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Absolutely continuous measures on the unit circle with sparse Verblunsky coefficients

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Orthogonal polynomials and measures on the unit circle are fully determined by their Verblunsky coefficients through the Szegő recurrences. We study measures μ from the Szegő class whose Verblunsky coefficients vanish off a sequence of positive integers with exponentially growing gaps. All such measures turn out to be absolutely continuous on the circle. We also gather some information about the density function μ'.

To Iossif Vladimirovich Ostrovskii on the occasion of his 70-th birthday

1. Introduction

Given a probability measure μ on the unit circle $\mathbb{T} = \{|\zeta| = 1\}$ with infinite support, supp μ, the polynomials $\varphi_n(z) = \varphi_n(\mu, z) = \kappa_n(\mu)z^n + \ldots$, orthonormal on $\mathbb{T}$ with respect to μ are uniquely determined by the conditions $\kappa_n = \kappa_n(\mu) > 0$ and

$$\int_{\mathbb{T}} \varphi_n(\zeta)\overline{\varphi_m(\zeta)}\, d\mu = \delta_{n,m}, \qquad n, m \in \mathbb{Z}_+ \stackrel{\text{def}}{=} \{0, 1, \ldots\}. \tag{1}$$

The monic orthogonal polynomials Φ_n are $\Phi_n(z) = \Phi_n(\mu, z) = \kappa_n^{-1}\varphi_n = z^n + \ldots$.

The numbers $\alpha_n \stackrel{\text{def}}{=} -\overline{\Phi_{n+1}(0)}$, $n \in \mathbb{Z}_+$, known as the *Verblunsky coefficients*, define completely both orthonormal and monic orthogonal polynomials.

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Let $\mu = \mu' dm + \mu_s$ be the Lebesgue decomposition with respect to the normalized Lebesgue measure dm on $\mathbb{T}$. One of the highlights of the theory of orthogonal polynomials on the unit circle — Geronimus' theorem — states that

$$\log \mu' \in L^1 \quad \iff \quad \sum_{k=0}^{\infty} |\alpha_k|^2 < \infty. \tag{2}$$

The measures with property (2) constitute the Szegő class. It is crystal-clear from (2) that $\{\alpha_n\} \in \ell^2$ makes no effect on the singular component μ_s of the measure μ.

The situation changes substantially when we deal with certain subclasses of the Szegő class. One of them, the measures with sparse Verblunsky coefficients, is the main object under investigation in the present paper.

Let $\Lambda \stackrel{\text{def}}{=} \{n_1 < n_2 < \dots\}$ be a sequence of positive integers with the Hadamard gaps

$$\lambda = \deg \Lambda \stackrel{\text{def}}{=} \inf_k \frac{n_{k+1}}{n_k} > 1. \tag{3}$$

The Verblunsky coefficients $\{\alpha_n\}$ are said to be *sparse* if

$$\alpha_n = 0, \qquad n \notin \Lambda \tag{4}$$

for some Λ (3). The main result of the paper claimes that measures from the Szegő class with sparse Verblunsky coefficients are absolutely continuous. In other words, (2) coupled with (4) suppress the singular component of μ.

The Nikishin–Prüfer variables and equations for them play a crucial role throughout the whole paper. In [7] E.M. Nikishin came up with the idea to study the interplay between absolute value and argument of orthogonal polynomials on the circle (see also [6]). The name Prüfer is related to a long and rich history of continuum and discrete Schrödinger operators, where the similar variables arise naturally. B. Simon [8] made this idea explicit and suggested the corresponding equations for the *real* Verblunsky coefficients (symmetric measures on $\mathbb{T}$), the case which arises in regard with the spectral theory of Jacobi matrices. We derive the Nikishin–Prüfer equations for general measures and also prove Nikishin's inequality and some of its consequences in Section 2. The sparse Verblunsky coefficients are studied in Sections 3 and 4.

Our work was obviously very strongly influenced and inspired by a lovely paper [5], wherein the Schrödinger operators with sparse potentials are examined. The similarity of methods is quite conspicuous. The only difference is that we were able to prove absolute continuity on the *whole* unit circle and also gather some information about the density μ'. On the other hand, Theorem 1.7 from [5] is much stronger in a sense that it can manage sparse potentials off ℓ^2 (the measure turns out to be pure singular). At the moment we can only conjecture that the similar result holds in the unit circle setting as well.*

* This conjecture is proved in the book [11].

2. Nikishin–Prüfer variables and equations

For the monic orthogonal polynomials Φ_n we write for $n \in \mathbb{Z}_+$

$$\Phi_n(\zeta) = R_n(\zeta) e^{i\lambda_n(t)}, \qquad R_n(\zeta) = |\Phi_n(\zeta)|\,, \quad \zeta = e^{it},$$

with $\lambda_0 = 0$. The value λ_n is only determined $\mod (2\pi)$, that will be fixed later on. As all zeros of Φ_n lie inside the disk $\mathbb{D}$, by the Argument Principle $\lambda_n(2\pi) - \lambda_n(0) = 2\pi n$. It seems reasonable to introduce

$$\vartheta_n \overset{\text{def}}{=} nt - \lambda_n(t), \qquad \vartheta_n(2\pi) = \vartheta_n(0), \quad \vartheta_0 = 0.$$

The values $\{R_n, \vartheta_n\}$ are called the *Nikishin–Prüfer variables.*

For $*$-reversed polynomials we have

$$\Phi_n^*(\zeta) = \zeta^n \overline{\Phi_n(\zeta)} = R_n(\zeta) e^{i\vartheta_n(t)},$$
$$\frac{\Phi_n(\zeta)}{\Phi_n^*(\zeta)} = e^{i(\lambda_n(t) - \vartheta_n(t))} = e^{i(nt - 2\vartheta_n(t))}.$$

It follows from the Szegő recurrences for Φ_n^* [9, Theorem 11.4.2] that

$$\frac{\Phi_{n+1}^*(\zeta)}{\Phi_n^*(\zeta)} = \frac{R_{n+1}(\zeta)}{R_n(\zeta)}\, e^{i(\vartheta_{n+1}(t) - \vartheta_n(t))} = 1 + \zeta\alpha_n \frac{\Phi_n(\zeta)}{\Phi_n^*(\zeta)} = 1 + w_n(t), \tag{5}$$

where

$$w_n(t) = \alpha_n e^{i\omega_n(t)}, \qquad \omega_n(t) = (n+1)t - 2\vartheta_n(t),$$

which gives

$$\frac{R_{n+1}(\zeta)}{R_n(\zeta)} \cos(\vartheta_{n+1}(t) - \vartheta_n(t)) = 1 + \Re w_n > 0$$

and so $\cos(\vartheta_{n+1} - \vartheta_n) > 0$. The ambiguity in ϑ can now be fixed by demanding

$$|\vartheta_{n+1}(t) - \vartheta_n(t)| < \frac{\pi}{2}.$$

As a matter of fact, taking an imaginary part in (5) gives

$$|\sin(\vartheta_{n+1}(t) - \vartheta_n(t))| = \frac{R_n(\zeta)}{R_{n+1}(\zeta)}\, |\Im w_n(t)| = \frac{|\Im w_n(t)|}{|1 + w_n(t)|} \le \frac{|\alpha_n|}{1 - Q},$$

so that

$$|\vartheta_{n+1}(t) - \vartheta_n(t)| \le \frac{\pi}{2} \frac{|\alpha_n|}{1 - Q}\,. \tag{6}$$

The above formulae assembled together lead to two equalities which will be referred to as the *Nikishin–Prüfer equations*

$$\frac{R_{n+1}^2}{R_n^2} = |1 + w_n|^2 = 1 + |\alpha_n|^2 + 2\Re\left\{\alpha_n e^{i\omega_n(t)}\right\}, \tag{7}$$

$$e^{2i(\vartheta_{n+1}(t)-\vartheta_n(t))} = \frac{(1+w_n)^2}{|1+w_n|^2} = \frac{1+2w_n+w_n^2}{|1+w_n|^2}. \tag{8}$$

We are now within an easy reach from the Nikishin inequality [7, Theorem 1]. Indeed,

$$\log|R_n(\zeta)| = \Re\sum_{k=0}^{n-1}\log(1+w_k)$$

and since $|\log(1+z)-z| \le |z|^2(1-|z|)^{-1}$ for $|z|<1$, we see that

$$|\log|\Phi_n(\zeta)|| \le \left|\sum_{k=0}^{n-1}\Re\left\{\alpha_k e^{i\omega_k(t)}\right\}\right| + \frac{1}{1-Q}\sum_{k=0}^{n-1}|\alpha_k|^2. \tag{9}$$

One of the conclusions we can draw from (9) is the following

Theorem 1. *Let $\alpha = \{\alpha_n\} \in \ell^p$ for some $1 < p \le 2$ and q be a conjugate exponent. Then*

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\log\left|\Phi_n\left(e^{it}\right)\right|\right|^q dt \le C\left\{\|\alpha\|_p^q\sum_{k=0}^{n-1}\left\{\sum_{j=k}^{\infty}|\alpha_j|^p\right\}^{q/p}+1\right\}. \tag{10}$$

In particular, if

$$\sum_{k=0}^{\infty}\left\{\sum_{j=k}^{\infty}|\alpha_j|^p\right\}^{q/p} < \infty \tag{11}$$

then $\log\mu' \in L^q(\mathbb{T})$.

P r o o f. Put

$$\beta_k(e^{it}) = \sum_{j=k}^{\infty}\alpha_j e^{i(j+1)t}, \qquad \alpha_k e^{i(k+1)t} = \beta_k - \beta_{k+1}$$

(the latter series converges a.e.). The Abel transformation gives

$$\begin{aligned}\sum_{k=0}^{n-1}\alpha_k e^{i\omega_k(t)} &= \sum_{k=0}^{n-1}\left(\beta_k(e^{it})-\beta_{k+1}(e^{it})\right)e^{-2i\vartheta_k(t)} \\ &= \sum_{k=1}^{n-1}\beta_k(e^{it})\left(e^{-2i\vartheta_k(t)}-e^{-2i\vartheta_{k-1}(t)}\right)+\beta_0-\alpha_n e^{i\omega_n(t)}\end{aligned}$$

and so by (6)

$$\left|\sum_{k=0}^{n-1}\alpha_k e^{i\omega_k(t)}\right| \le 2\sum_{k=1}^{n-1}\left|\beta_k(e^{it})\right| |\vartheta_k(t)-\vartheta_{k-1}(t)| + |\beta_0| + |\alpha_n|$$

$$\le \frac{\pi}{1-Q}\sum_{k=0}^{n-1}\left|\beta_k(e^{it})\right| |\alpha_{k-1}| + |\alpha_n|, \quad |\alpha_{-1}| = 1.$$

Nikishin's inequality (9) along with the Hölder inequality imply

$$|\log|\Phi_n(\zeta)||^q \le C^q\left(\left\{\sum_{k=0}^{n-1}\left|\beta_k(e^{it})\right| |\alpha_{k-1}|\right\}^q + 1\right)$$

$$\le C^q\left(\left\{\sum_{k=0}^{n-1}|\alpha_{k-1}|^p\right\}^{q/p}\sum_{k=0}^{n-1}\left|\beta_k(e^{it})\right|^q + 1\right)$$

so that

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\log\left|\Phi_n\left(e^{it}\right)\right|\right|^q\, dt \le C^q\|\alpha\|_p^q\frac{1}{2\pi}\int_0^{2\pi}\sum_{k=0}^{n-1}\left|\beta_k\left(e^{it}\right)\right|^q\, dt + C^q.$$

According to the Hausdorff–Young theorem [10]

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\beta_j\left(e^{it}\right)\right|^q\, dt \le \left\{\sum_{r=j}^{\infty}|\alpha_r|^p\right\}^{q/p},$$

which gives (10).

Next, under (11) the sequence $\{\log|\varphi_n|\}$ is bounded in $L^q(\mathbb{T})$. On the other hand, by [4, Theorem 2.5] $\log|\varphi_n|^{-2}$ converges to $\log\mu'$ in L^1, and so the Fatou lemma does the rest. ■

For instance, (11) is true as long as

$$\sum_{n=1}^{\infty} n^s|\alpha_n|^p < \infty, \qquad s > p-1 = \frac{p}{q}.$$

Another consequence of Nikishin's inequality provides a uniform bound $|\log|\Phi_n|| = O(1)$, $n\to\infty$, for monic orthogonal polynomials from the Szegő class whenever

$$\sum_{k=0}^{\infty}\left|\alpha_{k-1}\beta_k(e^{it})\right| < \infty. \tag{12}$$

Hence, if (12) holds on a closed arc Δ, then by [3] the measure μ is absolutely continuous on this arc.

3. Bounds for derivatives of the angular variable

We refer to ϑ_n as the *angular* variable. By differentiating (8) with respect to t and using

$$\omega_n'(t) = n + 1 - 2\vartheta_n'(t), \tag{13}$$

we come to

$$\vartheta_{n+1}'(t) = \vartheta_n'(t) u_n + v_n, \qquad u_n = \frac{1 - |\alpha_n|^2}{|1 + w_n(t)|^2}, \quad v_n = (n+1)\frac{\Re w_n(t) + |\alpha_n|^2}{|1 + w_n(t)|^2}. \tag{14}$$

The coefficients u_n, v_n satisfy

$$0 < \frac{1-Q}{1+Q} \le u_k \le 1, \qquad |v_k| \le 2(k+1)\,|\alpha_k|\,. \tag{15}$$

Lemma 2. *For an arbitrary measure μ with the Verblunsky coefficients α_n we have*

$$\left|\vartheta_{n+1}'(t)\right| \le 2\sum_{k=0}^{n}(k+1)\,|\alpha_k|\,, \qquad n \in \mathbb{Z}_+. \tag{16}$$

P r o o f. The relation (14) is the first order linear difference equation, which can be easily solved explicitly

$$U_{n+1}\vartheta_{n+1}'(t) = U_n\vartheta_n'(t) + U_{n+1}v_n, \qquad U_n \stackrel{\text{def}}{=} \prod_{l=0}^{n-1} u_l^{-1},$$

and since $\vartheta_0' = 0$ we see that

$$\vartheta_{n+1}'(t) = \sum_{k=0}^{n} v_k\,\frac{U_{k+1}}{U_{n+1}}\,.$$

But

$$\frac{U_{k+1}}{U_{n+1}} = \prod_{l=k+1}^{n} u_l \le 1$$

and (15) yields (16). ∎

From now on we will focus on sparse Verblunsky coefficients (3)–(4). By C we always denote positive constants which depend on α_n (and so on Λ) whose value can vary from one equation to the next.

Lemma 3. *For any sequence of sparse parameters* (3)–(4) *we have*

$$\left|\vartheta'_{n_j}(t)\right| \le Cn_j, \quad \left|\omega'_{n_j}(t)\right| \le Cn_j, \qquad j \in \mathbb{N}, \tag{17}$$

and

$$\left|\vartheta''_{n_j}(t)\right| \le C \sum_{k=1}^{j} n_k^2 \left|\alpha_{n_k}\right| \le Cn_j^2. \tag{18}$$

P r o o f. It is clear that now

$$u_{n_j+1} = u_{n_j+2} = \ldots = u_{n_{j+1}-1} = 1, \qquad v_{n_j+1} = v_{n_j+2} = \ldots = v_{n_{j+1}-1} = 0$$

and

$$\vartheta'_{n_j+1}(t) = \vartheta'_{n_j+2}(t) = \ldots = \vartheta'_{n_{j+1}}(t). \tag{19}$$

In our case (16) has the form

$$\left|\vartheta'_{n_j+1}(t)\right| \le 2 \sum_{k=1}^{j} (n_k+1) \left|\alpha_{n_k}\right|,$$

and hence

$$\left|\vartheta'_{n_j+1}(t)\right| \le 4n_j \sum_{k=1}^{j} \frac{n_k}{n_j} \left|\alpha_{n_k}\right| \le 4n_j \sum_{k=1}^{j} \left(\frac{1}{\lambda}\right)^{j-k} \left|\alpha_{n_k}\right| \le \frac{4\lambda}{\lambda-1} n_j,$$

which is the first inequality in (17). The second one follows now from (13).

The repeated differentiation of (14) leads to

$$\vartheta''_{n+1} = \vartheta''_n u_n + s_n, \qquad s_n = \left(\omega'_n\right)^2 \Re(iw_n) u_n.$$

By (17) $\left|s_{n_j}\right| \le Cn_j^2 \left|\alpha_{n_j}\right|$, and the same sort of argument (with the only difference that now $n = n_j \in \Lambda$) gives (18). ■

Under certain restrictions on α_n relation (17) can be refined as follows.

Lemma 4. *Let in the premises of the above lemma* $\lim_{n\to\infty} \alpha_n = 0$. *Then*

$$\lim_{n\in\Lambda} \max_t \left|\frac{\vartheta'_n(t)}{n}\right| = 0. \tag{20}$$

P r o o f. Write (14) as

$$\vartheta'_{n+1}(t) - \vartheta'_n(t) = \vartheta'_n(t)\,(u_n - 1) + v_n,$$

which implies $\left|\vartheta'_{n_{j+1}} - \vartheta'_{n_j}\right| \le Cn_j \left|\alpha_{n_j}\right|$ or

$$\left|\vartheta'_{n_{j+1}}(t)\right| \le C\left(\left|\vartheta'_{n_1}(t)\right| + \sum_{k=1}^{j} n_k \left|\alpha_{n_k}\right|\right),$$

$$\left|\frac{\vartheta'_{n_{j+1}}(t)}{n_{j+1}}\right| \le C\left(\left|\frac{\vartheta'_{n_1}}{n_{j+1}}\right| + \sum_{k=1}^{j} \frac{n_k}{n_{j+1}} \left|\alpha_{n_k}\right|\right). \tag{21}$$

It is fairly easy to make sure that the sum in (21) goes to zero as long as α_n does (for Hadamard sequences (3)), that proves (20). ■

Note that (20) can be expressed as

$$\lim_{n\in\Lambda} \max_t \left|\frac{\omega'_n(t)}{n} - 1\right| = 0,$$

which means that

$$\omega'_{n_j}(t) \ge \frac{1}{2}\, n_j, \qquad j \ge j_0. \tag{22}$$

4. Sparse Verblunsky coefficients from the Szego class

Let us turn to the first Nikishin–Prüfer equation (7) written in the form

$$\log R^2_{n_{j+1}}(\zeta) = \log R^2_{n_1}(\zeta) + \sum_{k=1}^{j} \log\left|1 + w_{n_k}(t)\right|^2$$

and differentiate it with respect to t

$$\left(\log R^2_{n_{j+1}}(\zeta)\right)' = \left(\log R^2_{n_1}(\zeta)\right)' + \sum_{k=1}^{j} \frac{\omega'_{n_k}(t)\Re\left(iw_{n_k}\right)}{\left|1 + w_{n_k}(t)\right|^2},$$

whence it follows from (22) that

$$\left|\left(\log R^2_{n_{j+1}}(\zeta)\right)'\right| \le C\left(1 + \sum_{k=1}^{j} n_k \left|\alpha_{n_k}\right|\right). \tag{23}$$

Theorem 5. *Let* $\alpha = \{\alpha_n\}$ *be a sparse sequence* (3)–(4) *of the Verblunsky coefficients from the Szegő class. Then for each closed interval* $I \subset (0, 2\pi)$ *and* $r \in \mathbb{N}$

$$\sup_n \int_I \left|\Phi_n(e^{it})\right|^{2r}\, dt < \infty. \tag{24}$$

P r o o f. It suffices to prove (24) for $n \in \Lambda$. Let $g \in C_\infty$ be an infinitely differentiable function on $[0, 2\pi]$ such that $0 \le g \le 1$, $g = 1$ on I and $g = 0$ near the endpoints. Then

$$\int_I \left|\Phi_{n_j}(e^{it})\right|^{2r} dt = \int_I R_{n_j}^{2r}(e^{it})\, dt \le \int_0^{2\pi} R_{n_j}^{2r}(e^{it}) g(t)\, dt = E_j.$$

By (7)

$$R_{n_{j+1}}^{2r}(\zeta) = R_{n_j}^{2r}(\zeta)\left(1 + 2r\Re w_{n_j}(t) + O(\alpha_{n_j}^2)\right)$$

and so

$$E_{j+1} = \left(1 + O(\alpha_{n_j}^2)\right) E_j + 2r \int_0^{2\pi} R_{n_j}^{2r}(e^{it})\, g(t)\, \Re w_{n_j}(t)\, dt.$$

Let us focus on the latter integral

$$F_j = \Re\alpha_{n_j} \int_0^{2\pi} R_{n_j}^{2r}\, g \cos\omega_{n_j}\, dt - \Im\alpha_{n_j} \int_0^{2\pi} R_{n_j}^{2r}\, g \sin\omega_{n_j}\, dt = \Re\alpha_{n_j} \cdot F_{j1} - \Im\alpha_{n_j} \cdot F_{j2}.$$

The key tool in what follows is integration by parts. Since $g = 0$ near the endpoints, we have

$$F_{j1} = -\int_0^{2\pi} \sin\omega_{n_j} \frac{d}{dt}\left(R_{n_j}^{2r}\, g \frac{1}{\omega'_{n_j}}\right),$$

which consists of three terms. The first one is

$$F_{j1}^{(1)} = \int_0^{2\pi} R_{n_j}^{2r}\, g \frac{\sin\omega_{n_j}}{\omega'_{n_j}}\, dt.$$

It is clear form (7) that

$$R_{n_j}^{2r} \le C \exp\left\{2r \sum_{k=1}^{j-1} |\alpha_{n_k}|\right\}$$

and by (22) for $j \ge j_0$

$$\frac{1}{\omega'_{n_j}} < 2e^{-\log n_j} < Ce^{-j\log\lambda}.$$

Next, $\alpha_n \to 0$ implies $2r\sum_{k=1}^{j}|\alpha_{n_k}| < \frac{1}{2}j\log\lambda$ for $j \ge j_1$ and hence

$$\left|F_{j1}^{(1)}\right| < C\exp\left\{-\frac{\log\lambda}{2}j\right\}.$$

The second term is

$$F_{j1}^{(2)} = 2r\int_0^{2\pi} R_{n_j}^{2r-1}\left(R_{n_j}\right)' g\,\frac{\sin\omega_{n_j}}{\omega'_{n_j}}\,dt = 2r\int_0^{2\pi} R_{n_j}^{2r}\left(\log R_{n_j}\right)' g\,\frac{\sin\omega_{n_j}}{\omega'_{n_j}}\,dt.$$

Now (23) and (22) give

$$\left|F_{j1}^{(2)}\right| \le C\frac{E_j}{n_j}\left\{1+\sum_{k=1}^{j} n_k\left|\alpha_{n_k}\right|\right\}.$$

As for the third term, the bound for the second derivative of the angular variable comes in

$$F_{j1}^{(3)} = \int_0^{2\pi} R_{n_j}^{2r}\, g\sin\omega_{n_j}\left(-\frac{\omega''_{n_j}}{\left(\omega'_{n_j}\right)^2}\right)dt$$

and by (18)

$$\left|F_{j1}^{(3)}\right| \le CE_j\sum_{k=1}^{j}\frac{n_k^2}{n_j^2}\left|\alpha_{n_k}\right|.$$

So we end up with

$$|F_{j1}| \le CE_j\left\{\frac{1}{n_j}+\sum_{k=1}^{j}\frac{n_k}{n_j}\left|\alpha_{n_k}\right|+\sum_{k=1}^{j}\frac{n_k^2}{n_j^2}\left|\alpha_{n_k}\right|\right\}+C\exp(-Cj).$$

Clearly, the same inequality holds for F_{j2} as well, and we complete with the bound

$$E_{j+1} \le \left(1+O(\alpha_{n_k}^2)+\beta_j\right)E_j + C\exp(-Cj),$$

$$\beta_j = \left|\alpha_{n_j}\right|\left\{\frac{1}{n_j}+\sum_{k=1}^{j}\left(\frac{n_k}{n_j}+\frac{n_k^2}{n_j^2}\right)\left|\alpha_{n_k}\right|\right\}.$$

It is shown in [5, Lemma 5.3] that for Hadamard sequences (3)

$$\sum_{j=1}^{\infty}\beta_j \le C\sum_{j=1}^{\infty}\left|\alpha_{n_j}\right|^2,$$

which yields $E_j = O(1)$, $j \to \infty$, as claimed. ■

We are in a position now to prove the main result of the paper.

Theorem 6. *Let* $\alpha = \{\alpha_n\}$ *be a sparse sequence* (3)–(4) *of the Verblunsky coefficients of a measure* μ *from the Szegő class. Then* μ *is absolutely continuous on* $\mathbb{T}$. *Furthermore, for each closed arc* $\Gamma \subset \mathbb{T}\backslash\{1\}$ *and* $p > 0$ *we have* $(\mu')^{\pm 1} \in L^p(\Gamma)$.

P r o o f. We can apply Theorem 5 to both polynomials Φ_n and Ψ_n of the first and second kind to get in addition to (24)

$$\sup_n \int_I \left|\Psi_n(e^{it})\right|^{2r} dt < \infty.$$

Hence the same is true for the norm of transfer matrices [2]

$$\sup_n \int_\Gamma \|\mathcal{T}_n(\zeta)\|^{2r} dm < \infty,$$

where Γ is a closed arc in $\mathbb{T}\backslash\{1\}$ and dm is the normalized Lebesgue measure on $\mathbb{T}$.

In view of [2, Theorem 15] the measure μ is absolutely continuous on $\mathbb{T}\backslash\{1\}$ and it remains only to rule out the possibility of a masspoint at $\zeta = 1$.

We proceed along the line of [1] and write

$$\sum_{n=n_1}^{\infty} \exp\left\{-2\sum_{k=0}^{n-1}|\alpha_k|\right\} = \sum_{j=1}^{\infty}(n_{j+1}-n_j)\exp\left\{-2\sum_{k=1}^{j}|\alpha_{n_k}|\right\}.$$

By Schwarz's inequality

$$\left(\sum_{k=1}^{j}|\alpha_{n_k}|\right)^2 \le j\sum_{k=1}^{j}|\alpha_{n_k}|^2 \le C^2 j, \qquad C^2 \stackrel{\text{def}}{=} \sum_{k=1}^{\infty}|\alpha_{n_k}|^2,$$

so that

$$\sum_{j=1}^{\infty}(n_{j+1}-n_j)\exp\left\{-2\sum_{k=1}^{j}|\alpha_{n_k}|\right\} \ge \sum_{j=1}^{\infty}(n_{j+1}-n_j)\,e^{-2C\sqrt{j}}.$$

By [1, Theorem 6] the corresponding measure μ has no mass points on $\mathbb{T}$, as needed.

As for the second statement, note that $|\Phi_n\Psi_n| \ge C > 0$ for all n gives

$$|\Phi_n|^{-2r} \le C^{-2r}\,|\Psi_n|^{2r}$$

and so (24) is actually true for all integer r. Next, within the Szegő class the relation

$$\lim_{k\in\widetilde{\Lambda}} |\varphi_k|^{-2} = \mu'$$

holds a.e. on $\mathbb{T}$ for some subsequence $\widetilde{\Lambda} \subset \mathbb{N}$. The result now drops out immediately from the Fatou lemma. ∎

R e m a r k. The absolute continuity of the measure on the whole circle can be proved by using the rotation μ_γ of the original measure and taking into account that its Verblunsky coefficients $\alpha_n(\gamma) = e^{in\gamma}\alpha_n$ are also sparse with the same gaps, so both μ and μ_γ are absolutely continuous on $\mathbb{T}\backslash\{1\}$.

Much the same method can be implemented in a slightly more general situation when clusters of nonzero parameters are allowed.

Theorem 7. *Let*

$$\Delta = \bigcup_{k\geq 1} [n_k, n_k + m_k], \qquad n_k + m_k < n_{k+1}$$

be a sequence of positive integers which satisfies the following conditions

(i) $m_k = O(1)$, $k \to \infty$;

(ii) $\inf_k n_{k+1}/n_k > 1$.

Suppose that the Verblunsky coefficients of a measure μ are all zeros off Δ. Then μ is absolutely continuous.

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References

[1] *L. Golinskii*, Mass points of measures and orthogonal polynomials on the unit. — *J. Approx. Theory* (2002), v. 118, p. 257–274.

[2] *L. Golinskii and P. Nevai*, Szegő difference equations, transfer matrices and orthogonal polynomials on the unit circle. — *Comm. Math. Phys.* (2001), v. 223, p. 223–259.

[3] *L. Golinskii, P. Nevai, and W. Van Assche*, Perturbation of orthogonal polynomials on an arc of the unit circle. — *J. Approx. Theory* (1995), v. 83, p. 392–422.

[4] *S. Khrushchev*, Schur's algorithm, orthogonal polynomials and convergence of Wall's continued fractions in $L^2(\mathbb{T})$. — *J. Approx. Theory* (2001), v. 108, p. 161–248.

[5] *A. Kiselev, Y. Last, and B. Simon*, Modified Prüfer and EFGP transforms and the spectral analysis of the one-dimensional Schrödinger operators. — *Comm. Math. Phys.* (1998), v. 194, p. 1–45.

[6] *P. Nevai*, Orthogonal polynomials, measures and recurrences on the unit circle. — *Trans. Amer. Math. Soc.* (1987), v. 300, p. 175–189.

[7] *E.M. Nikishin*, On an estimate for orthogonal polynomials. — *Acta Sci. Math.* (Szeged) (1985), v. 48, p. 395–399 (Russian); Engl. transl.: Amer. Math. Soc. Transl., Ser. 2, (1989), v. 143, p. 129–132.

[8] *B. Simon*, The Golinskii–Ibragimov method and a theorem of Damanik and Killip. — *IMRN* (2003), v. 36, p. 1973–1986.

[9] *G. Szegő*, Orthogonal polynomials. AMS Colloq. Publ. (4th edition). AMS, Providence, RI (1975), v. 23.

[10] *A. Zygmund*, Trigonometric series. Cambridge Univ. Press, Cambridge (1977).

[11] *B. Simon*, Orthogonal polynomials on the unit circle, V. 2: Spectral theory. AMS Colloq. Series. AMS, Providence, RI. (In press)