integrals). Under assumptions that the order of v is a noninteger number and the Riesz measure of v is supported by a ray we obtain sharp estimates of asymptotical behavior of v at infinity along rays.

Let f be a Weierstrass canonical product of noninteger order ρ. Assume that zeros of f are situated on the negative ray and denote by $n(r)$ the number of the zeros in the disc $\{z : |z| \leq r\}$. In [3] it had been shown that

$$\limsup_{r\to\infty} \frac{\log|f(re^{i\theta})|}{n(r)} \geq \frac{\pi\cos\theta\rho}{\sin\pi\rho} \geq \liminf_{r\to\infty} \frac{\log|f(re^{i\theta})|}{n(r)}, \quad \theta \in (-\pi, \pi), \qquad (1)$$

and both inequalities are sharp.

This paper is devoted to extension of this result to functions subharmonic in $\mathbf{R}^n$, $n \geq 2$.

We denote by $|x|$ the euclidian norm of a vector $x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n$, by $(\widehat{x,y})$ the angle between vectors $x, y \in \mathbf{R}^n$, by S_n the unit sphere of $\mathbf{R}^n$, by $l_\pm$ the ray $\{x : \pm x_1 > 0, 0, \dots, 0\}$. Each vector x can be written in the form $x = r\xi$, $r \geq 0$, $\xi \in S_n$.

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We will follow terminology of the subharmonic function theory accepted in [2]. Let us remind the definition of Weierstrass canonical products of genus $q = 0, 1, 2, \dots$ in $\mathbf{R}^n$, $n \geq 2$.

Following [2, Ch. 4], up to notations, we set for $n = 2$, $q = 0, 1, 2, \dots,$

$$K_q^{(2)}(x, y) = \log \left| E\left(\frac{|x|}{|y|} e^{i(\widehat{x,y})}, q\right) \right|,$$

where $E(w, q)$ is the Weierstrass primary factor of genus q, and set for $n \geq 3$, $q = 0, 1, 2, \dots,$

$$K_q^{(n)}(x, y) = -(|x|^2 + |y|^2 - 2|x||y| \cos(\widehat{x,y}))^{(2-n)/2}$$

$$+ |y|^{2-n} \sum_{j=0}^{q} \left(\frac{|x|}{|y|}\right) G_j^{(n)}(\cos(\widehat{x,y})),$$

where $G_j^{(n)}(w)$'s are Gegenbauer polynomials with the generating function $(1 + z^2 - 2zw)^{(2-n)/2}$.

Let μ be a locally finite measure in $\mathbf{R}^n$ and let

$$n(r) = r^{2-n} \mu(\{x : |x| \leq r\}), \quad r \geq 0.$$

It is known ([2, Ch. 4]) that if

$$\int_1^\infty \frac{n(r)}{r^{q+n}} dr < \infty,$$

then the integral

$$v(x) = \int_{|y| \geq 1} K_q^{(n)}(x, y) d\mu(y) \tag{2}$$

converges and $v(x)$ is a subharmonic function in $\mathbf{R}^n$ whose order coincides with that of the function $n(r)$. A function of form (2) is called Weierstrass canonical integral of genus q. It is a counterpart of Weierstrass canonical product for $\mathbf{R}^n, n \geq 2$.

R e m a r k. By the Hadamard theorem (cf.[2, p. 146]) each function u subharmonic in $\mathbf{R}^n, n \geq 2$, and of finite order ρ can be represented in the form $u = v + h$ where v is a Weierstrass canonical integral and h is a harmonic polynomial of degree not greater than $q := [\rho]$. Hence, if ρ is noninteger, then $u(x) = v(x) + o(|x|^\rho)$, $|x| \to \infty$.

Now we introduce a function $I(\rho, n, \theta)$, $\rho > 0$, $n = 2, 3, 4, \dots$, $0 \le \theta < \pi$, which will play a role similar to that of $(\pi \cos\theta\rho)/(\sin\pi\rho)$ in (1). For $n = 2$ it is the same as in (1) that is

$$I(\rho, n, \theta) = \frac{\pi \cos\theta\rho}{\sin\pi\rho}. \tag{3}$$

If $n \ge 3$, then we firstly define $I(\rho, n, \theta)$ for $0 < \rho < 1$ as follows:

$$I(\rho, n, \theta) = -(\rho + n - 2) \int_0^\infty \frac{\partial}{\partial u}(1 + u^2 + 2u\cos\theta)^{(2-n)/2} \frac{du}{u^\rho}. \tag{4}$$

The integral in the right hand side absolutely converges even in the strip $\{\rho : 2 - n < \Re\rho < 1\}$ and is an analytic function there. It is easy to show that this function can be analytically continued into the half-plane $\{\rho : \Re\rho > 2 - n\}$ as a meromorphic function with poles in $\{\rho = 1, 2, \dots\}$. Indeed, taking an arbitrary $q \in \{0, 1, 2, \dots\}$ and integrating by parts q times, we get

$$I(\rho, n, \theta)$$
$$= -\frac{\rho + n - 2}{(\rho - 1)(\rho - 2)\dots(\rho - q)} \int_0^\infty \left(\frac{\partial}{\partial u}\right)^{q+1} (1 + u^2 + 2u\cos\theta)^{(2-n)/2} \frac{du}{u^{\rho - q}}. \tag{5}$$

The integral in the right hand side absolutely converges and is analytic in the strip $\{\rho : 2 - n < \Re\rho < q + 1\}$. In this way we define $I(\rho, n, \theta)$ for all noninteger $\rho > 0$.

R e m a r k. The function $I(\rho, n, \theta)$ was introduced in [2, p. 160], by the following way:

$$I(\rho, n, \theta) = \int_0^\infty [y^{2-n} - (1 + y^2 + 2y\cos\theta)^{(2-n)/2}] y^{\rho + n - 3} dy, \quad 0 < \rho < 1.$$

Changing variable $y = 1/u$ and integrating by parts, we see that this definition coincides with (4). In [2, p. 160], it was shown that $I(\rho, n, \theta)$ can be analytically extended into whole complex plane as a meromorphic function of ρ with poles in $\{0, \pm 1, \pm 2, \dots\}$.

Now we are ready to state the main result of the paper.

Theorem. *Let μ be a locally finite measure in $\mathbf{R}^n$, $n \ge 2$, supported by l_- and such that the function $n(r)$ has noninteger order ρ. Let v be a Weierstrass canonical integral in $\mathbf{R}^n$, $n \ge 2$, of genus $q = [\rho]$. Then the inequality holds*

$$\limsup_{r\to\infty} \frac{v(r\xi)}{n(r)} \ge I(\rho, n, (\widehat{\xi, l_+})) \ge \liminf_{r\to\infty} \frac{v(r\xi)}{n(r)}, \quad \xi \in S_n \setminus l_-. \tag{6}$$

Both inequalities in (6) are sharp.

Corollary 1. *For $n = 2$ we have*

$$\limsup_{r\to\infty} \frac{v(re^{i\theta})}{n(r)} \geq \frac{\pi\cos\theta\rho}{\sin\pi\rho} \geq \liminf_{r\to\infty} \frac{v(re^{i\theta})}{n(r)}, \quad -\pi < \theta < \pi.$$

If v is a logarithm of modulus of a Weierstrass canonical product, then it is a Weierstrass canonical integral, therefore corollary 1 contains the result of [3] mentioned before.

Corollary 2. *For $n \geq 3$ we have*

$$\limsup_{r\to\infty} \frac{v(r\xi_+)}{n(r)} \geq \frac{\pi\rho(\rho+1)(\rho+2)\dots(\rho+n-2)}{(n-1)!\,\sin\pi\rho} \geq \liminf_{r\to\infty} \frac{v(r\xi_+)}{n(r)}, \tag{7}$$

where $\xi_+ = (1, 0, 0, \dots, 0)$.

To derive the latter corollary from the theorem it suffices to calculate $I(\rho, n, 0)$. Let $q < \rho < q+1$. Using (5) with $\theta = 0$, we obtain

$$I(\rho, n, 0) = -\frac{\rho+n-2}{(\rho-1)(\rho-2)\dots(\rho-q)} \int_0^\infty [(1+u)^{2-n}]^{(q+1)} \frac{du}{u^{\rho-q}}$$

$$= \frac{(-1)^q(n-2)(n-1)\dots(n+q-2)}{(\rho-1)(\rho-2)\dots(\rho-q)} \int_0^\infty \frac{du}{(1+u)^{n+q-1}u^{\rho-q}}.$$

Calculating the integral, we obtain the desired result.

Before starting with proof of the theorem we get a representation for $I(\rho, n, \theta)$ different of previous ones. Set

$$h_2(u, \theta, q) = \log|E(ue^{i\theta}, q)|,$$

and, for $n \geq 3$,

$$h_n(u, \theta, q) = -(1 + u^2 + 2u\cos\theta)^{(2-n)/2} + \sum_{j=0}^{q} (-1)^j u^j G_j^{(n)}(\cos\theta),$$

where $G_j^{(n)}(w)$'s are Gegenbauer polynomials with generating function $(1 + u^2 - 2uw)^{(2-n)/2}$. It is easy to see that for $\theta \in [0, \pi)$ and all $n \geq 2$, $q = 0, 1, 2, \dots$, except the case $n = 2$, $q = 0$, the following estimate holds

$$|h_n(u, \theta, q)| \leq C\min(u^q, u^{q+1}), \quad u > 0,$$

and

$$|h_2(u,\theta,0)| \le C\min(|\log u|, u), \quad u>0,$$

where $C>0$ is a constant not depending on u.

We will need the representation

$$I(\rho,n,\theta) = (\rho+n-2)\int\limits_0^\infty \frac{h_n(u,\theta,q)}{u^{1+\rho}}du, \quad q<\rho<q+1. \tag{8}$$

To prove it denote for the expression in the right hand side of (8) by J for a moment. Integrating by parts $q+1$ times, we obtain for $n\ge 3$

$$J = \frac{\rho+n-2}{\rho(\rho-1)(\rho-2)\dots(\rho-q)}\int\limits_0^\infty \left(\frac{\partial}{\partial u}\right)^{q+1} h_n(u,\theta,q)\,\frac{du}{u^{\rho-q}}$$

$$= -\frac{\rho+n-2}{\rho(\rho-1)(\rho-2)\dots(\rho-q)}\int\limits_0^\infty \left(\frac{\partial}{\partial u}\right)^{q+1} (1+u^2+2\cos\theta)^{(2-n)/2}\,\frac{du}{u^{\rho-q}}.$$

Comparing with (5), we see that $J=I(\rho,n,\theta)$. If $n=2$, then integrating in (8) $q+1$ times, we get

$$J = \frac{\rho}{(\rho-1)(\rho-2)\dots(\rho-q)}\int\limits_0^\infty \left(\frac{\partial}{\partial u}\right)^{q+1} \log|u+e^{i\theta}|\,\frac{du}{u^{\rho-q}}$$

$$= \frac{(-1)^q q!}{(\rho-1)(\rho-2)\dots(\rho-q)}\,\Re\int\limits_0^\infty \frac{du}{(u+e^{i\theta})^{q+1}u^{\rho-q}}.$$

Calculating the integral and comparing with (3), we see that $J=I(\rho,2,\theta)$.

Let us start with proof of the theorem.

Since μ is supported by l_-, (2) can be rewritten in the form

$$v(r\xi) = \int\limits_1^\infty K_q^{(n)}(r\xi, t\xi_-)d(t^{n-2}n(t)), \quad \xi\in S_n,\ r>0, \tag{9}$$

where $q=[\rho]$, $\xi_-=(-1,0,0,\dots,0)$. The function v has order ρ and is harmonic in $\mathbf{R}^n\setminus l_-$. Let us fix $\xi\in S_n\setminus l_-$ and take $\sigma\in(\rho,q+1)$. Dividing both sides of (9) over $r^{1+\sigma}$, integrating from 0 to ∞ and changing order of integration, we obtain

$$\int\limits_0^\infty \frac{v(r\xi)}{r^{1+\sigma}}dr = \int\limits_1^\infty \left\{\int_0^\infty \frac{K_q^{(n)}(r\xi,t\xi_-)}{r^{1+\sigma}}dr\right\} d(t^{n-2}n(t)). \tag{10}$$

It is easy to see that

$$K_q^{(n)}(r\xi, t\xi_-) = t^{2-n} h_n(r/t, \theta, q), \quad \text{for} \quad \theta = (\widehat{\xi, l_+}). \tag{11}$$

Using (8), we obtain

$$\int_0^\infty \frac{K_q^{(n)}(r\xi, t\xi_-)}{r^{1+\sigma}} dr = t^{2-n-\sigma} \int_0^\infty \frac{h_n(u, \theta, q)}{u^{1+\sigma}} du = t^{2-n-\sigma} \frac{I(\sigma, n, \theta)}{n + \sigma - 2}.$$

Substituting this into (10), we get

$$\int_0^\infty \frac{v(r\xi)}{r^{1+\sigma}} dr = \frac{I(\sigma, n, \theta)}{n + \sigma - 2} \int_1^\infty \frac{d(t^{n-2} n(t))}{t^{n+\sigma-2}}.$$

Let us extend $n(t)$ to $[0, \infty)$ by putting $n(t) = 0$ for $0 \le t < 1$. Then integration by parts implies

$$\int_0^\infty \frac{v(r\xi)}{r^{1+\sigma}} dr = I(\sigma, n, \theta) \int_0^\infty \frac{n(r)}{r^{1+\sigma}} dr. \tag{12}$$

Further we will use the following result from [1] which is a version of a theorem of Pólya [4] and can be found in an implicit form in [5, Sect. 8.74].

Lemma. *Let φ_1, φ_2 be two functions on $[0, \infty)$ and $\varphi_2(r) \ge 0$. Let $\rho \ge 0$, $\varepsilon > 0$ be two numbers such that both integrals*

$$I_1(\sigma) := \int_0^\infty \frac{\varphi_1(r)}{r^{1+\sigma}} dr, \ I_2(\sigma) := \int_0^\infty \frac{\varphi_2(r)}{r^{1+\sigma}} dr$$

converge for $\rho < \sigma < \rho + \varepsilon$, meanwhile $I_2(\sigma)$ diverges for $\sigma < \rho$. Assume that the function

$$\Psi(\sigma) := I_1(\sigma)/I_2(\sigma)$$

can be extended to an analytic function in the disc $\{z : |z - \rho| < \varepsilon\}$. Then

$$\limsup_{r\to\infty} \frac{\varphi_1(r)}{\varphi_2(r)} \ge \Psi(\rho) \ge \liminf_{r\to\infty} \frac{\varphi_1(r)}{\varphi_2(r)}. \tag{13}$$

Taking $\varphi_1(r) = v(r\xi)$, $\varphi_2(r) = n(r)$, $\Psi(\sigma) = I(\sigma, n, \theta)$, we see that all conditions of the lemma are satisfied for $0 < \varepsilon < \min(\rho - q, q + 1 - \rho)$. Therefore (13) implies (6).

To prove the sharpness of (6) consider the Weierstrass canonical integral (2) with μ supported by l_- and such that

$$n(r) = r^\rho, \ r \ge 1. \tag{14}$$

Evidently,

$$v(r\xi) = v_0(r\xi) + O(r^q), \quad r \to \infty, \tag{15}$$

where

$$v_0(r\xi) = \int_0^\infty K_q^{(n)}(r\xi, t\xi_-)d(t^{\rho+n-2}).$$

Using (11), we obtain

$$v_0(r\xi) = (\rho + n - 2)\int_0^\infty h_n(r/t, \theta, q)t^{\rho-1}dt,$$

where $\theta = (\widehat{\xi, l_+})$. Changing variable $t = r/u$ and taking into account (8), we get

$$v_0(r\xi) = r^\rho I(\sigma, n, \theta), \quad \theta = (\widehat{\xi, l_+}).$$

The equations (14) and (15) imply that the equality sign takes place for the function v in both inequalities in (2). ■

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