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On entire functions having Taylor sections with only real zeros

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We investigate power series with positive coefficients having sections with only real zeros. For an entire function $f(z) = \sum_{k=0}^{\infty} a_k z^k,\ \ a_k > 0$, we denote by $q_n(f) := \frac{a_{n-1}^2}{a_{n-2}a_n},\ n \geq 2$. The following problem remains open: which entire function with positive coefficients and sections with only real zeros has the minimal possible $\liminf_{n\to\infty} q_n(f)$? We prove that the extremal function in the class of such entire functions with additional condition $\exists \lim_{n\to\infty} q_n(f)$ is the function of the form $f_a(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!\ a^{k^2}}$. We answer also the following questions: for which a do the function $f_a(z)$ and the function $y_a(z) := 1 + \sum_{k=1}^{\infty} \frac{z^k}{(a^k-1)(a^{k-1}-1)\cdots(a-1)}$, $a > 1$, have sections with only real zeros?

To Iossif Vladimirovich Ostrovskii on the occasion of his 70-th birthday

1. Introduction and statement of results

There are many papers concerning the zero distribution of sections (and tails) of power series, see for example a very interesting survey of the topic in [8]. In

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this paper we investigate power series with nonnegative coefficients having sections with only real zeros.

By R^* we will denote the set of real polynomials having only real zeros. The following fact was mentioned by Pólya in [10]:

Theorem A. *Let $P(z) = a_0 + a_1 z + \ldots + a_n z^n \in R^*$, $a_j > 0$, $j = 0, 1, \ldots, n$ and $n \geq 2$. Then*

$$\frac{a_{n-1}}{a_n} \geq \frac{2n}{n-1} \cdot \frac{a_{n-2}}{a_{n-1}}. \tag{1}$$

Let

$$\sum_{k=0}^{\infty} a_k z^k, \quad a_k > 0 \quad \text{for } k \in \mathbf{N} \cup \{\mathbf{0}\}, \tag{2}$$

be a formal power series and let

$$S_n(z) = \sum_{k=0}^{n} a_k z^k, \ n \in \mathbf{N} \cup \{\mathbf{0}\} \tag{3}$$

be its sections.

The following theorem is a corollary of Theorem A.

Theorem B. *Let the formal power series* (2) *have the property:* $\exists N \in \mathbf{N}$: $\forall n \geq N \ S_n \in R^*$. *Then this series is absolutely convergent in* $\mathbf{C}$, *i.e., its sum is an entire function.*

We will consider three classes of entire functions:

$S^* := \Big\{ f(z) = \sum_{k=0}^{\infty} a_k z^k : a_k > 0, \ \forall k; \ \exists \{n_k\} \subset \mathbf{N}, \ n_k \to \infty$, such that $\forall k \in \mathbf{N} \ S_{n_k} \in R^* \Big\}$;

$A^* := \Big\{ f(z) = \sum_{k=0}^{\infty} a_k z^k : a_k > 0, \ \forall k; \ \exists N = N(f) \in \mathbf{N}, \ \forall n \geq N \ S_n \in R^* \Big\}$;

$B^* := \Big\{ f(z) = \sum_{k=0}^{\infty} a_k z^k \ : \ a_k > 0, \ \forall k; \ \forall n \in \mathbf{N} \ S_n \in R^* \Big\}$.

Obviously, $B^* \subset A^* \subset S^*$. We need also two notations:

$$p_n = p_n(f) := \frac{a_{n-1}}{a_n}, \ n \geq 1; \quad q_n = q_n(f) := \frac{p_n}{p_{n-1}} = \frac{a_{n-1}^2}{a_{n-2} a_n}, \ n \geq 2. \tag{4}$$

Note that

$$a_n = \frac{a_0}{p_1 p_2 \ldots p_n}, \ n \geq 1 \ ; \quad a_n = \frac{a_1}{q_2^{n-1} q_3^{n-2} \ldots q_{n-1}^2 q_n} \left(\frac{a_1}{a_0} \right)^{n-1}, \ n \geq 2. \tag{5}$$

Using these notations and Theorem A, we can state

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \in A^* \ \Rightarrow \ q_n(f) \geq 2, \ \forall n \geq N(f). \tag{6}$$

In 1926, Hutchinson [5, p. 327] extended the work of Petrovitch [9] and Hardy [3] or [4, p. 95–100] and proved the following theorem.

Theorem C. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k, \ \ a_k > 0, \ \ \forall k.$ *Inequality* $q_n(f) \geq 4,$ $\forall n \geq 2$ *holds if and only if the following two properties hold:*

(i) the zeros of f *are all real, simple and negative and*

(ii) the zeros of any polynomial $\sum_{k=m}^{n} a_k z^k$, *formed by taking any number of consecutive terms of* f, *are all real and nonpositive.*

For some extensions of Hutchinson's results see, for example, [1, § 4]. The following statement is a corollary of Theorem C.

Theorem D. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$, $\forall k$, *and* $q_n(f) \geq 4$, $\forall n \geq 2$. *Then* $f \in B^*$.

We obtain from (6) that for every $f \in A^*$

$$\liminf_{n\to\infty} q_n(f) \geq 2. \tag{7}$$

In [6] it is proved that the constant 2 in (7) can be increased and the constant 4 can not be decreased even in the statement

$$q_n(f) \geq 4 \ \forall n \geq 2 \Rightarrow f \in S^*.$$

Theorem E. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k \in A^*$. *Then* $\liminf_{n\to\infty} q_n(f) \geq 1 + \sqrt{3}$.

R e m a r k. Using the same method as in the proof of Theorem E after cumbersome calculations, we can prove that

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \in A^* \Rightarrow \liminf_{n\to\infty} q_n(f) > 2.9.$$

Theorem F. *For every* $\varepsilon > 0$ *there exists* $f_\varepsilon(z) = \sum_{k=0}^{\infty} a_k(\varepsilon) z^k$ *such that* $\forall k \in \mathbf{N} \cup \{\mathbf{0}\}$ $\ a_k(\varepsilon) > 0$ *and* $\forall n \geq 2 \ \ q_n(f_\varepsilon) > 4 - \varepsilon$ *but* $f_\varepsilon \notin S^*$.

In connection with the above mentioned theorems it is natural to investigate the function

$$g_a(z) := \sum_{k=0}^{\infty} \frac{z^k}{a^{k^2}}, \ \ a > 1, \tag{8}$$

with the property $q_n(g_a) = a^2$ for all $n \geq 2$. In [4, p. 95–100] it is shown that $g_a(z)$ has only real zeros if $a^2 \geq 9$. In [14, Problem 176, p. 66] it is proved that $g_a(z)$ has only real zeros if $a^2 \geq 4$. The question about the smallest value of a for which $g_a(z)$ has only real zeros was discussed by T. Craven and G. Csordas in [2]. T. Craven and G. Csordas have improved the method of [14, Problem 176, p. 66] and have shown that $a^2 \geq 3.4225$ is enough see [2, Examples 4.10,4.11]. In [6] it is given the answer to the question: for which a does the function $g_a(z)$ have only real zeros?

Theorem G. *There exists a constant q_∞ ($q_\infty \approx 3.23$) such that:*

1. $S_{2k+1}(z, g_a) := \sum_{j=0}^{2k+1} \frac{z^j}{a^{j^2}} \in R^*$ *for every* $k \in \mathbf{N} \ \Leftrightarrow\ a^2 \geq q_\infty$;
2. $\exists N_0 \in \mathbf{N} \quad \forall k \geq N_0 \quad S_{2k}(z, g_a) := \sum_{j=0}^{2k} \frac{z^j}{a^{j^2}} \in R^* \ \Leftrightarrow\ a^2 > q_\infty$;
3. $g_a(z)$ *has only real zeros* $\Leftrightarrow\ a^2 \geq q_\infty$.

In [6] it is noted also that

$$S_{2k}(z, g_a) \in R^* \Rightarrow \forall m \geq k \quad S_{2m}(z, a) \in R^*; \tag{9}$$

$$S_{2k+1}(z, g_a) \in R^* \Rightarrow \forall m \leq k \quad S_{2m+1}(z, a) \in R^* \tag{10}$$

and

$$S_n(z, g_a) \in R^* \Leftrightarrow \exists x_n \in [a, a^3] : S_n(-x_n, g_a) \leq 0. \tag{11}$$

Using Theorem D and considering $S_2(z, g_a)$, it is easy to see that $g_a(z) \in B^* \ \Leftrightarrow\ a^2 \geq 4$.

The question about the sharp constant in Theorem E is open. Theorem G shows that this sharp constant is less than or equal to q_∞.

Definition 1. *A sequence $\{\gamma_k\}_{k=0}^\infty$ of real numbers is called a multiplier sequence if, whenever the real polynomial $P(x) = \sum_{k=0}^n a_k z^k \in R^*$, the polynomial $\sum_{k=0}^n \gamma_k a_k z^k \in R^*$. The class of multiplier sequences we will denote by MS.*

The following famous theorem by G. Pólya and J. Schur gives the complete characterization of multiplier sequences:

Theorem H (see [13], [12] or [7, Ch. VIII, Sect. 3]). *A sequence $\{\gamma_k\}_{k=0}^\infty$ is a multiplier sequence if and only if the power series $\Phi(z) := \sum_{k=0}^\infty \frac{\gamma_k}{k!} z^k$ converges absolutely in the whole complex plane and the entire function $\Phi(z)$ or the entire function $\Phi(-z)$ admit the representation*

$$C e^{\sigma z} z^m \prod_{k=1}^\infty (1 + \frac{z}{x_k}), \tag{12}$$

where $C \in \mathbf{R}, \sigma \geq 0, m \in \mathbf{N} \cup \{0\}, 0 < x_k \leq \infty,\ \sum_{k=1}^\infty \frac{1}{x_k} < \infty$.

The simple consequence of Theorem H is that the sequence $\{\gamma_0, \gamma_1, \ldots, \gamma_l, 0, 0, \ldots\}$ is a multiplier sequence if and only if the polynomial $P(z) = \sum_{k=0}^{l} \frac{\gamma_k}{k!} z^k$ has only real zeros of the same sign.

For a real polynomial P we will denote by $Z_c(P)$ the number of nonreal zeros of P, counting multiplicities.

Definition 2. *A sequence $\{\gamma_k\}_{k=0}^{\infty}$ of real numbers is said to be a complex zero decreasing sequence if*

$$Z_c(\sum_{k=0}^{n} \gamma_k a_k z^k) \leq Z_c(\sum_{k=0}^{n} a_k z^k), \tag{13}$$

for any real polynomial $\sum_{k=0}^{n} a_k z^k$. We will denote the class of complex zero decreasing sequences by CZDS.

Obviously, $CZDS \subset MS$. The existence of nontrivial CZDS sequences is a consequence of the following remarkable theorem proved by Laguerre and extended by Pólya (see [11] or [12, p. 314–321]).

Theorem I. *Suppose an entire function $f(z)$ can be expressed in the form*

$$f(z) = Cz^m e^{-\alpha z^2 + \beta z} \prod_{k=1}^{\infty} (1 + \frac{z}{x_k}) e^{-\frac{z}{x_k}}, \tag{14}$$

where $m \in \mathbf{N} \cup \{0\}$, C, $\beta \in \mathbf{R}$, $\alpha \geq 0$ and $0 < x_k \leq \infty$, $\sum_{k=1}^{\infty} \frac{1}{x_k^2} < \infty$. Then the sequence $\{f(k)\}_{k=0}^{\infty}$ is a complex zero decreasing sequence.

As it follows from the above theorem,

$$\{a^{-k^2}\}_{k=0}^{\infty} \in CZDS,\ a \geq 1, \quad \{\frac{1}{k!}\}_{k=0}^{\infty} \in CZDS. \tag{15}$$

Denote by

$$q_{\text{inf}} := \inf_{f \in A^*} \liminf_{n \to \infty} q_n(f).$$

Theorem G shows that $q_\infty \geq q_{\text{inf}}$. The problem about the precise value of q_{inf} remains open, and this problem is of interest for the authors. It is also unknown whether or not does there exist the "extremal function" in A^*, namely such function $f_{\text{inf}} \in A^*$ that $\liminf_{n \to \infty} q_n(f_{\text{inf}}) = q_{\text{inf}}$.

In this paper we will investigate the function

$$f_a(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!\ a^{k^2}},\ a > 1,$$

with the property

$$q_n(f_a) = \frac{n}{n-1}a^2 \ \to a^2, \ n \to \infty. \tag{16}$$

Since $\{a^{-k^2}\}_{k=0}^{\infty} \in CZDS$ for $a \geq 1$, this sequence is a multiplier sequence. Hence, by Theorem H, $f_a(z)$ has only real (and negative) zeros. We will answer the following question: for which a does the function $f_a(z)$ have sections with only real zeros?

Theorem 1.

$$f_a \in S^* \Leftrightarrow f_a \in B^* \Leftrightarrow a^2 \geq q_\infty,$$

where the constant q_∞ *was introduced in Theorem G.*

To motivate this investigation we prove the following statement.

Theorem 2. *Let* $f(z) \in S^*$. *If there exists* $\lim_{n\to\infty} q_n(f)$ *and* $q_0 := \lim_{n\to\infty} q_n(f)$ *then for every* $m \in \mathbf{N}$ *we have* $\sum_{k=0}^{m} \frac{z^k}{k!\ (\sqrt{q_0})^{k^2}} \in R^*$.

The following statement is a corollary of Theorems 1 and 2.

Corollary. *If* $f(z) \in S^*$ *and* $\exists \lim_{n\to\infty} q_n(f)$ *then* $\lim_{n\to\infty} q_n(f) \geq q_\infty$.

Denote by $L^* = \{f \in S^* : \exists \lim_{n\to\infty} q_n(f)\}$. Corollary shows that

$$\inf_{f\in L^*} \lim_{n\to\infty} q_n(f) = q_\infty$$

and the "extremal function" in the class L^* is f_{q_∞}. At the moment we do not know whether or not the function f_{q_∞} is the "extremal function" in the class A^*.

The following identity I.J. Schoenberg attributed to Gauss:

$$\prod_{k=1}^{\infty}(1+\frac{z}{q^k}) = 1 + \sum_{k=1}^{\infty} \frac{z^k}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}, \quad q > 1. \tag{17}$$

So the entire function $y_q(z) := 1 + \sum_{k=1}^{\infty} \frac{z^k}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}, \quad q > 1$, with the property

$$q_n(y_q) = \frac{q^n-1}{q^{n-1}-1} \ \to q, \ n \to \infty, \tag{18}$$

has only real zeros.

Prof. I.V. Ostrovskii posed the problem: for which q does the function $y_q(z)$ have sections with only real zeros?

Theorem 3. *1.* $q > q_\infty \Rightarrow \ \exists N_0 \in \mathbf{N} \quad \forall n \geq N_0 \quad S_n(z, y_q) \in R^*$;

2. $q < q_\infty \Rightarrow \ \exists N_0 \in \mathbf{N} \quad \forall n \geq N_0 \quad S_n(z, y_q) \notin R^*$.

2. Proof of Theorem 1

The following identity

$$\frac{d}{dz}S_n(z,f_a)=\frac{1}{a}S_{n-1}(\frac{z}{a^2},f_a) \tag{19}$$

shows that

$$S_n(z,f_a)\in R^* \Longrightarrow S_{n-1}(z,f_a)\in R^*,$$

or

$$f_a\in S^* \Longleftrightarrow f_a\in B^*. \tag{20}$$

By Theorem G (1)

$$\sum_{j=0}^{2k+1}\frac{z^j}{a^{j^2}}\in R^* \quad \text{for every} \quad k\in \mathbf{N} \iff a^2\geq q_\infty. \tag{21}$$

Since $\{\frac{1}{k!}\}_{k=0}^\infty \in CZDS$ (see (15)) and by (20) we obtain

$$a^2\geq q_\infty \Longrightarrow f_a\in B^*. \tag{22}$$

It remains to prove that

$$f_a\in A^* \Longrightarrow a^2\geq q_\infty. \tag{23}$$

Note that since $\{a^{-k^2}\}_{k=0}^\infty \in CZDS,\ \forall a\geq 1$ (see (15)) we have

$$S_n(z,f_{\tilde a})\in R^* \Longrightarrow \forall a\geq \tilde a \quad S_n(z,f_a)\in R^*.$$

Let

$$k_n:=\inf\{a>1: S_n(z,f_a)\in R^*\}. \tag{24}$$

By (19) we have

$$k_2\leq k_3\leq k_4\leq \ldots,$$

and so

$$\exists \lim_{n\to\infty} k_n.$$

Denote by $k_\infty=\lim_{n\to\infty}k_n$. We know that $k_\infty\leq q_\infty$, and we are going to prove that $k_\infty=q_\infty$.

We will consider polynomials

$$F_n(z,a):=S_n(-z,f_a)=\sum_{k=o}^{n}\frac{(-1)^k z^k}{k!a^{k^2}}.$$

Obviously,

$$F_n(z,a)\in R^* \Leftrightarrow S_n(z,f_a)\in R^*.$$

We will answer the following question: for which a do polynomials $F_n(z,a)$ have only real zeros? We need the following Lemma.

Lemma 1. *Suppose $a^2 \geq 3$. Then $\exists\ n_0 \in \mathbf{N} \quad \forall n \geq n_0$ polynomial $F_n(z,a)$ has exactly two roots in the domain $\{z : |z| > na^{2n-3}\}$.*

P r o o f o f L e m m a 1. We have

$$F_n(z,a) = \frac{(-1)^n z^n}{n!\ a^{n^2}} \sum_{k=o}^{n} (-1)^{k-n} \frac{n!}{k! z^{n-k} a^{k^2-n^2}}.$$

Denote by $t := \frac{na^{2n}}{z}$. For $|z| > na^{2n-3}$ we have $|t| < a^3$. We obtain for $n \geq 4$

$$\begin{gathered} F_n(z,a) = \frac{(-1)^n z^n}{n!\ a^{n^2}} \sum_{k=o}^{n} (-1)^{k-n}\ \frac{(n-1)!}{k!\ n^{n-k-1}} \frac{t^{n-k}}{a^{(n-k)^2}} \\ = \frac{(-1)^n\ z^n}{n!\ a^{n^2}} \sum_{j=o}^{n} (-1)^j\ \frac{(n-1)!}{(n-j)!\ n^{j-1}} \frac{t^j}{a^{j^2}} \\ = \frac{(-1)^n z^n}{n!\ a^{n^2}} \Big((1 - \frac{t}{a} + \frac{n-1}{n} \frac{t^2}{a^4} - \frac{(n-1)(n-2)}{n^2} \frac{t^3}{a^9} + \frac{(n-1)(n-2)(n-3)}{n^3} \frac{t^4}{a^{16}}) \\ + \sum_{j=5}^{n} (-1)^j\ \frac{(n-1)!}{(n-j)!\ n^{j-1}} \frac{t^j}{a^{j^2}} \Big). \qquad (25) \end{gathered}$$

Further we need two lemmas from [6] concerning the polynomial $S_4(-z, g_a) = 1 - \frac{t}{a} + \frac{t^2}{a^4} - \frac{t^3}{a^9} + \frac{t^4}{a^{16}}$. For the sake of completeness we will present the short proofs of these lemmas.

Lemma 2. *For $a^2 \geq 3$ the inequality holds*

$$|S_4(-a^3 e^{i\varphi}, g_a)| \geq a^{-4}, \quad \forall \varphi \in [0, 2\pi]. \qquad (26)$$

P r o o f o f L e m m a 2. We have

$$\begin{gathered} S_4(-a^3 e^{i\varphi}, g_a) = 1 - a^2 e^{i\varphi} + a^2 e^{2i\varphi} - e^{3i\varphi} + a^{-4} e^{4i\varphi} \\ = -ie^{3i\varphi/2} \left((2\sin(3\varphi/2) - 2a^2 \sin(\varphi/2)) + ia^{-4} e^{5i\varphi/2} \right), \qquad (27) \end{gathered}$$

whence

$$\begin{gathered} |S_4(-a^3 e^{i\varphi}, g_a)|^2 = 4 \left(\sin(3\varphi/2) - a^2 \sin(\varphi/2) \right)^2 \\ -4a^{-4} \sin(5\varphi/2) \left(\sin(3\varphi/2) - a^2 \sin(\varphi/2) \right) + a^{-8}. \qquad (28) \end{gathered}$$

After simple transformation we obtain

$$\begin{gathered} |S_4(-a^3 e^{i\varphi}, g_a)|^2 = 4 \sin^2(\varphi/2) \left((a^2 - 3) + 4 \sin^2(\varphi/2) \right)^2 \\ +4a^{-4} \sin(5\varphi/2) \sin(\varphi/2) \left((a^2 - 3) + 4 \sin^2(\varphi/2) \right) + a^{-8}. \qquad (29) \end{gathered}$$

For $a^2 \geq 3$ and $\varphi \in [0, 2\pi]$ (26) is a consequence of

$$\sin(\varphi/2)\left((a^2-3)+4\sin^2(\varphi/2)\right)+a^{-4}\sin(5\varphi/2) \geq 0.$$

The last inequality follows from

$$4a^4 \sin^3 \varphi/2 + \sin(5\varphi/2) \geq 0 \tag{30}$$

for $\varphi \in [0, 2\pi]$. If $\varphi \in [0, 2\pi/5] \cup [8\pi/5, 2\pi]$, then $\sin(5\varphi/2) \geq 0$ and (30) holds. If $\varphi \in [2\pi/5, 8\pi/5]$, then (30) follows from

$$4a^4 \sin^3 \pi/5 - 1 \geq 0,$$

that is true since $\sin^3 \pi/5 \geq \sin^3 \pi/6 = 1/8$.

Lemma 2 is proved.

Lemma 3. *If $a^2 \geq 3$ then $S_4(-z, g_a)$ has exactly two zeros in $\{z : |z| < a^3\}$ and has no zeros in $\{z : |z| = a^3\}$.*

P r o o f o f L e m m a 3. Denote by $P_a(t) := S_4(-a^4 t, a) = 1 - a^3 t + a^4 t^2 - a^3 t^3 + t^4$. We are going to show that $P_a(t)$ has exactly two zeros in $\{t : |t| < a^{-1}\}$ (and exactly two zeros in $\{t : |t| \leq a^{-1}\}$.) Let $w(t) := t + t^{-1}$. The function $w(t)$ maps conformally $\{t : |t| < a^{-1}\}$ on a domain Ω such that $\{w : |w| > a + a^{-1}\} \subset \Omega$. We have $P_a(t) = t^2(w^2 - 2 - a^3 w + a^4)$. Let us show that $Q_a(w) := (w^2 - 2 - a^3 w + a^4)$ has exactly two zeros in $\{w : |w| > a + a^{-1}\}$. Let w_1, w_2 be the zeros of $Q_a(z)$ and let D be the discriminant of $Q_a(z)$. If $D \leq 0$, then $|w_j| \geq \text{Re } w_j = \frac{a^3}{2} \geq \frac{3a}{2} > a + a^{-1}$ for $j = 1, 2$. If $D > 0$, then

$$|w_j| \geq \frac{a^3 - \sqrt{a^6 - 4a^4 + 8}}{2} > a + a^{-1}, \quad j = 1, 2.$$

So for $a^2 \geq 3$ the polynomial $Q_a(w)$ has exactly two zeros in $\{w : |w| > a + a^{-1}\}$. Therefore $P_a(t)$ has exactly two zeros in $\{t : |t| < a^{-1}\}$ and has no zeros in the boundary of this circle.

Lemma 3 is proved.

Let's continue the proof of Lemma 1. Since

$$\begin{aligned} Q_4(t,a) &:= (1 - \tfrac{t}{a} + \tfrac{n-1}{n}\tfrac{t^2}{a^4} - \tfrac{(n-1)(n-2)}{n^2}\tfrac{t^3}{a^9} \\ &+ \tfrac{(n-1)(n-2)(n-3)}{n^3}\tfrac{t^4}{a^{16}}) \longrightarrow S_4(-t, g_a), \quad n \to \infty, \end{aligned} \tag{31}$$

and this limit is uniform on the compact sets, we have by Lemmas 2 and 3 and Hurwitz theorem that

$$\exists n_0 \quad \forall n \geq n_0 \quad \forall \phi \in [0, 2\pi] \quad |Q_4(a^3 e^{i\phi}, a)| \geq \frac{1}{2} a^{-4}, \tag{32}$$

and $\forall n \geq n_0$ polynomial $Q_4(t,a)$ has exactly two roots in the circle $\{t : |t| < a^3\}$. By (25) we have

$$F_n(z,a) = \tfrac{(-1)^n z^n}{n!\ a^{n^2}} \Big(Q_4(t,a) + \textstyle\sum_{j=5}^{n} (-1)^j \tfrac{(n-1)(n-2)\cdots(n-j+1)}{n^{j-1}} \tfrac{t^j}{a^{j^2}}\Big) =: \tfrac{(-1)^n z^n}{n!\ a^{n^2}} (Q_4(t,a) + T_n(t,a)). \quad (33)$$

Since

$$|T_n(a^3 e^{i\phi}, a)| \leq \sum_{j=5}^{n} \frac{(a^3)^j}{a^{j^2}} \leq a^{-10} \sum_{j=0}^{\infty} a^{-8j} = \frac{1}{a^2(a^8-1)},$$

and

$$\frac{1}{2} a^{-4} > \frac{1}{a^2(a^8-1)}, \quad a^2 \geq 3,$$

the statement of Lemma 1 follows.

Lemma 1 is proved.

Let us prove (23). Suppose $f_a \in A^*$ for some $a^2 \geq 3$. Then $\exists\ n_0 \in \mathbf{N}\ \ \forall n \geq n_0$ $F_n(z,a) \in R^*$. By Lemma 1 $\exists\ n_1 \in \mathbf{N}\ \ \forall n \geq n_0$ polynomial $F_n(z,a)$ has exactly two roots in the domain $\{z : |z| > na^{2n-3}\}$. Then for $n \geq \max(n_0, n_1) =: n_2$ $\exists x_n \in (na^{2n-3}, \infty) :\ F_n(x_n, a) = 0$. For $x \geq na^{2n-1}$ we have

$$1 < \frac{x}{a} < \frac{x^2}{2a^4} < \cdots < \frac{x^{n-1}}{(n-1)!\ a^{(n-1)^2}} \leq \frac{x^n}{n!\ a^{n^2}},$$

and so

$$x \geq na^{2n-1} \Rightarrow F_n(x,a) \neq 0.$$

Thus, $x_n \in (na^{2n-3}, na^{2n-1})$. Let us fix any $m \in \mathbf{N}$. We have for $n > \max(n_2, 2m+4)$

$$0 = (-1)^{n-1} F_n(x_n, a) = \textstyle\sum_{k=0}^{n-2m} (-1)^{k+n-1} \tfrac{x_n^k}{k!\ a^{k^2}} + \Big(\tfrac{x_n^{n-2m+1}}{(n-2m+1)!\ a^{(n-2m+1)^2}} - \cdots - \tfrac{x_n^{n-2}}{(n-2)!\ a^{(n-2)^2}} + \tfrac{x_n^{n-1}}{(n-1)!\ a^{(n-1)^2}} - \tfrac{x_n^{n}}{(n)!\ a^{(n)^2}}\Big). \quad (34)$$

For $x_n \in (na^{2n-3}, na^{2n-1})$ summands in $\sum_{k=0}^{n-2m} (-1)^{k+n-1} \frac{x_n^k}{k!\ a^{k^2}}$ are alternating in sign and their moduli are increasing. So

$$\sum_{k=0}^{n-2m} (-1)^{k+n-1} \frac{x_n^k}{k!\ a^{k^2}} < 0,$$

and by (34) we obtain

$$\Big(\frac{x_n^{n-2m+1}}{(n-2m+1)!\ a^{(n-2m+1)^2}} - \cdots - \frac{x_n^{n-2}}{(n-2)!\ a^{(n-2)^2}}$$

$$+\frac{x_n^{n-1}}{(n-1)!\ a^{(n-1)^2}} - \frac{x_n^n}{(n)!\ a^{(n)^2}}\Bigg) > 0.$$

Dividing this inequality by $-\frac{x_n^n}{(n)!\ a^{(n)^2}}$ and rewriting it from right to left, we obtain

$$\begin{aligned} 1 - \tfrac{n}{x_n}a^{2n-1} + \tfrac{n(n-1)}{x_n^2}a^{2(2n-2)} - \tfrac{n(n-1)(n-2)}{x_n^3}a^{3(2n-3)} \\ + \cdots + \tfrac{n(n-1)\cdots(n-2m+3)}{x_n^{2m-2}}a^{(2m-2)(2n-2m+2)} \\ - \tfrac{n(n-1)\cdots(n-2m+2)}{x_n^{2m-1}}a^{(2m-1)(2n-2m+1)} < 0. \end{aligned} \tag{35}$$

Denote by $y_n = \frac{na^{2n}}{x_n}$. Since $x_n \in (na^{2n-3}, na^{2n-1})$ we have $y_n \in (a, a^3)$. In this notation we rewrite (35) in the form

$$\begin{aligned} 1 - \tfrac{y_n}{a} + \tfrac{(n-1)}{n}\tfrac{y_n^2}{a^4} - \tfrac{(n-1)(n-2)}{n^2}\tfrac{y_n^3}{a^9} + \cdots \\ + \tfrac{(n-1)(n-2)\cdots(n-2m+3)}{n^{2m-3}}\tfrac{y_n^{2m-2}}{a^{(2m-2)^2}} - \tfrac{(n-1)(n-2)\cdots(n-2m+2)}{n^{2m-2}}\tfrac{y_n^{2m-1}}{a^{(2m-1)^2}} < 0. \end{aligned} \tag{36}$$

Passing to the limit in this formula as $n \to \infty$, we obtain that there exists $y_0 \in [a, a^3]$ such that

$$S_{2m-1}(y_0, g_a) \le 0.$$

By (11) it means

$$S_{2m-1}(z, g_a) \in R^*.$$

Since m is an arbitrary positive integer we obtain by Theorem G (1) that

$$a^2 \ge q_\infty.$$

Using (21) and (20), we conclude that

$$\forall n \in \mathbf{N} \quad F_n(z,a) \in R^* \Longleftrightarrow a^2 \ge q_\infty.$$

Theorem 1 is proved.

3. Proof of Theorem 2

Let us fix an arbitrary $m \in \mathbf{N}$. For $n_k > m$ we have

$$S_{n_k}(x) \in R^* \Longrightarrow S_{n_k}^{(n_k-m)}(x) = \sum_{j=0}^{m} \frac{(n_k-m+j)!}{j!} a_{n_k-m+j}x^j \in R^*$$

$$\Longrightarrow \frac{1}{(n_k-m)!a_{n_k-m}} S_{n_k}^{(n_k-m)}\Big(\frac{a_{n_k-m}}{(n_k-m+1)a_{n_k-m+1}}x\Big) = 1 + x$$

$$+\sum_{j=2}^{m}\frac{(n_k-m+2)\cdot(n_k-m+3)\cdot\dots\cdot(n_k-m+j)}{(n_k-m+1)^{j-1}}\frac{1}{j!}\frac{a_{n_k-m+j}a_{n_k-m}^{j-1}}{a_{n_k-m+1}^{j}}x^j\in R^*.$$

Using (5), we can rewrite the last polynomial in the form

$$1+x+\sum_{j=2}^{m}(\frac{(n_k-m+2)\cdot(n_k-m+3)\cdot\dots\cdot(n_k-m+j)}{(n_k-m+1)^{j-1}}\frac{1}{j!}$$

$$\times\frac{x^j}{q_{n_k-m+2}(f)^{j-1}q_{n_k-m+3}(f)^{j-2}\cdot\dots\cdot q_{n_k-m+j}(f)})\in R^*.$$

Taking the limit as $n_k\to\infty$, we obtain

$$\sum_{j=0}^{m}\frac{x^j}{j!q_0^{j(j-1)/2}}\in R^*,$$

and putting $x=\frac{z}{\sqrt{q_0}}$, we have

$$\sum_{k=0}^{m}\frac{z^k}{k!\,(\sqrt{q_0})^{k^2}}\in R^*.$$

Theorem 2 is proved.

4. Proof of Theorem 3

The statement

$$q<q_\infty\Rightarrow\ \exists N_0\in\mathbf{N}\quad\forall n\geq N_0\quad S_n(z,y_q)\notin R^*$$

in Theorem 3 is a simple consequence of the Corollary formulated after Theorems 2 and 1. Let us prove that

$$q>q_\infty\Rightarrow\ \exists N_0\in\mathbf{N}\quad\forall n\geq N_0\quad S_n(z,y_q)\in R^*.$$

Denote by

$$S_n(x,q):=S_n(-x,y_q)=1-\frac{x}{q-1}+\frac{x^2}{(q^2-1)(q-1)}-\cdots$$

$$+(-1)^n\frac{x^n}{(q^n-1)(q^{n-1}-1)\cdots(q-1)}.$$

Obviously, $S_n(x,q)\in R^*\Leftrightarrow S_n(x,y_q)\in R^*$. We will investigate polynomials $S_n(x,q)$ for $q>q_\infty$. In this section we will use the following agreement: we mean that the expression $(q^j-1)(q^{j-1}-1)\cdots(q-1)$ is equal to 1 when $j=0$

We need the following two lemmas.

Lemma 4. *Suppose $q \geq 3$. There exist nonnegative numbers $0 = \xi_0 < \xi_1 < \xi_2 < \ldots < \xi_{n-2}$, such that:*
(i) $q^j - 1 < \xi_j < q^{j+1} - 1$;
(ii) $(-1)^j S_n(\xi_j, q) \geq 0$.

P r o o f o f L e m m a 4. Put $\xi_j = \sqrt{(q^{j+1}-1)(q^j-1)}, j = 1, 2, \ldots$. Obviously, (i) holds. For $j = 3, 4, \ldots, n-3$ we have

$$(-1)^j S_n(\xi_j, q) = \frac{\xi_j^j}{(q^j-1)(q^{j-1}-1)\cdots(q-1)}$$
$$-\left(\frac{\xi_j^{j-1}}{(q^{j-1}-1)(q^{j-2}-1)\cdots(q-1)} + \frac{\xi_j^{j+1}}{(q^{j+1}-1)(q^j-1)\cdots(q-1)}\right)$$
$$+\left(\frac{\xi_j^{j-2}}{(q^{j-2}-1)(q^{j-3}-1)\cdots(q-1)} + \frac{\xi_j^{j+2}}{(q^{j+2}-1)(q^{j+1}-1)\cdots(q-1)}\right)$$
$$-\left(\frac{\xi_j^{j-3}}{(q^{j-3}-1)(q^{j-4}-1)\cdots(q-1)} + \frac{\xi_j^{j+3}}{(q^{j+3}-1)(q^{j+2}-1)\cdots(q-1)}\right) + R_{j,n}(\xi_j, q), \tag{37}$$

where

$$R_{j,n}(\xi_j, q) = \sum_{k=0}^{j-4} (-1)^{k+j} \frac{\xi_j^k}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}$$
$$+\sum_{k=j+4}^{n} (-1)^{k+j} \frac{\xi_j^k}{(q^k-1)(q^{k-1}-1)\cdots(q-1)} =: \Sigma_1(\xi_j, q) + \Sigma_2(\xi_j, q). \tag{38}$$

For $\forall z \in (q^j - 1, q^{j+1} - 1)$

$$\frac{z^k}{(q^k-1)(q^{k-1}-1)\cdots(q-1)} \leq \frac{z^{k+1}}{(q^{k+1}-1)(q^k-1)\cdots(q-1)}$$

holds for $0 \leq k \leq j-1$. So for all $z \in (q^j - 1, q^{j+1} - 1)$ summands in $\Sigma_1(z, q)$ are alternating in sign and their moduli are increasing. Analogously $\forall z \in (q^j - 1, q^{j+1} - 1)$

$$\frac{z^k}{(q^k-1)(q^{k-1}-1)\cdots(q-1)} \geq \frac{z^{k+1}}{(q^{k+1}-1)(q^k-1)\cdots(q-1)}$$

holds for $j+1 \leq k \leq n$. So for all $z \in (q^j - 1, q^{j+1} - 1)$ summands in $\Sigma_2(z, q)$ are alternating in sign and their moduli are decreasing. Thus $\Sigma_1(z, q) \geq 0$, $\Sigma_2(z, q) \geq 0$ for all $z \in (q^j - 1, q^{j+1} - 1)$, and, in particular, it is true for

$z = \xi_j = \sqrt{(q^j - 1)(q^{j+1} - 1)}$. Hence we have

$$(-1)^j S_n(\xi_j, q) \geq \tfrac{(q^{j+1}-1)^{j/2}(q^j-1)^{j/2-1}}{(q^{j-1}-1)(q^{j-2}-1)\cdots(q-1)} - 2\tfrac{(q^{j+1}-1)^{(j-1)/2}(q^j-1)^{(j-1)/2}}{(q^{j-1}-1)(q^{j-2}-1)\cdots(q-1)}$$
$$+\left(\tfrac{(q^{j+1}-1)^{j/2-1}(q^j-1)^{j/2-1}}{(q^{j-2}-1)(q^{j-3}-1)\cdots(q-1)} + \tfrac{(q^{j+1}-1)^{j/2+1}(q^j-1)^{j/2+1}}{(q^{j+2}-1)(q^{j+1}-1)\cdots(q-1)}\right)$$
$$-\left(\tfrac{(q^{j+1}-1)^{(j-3)/2}(q^j-1)^{(j-3)/2}}{(q^{j-3}-1)(q^{j-4}-1)\cdots(q-1)} + \tfrac{(q^{j+1}-1)^{(j+3)/2}(q^j-1)^{(j+3)/2}}{(q^{j+3}-1)(q^{j+2}-1)\cdots(q-1)}\right)$$
$$= \tfrac{(q^{j+1}-1)^{j/2}(q^j-1)^{j/2-1}}{(q^{j-1}-1)(q^{j-2}-1)\cdots(q-1)}\left(1 - 2\sqrt{\tfrac{q^j-1}{q^{j+1}-1}} + (\tfrac{q^{j-1}-1}{q^{j+1}-1} + \tfrac{q^j-1}{q^{j+2}-1})\right.$$
$$\left. -(\tfrac{(q^{j-1}-1)(q^{j-2}-1)}{(q^{j+1}-1)^{3/2}(q^j-1)^{1/2}} + \tfrac{(q^{j+1}-1)^{1/2}(q^j-1)^{3/2}}{(q^{j+3}-1)(q^{j+2}-1)})\right). \tag{39}$$

Since $\frac{q^{j-1}-1}{q^j-1} \leq \frac{1}{q}$ we have

$$(-1)^j S_n(\xi_j, q) \geq \tfrac{(q^{j+1}-1)^{j/2}(q^j-1)^{j/2-1}}{(q^{j-1}-1)(q^{j-2}-1)\cdots(q-1)}\left(1 - 2\sqrt{\tfrac{q^j-1}{q^{j+1}-1}}\right.$$
$$\left.+(\tfrac{q^{j-1}-1}{q^{j+1}-1} + \tfrac{q^j-1}{q^{j+2}-1}) - \tfrac{2}{q^{9/2}}\right) =: \tfrac{(q^{j+1}-1)^{j/2}(q^j-1)^{j/2-1}}{(q^{j-1}-1)(q^{j-2}-1)\cdots(q-1)}F_j(q). \tag{40}$$

Reasoning analogously for $j = 2$ and $j = n - 2$, we obtain

$$(-1)^j S_n(\xi_j, q) \geq \tfrac{(q^{j+1}-1)^{j/2}(q^j-1)^{j/2-1}}{(q^{j-1}-1)(q^{j-2}-1)\cdots(q-1)}\left(1 - 2\sqrt{\tfrac{q^j-1}{q^{j+1}-1}}\right.$$
$$\left.+(\tfrac{q^{j-1}-1}{q^{j+1}-1} + \tfrac{q^j-1}{q^{j+2}-1}) - \tfrac{1}{q^{9/2}}\right) \geq \tfrac{(q^{j+1}-1)^{j/2}(q^j-1)^{j/2-1}}{(q^{j-1}-1)(q^{j-2}-1)\cdots(q-1)}F_j(q). \tag{41}$$

We will estimate $F_j(q)$ from below. We have for any $A \in \mathbf{R}$

$$\begin{aligned} F_j(q) &= \left(1 - 2\sqrt{\frac{q^j-1}{q^{j+1}-1}} + ((1-A)\frac{q^{j-1}-1}{q^{j+1}-1} + \frac{q^j-1}{q^{j+2}-1})\right) \\ &+ \left(A\frac{q^{j-1}-1}{q^{j+1}-1} - \frac{2}{q^{9/2}}\right). \end{aligned} \tag{42}$$

Since $\frac{q^{j+1}-1}{q^{j-1}-1} \leq q^2 + q + 1$ for $j \geq 2$, we obtain

$$A\frac{q^{j-1}-1}{q^{j+1}-1} - \frac{2}{q^{9/2}} = A\frac{q^{j-1}-1}{q^{j+1}-1}q^{-9/2}\left(q^{9/2} - \frac{2}{A}\frac{q^{j+1}-1}{q^{j-1}-1}\right)$$
$$\geq A\frac{q^{j-1}-1}{q^{j+1}-1}q^{-9/2}\left(q^{9/2} - \frac{2}{A}(q^2+q+1)\right) > 0$$

for $A > \frac{2(q^2+q+1)}{q^{9/2}}$. Since $q \geq 3$,

$$\frac{2(q^2+q+1)}{q^{9/2}} \leq \frac{2 \cdot 13}{3^{9/2}} \leq 0.2\ .$$

Put $A = 0.2$. Then by our estimates we have

$$F_j(q) \geq \left(1 - 2\sqrt{\frac{q^j - 1}{q^{j+1} - 1}} + (0.8\frac{q^{j-1} - 1}{q^{j+1} - 1} + \frac{q^j - 1}{q^{j+2} - 1})\right).$$

For $j \geq 2, q \geq 3$ the following inequality is true:

$$\frac{q^{j-1} - 1}{q^{j+1} - 1} \geq \frac{1}{q^2 + q + 1} \geq \frac{9}{13q^2} \geq \frac{9}{13}\frac{q^j - 1}{q^{j+2} - 1}.$$

Thus,

$$F_j(q) \geq 1 - 2\sqrt{\frac{q^j - 1}{q^{j+1} - 1}} + (1 + \frac{9}{13} \cdot 0.8)\frac{q^j - 1}{q^{j+2} - 1}$$

$$= \frac{q^j - 1}{q^{j+2} - 1}\left(\frac{q^{j+2} - 1}{q^j - 1} - 2\frac{(q^{j+2} - 1)^{1/2}}{(q^{j+1} - 1)^{1/2}}\frac{(q^{j+2} - 1)^{1/2}}{(q^j - 1)^{1/2}} + \frac{202}{130}\right)$$

$$\geq \frac{q^j - 1}{q^{j+2} - 1}\left(\frac{q^{j+2} - 1}{q^j - 1} - 2\frac{(q^{j+2} - 1)^{3/4}}{(q^j - 1)^{3/4}} + \frac{202}{130}\right).$$

Let us consider a polynomial

$$Q(t) := t^4 - 2t^3 + \frac{202}{130}.$$

Obviously, for $t \geq 3/2$ this polynomial is increasing in t and $Q(1.7) > 0$. Hence, $Q(t) > 0$ for $t \geq 1.7$. Since

$$\left(\frac{q^{j+2} - 1}{q^j - 1}\right)^{1/4} \geq q^{1/2} \geq \sqrt{3},$$

for $q \geq 3$ we obtain

$$F_j(q) \geq \frac{q^j - 1}{q^{j+2} - 1}Q\left(\frac{(q^{j+2} - 1)^{1/4}}{(q^j - 1)^{1/4}}\right) \geq \frac{q^j - 1}{q^{j+2} - 1}Q(\sqrt{3}) > 0.$$

Thus, by (40) and (41) we conclude that for $q \geq 3$ and $j = 2, 3, \ldots, n - 2$ the following inequality is true:

$$(-1)^j S_n(\xi_j, q) > 0.$$

It is obvious that $S_n(0, q) > 0$. It remains to prove that

$$S_n(\xi_1, q) = S_n(\sqrt{(q^2 - 1)(q - 1)}, q) \leq 0.$$

We have

$$-S_n(\xi_1, q) = -1 + \frac{\xi_1}{q-1} - \frac{\xi_1^2}{(q^2-1)(q-1)}$$

$$+\sum_{k=3}^{n}(-1)^{k+1}\frac{\xi_j^k}{(q^k-1)(q^{k-1}-1)\cdots(q-1)} \geq \frac{\xi_1}{q-1} - (1 + \frac{\xi_1^2}{(q^2-1)(q-1)}),$$

whence

$$-S_n(\xi_1, q) \geq \sqrt{\frac{q^2-1}{q-1}} - 2 > 0 \Leftrightarrow q^2 - 1 \geq 4(q-1).$$

So, for $q \geq 3$ we have $-S_n(\xi_1, q) \geq 0$.

Lemma 4 is proved.

Lemma 5. *Suppose* $P_{2k-2}(z,q) := \sum_{j=0}^{2k-2}(-1)^j \frac{z^j}{q^{j^2/2}} \in R^*$. *Then* $\exists N_0\ \forall n \geq N_0\ S_n(z,q) \in R^*$.

P r o o f o f L e m m a 5. We have

$$(-1)^n S_n(z,q) = \sum_{j=n-2k}^{n}(-1)^{n+j}\frac{z^j}{(q^j-1)(q^{j-1}-1)\cdots(q-1)}$$

$$+\sum_{j=0}^{n-2k-1}(-1)^{n+j}\frac{z^j}{(q^j-1)(q^{j-1}-1)\cdots(q-1)}$$

$$= \frac{z^{n-2k}}{(q^{n-2k}-1)(q^{n-2k-1}-1)\cdots(q-1)}\cdot\left(\frac{z^{2k}}{(q^n-1)(q^{n-1}-1)\cdots(q^{n-2k+1}-1)}\right.$$

$$\left.-\frac{z^{2k-1}}{(q^{n-1}-1)(q^{n-2}-1)\cdots(q^{n-2k+1}-1)} + \cdots - \frac{z}{q^{n-2k+1}-1} + 1\right) + R_{n,k}(z),$$

where $R_{n,k}(z) < 0$ for $z \geq q^{n-3}-1$. So for all $z \geq q^{n-3}-1$ we obtain

$$\begin{aligned}&(-1)^n S_n(z,q)\\ &< \tfrac{z^{n-2k}}{(q^{n-2k}-1)(q^{n-2k-1}-1)\cdots(q-1)}\left(\tfrac{z^{2k}}{(q^n-1)(q^{n-1}-1)\cdots(q^{n-2k+1}-1)}\right.\\ &\left.-\tfrac{z^{2k-1}}{(q^{n-1}-1)(q^{n-2}-1)\cdots(q^{n-2k+1}-1)} + \cdots - \tfrac{z}{q^{n-2k+1}-1} + 1\right)\\ &=: \tfrac{z^{n-2k}}{(q^{n-2k}-1)(q^{n-2k-1}-1)\cdots(q-1)}F_{n,k}(z).\end{aligned} \tag{43}$$

Denote by $u = \frac{z\sqrt{q}}{q^{n-2k+1}-1}$. We have

$$\begin{aligned}&F_{n,k}(z) = \Phi_{n,k}(u) = \left(\tfrac{(q^{n-2k+1}-1)^{2k-1}}{(q^n-1)(q^{n-1}-1)\cdots(q^{n-2k+2}-1)}\cdot\tfrac{u^{2k}}{(\sqrt{q})^{2k}}\right.\\ &\left.-\tfrac{(q^{n-2k+1}-1)^{2k-2}}{(q^{n-1}-1)(q^{n-2}-1)\cdots(q^{n-2k+2}-1)}\cdot\tfrac{u^{2k-1}}{(\sqrt{q})^{2k-1}} + \cdots - \tfrac{u}{\sqrt{q}} + 1\right)\\ &\longrightarrow_{n\to\infty} \tfrac{u^{2k}}{(\sqrt{q})^{(2k)^2}} - \tfrac{u^{2k-1}}{(\sqrt{q})^{(2k-1)^2}} + \cdots - \tfrac{u}{\sqrt{q}} + 1 = P_{2k}(u,q).\end{aligned} \tag{44}$$

By the condition of Lemma $P_{2k-2}(u,q) \in R^*$. As it was mentioned above (see (11)), in [6, Lemma 4] it was proved that

$$P_{2k-2}(u,q) \in R^* \Leftrightarrow \exists u_0 \in (\sqrt{q}, (\sqrt{q})^3), \quad P_{2k-2}(u_0,q) \le 0,$$

and consequently

$$P_{2k}(u_0,q) = P_{2k-2}(u_0,q) - \left(\frac{u_0^{2k-1}}{(\sqrt{q})^{(2k-1)^2}} - \frac{u_0^{2k}}{(\sqrt{q})^{(2k)^2}} \right) < 0.$$

Since

$$P_{2k}(u,q) \equiv \frac{u^{2k}}{(\sqrt{q})^{(2k)^2}} P_{2k}\left(\frac{(\sqrt{q})^{4k}}{u}, q \right),$$

we have

$$\exists u_1 \in \left((\sqrt{q})^{4k-3}, (\sqrt{q})^{4k-1} \right), \quad P_{2k}(u_1,q) < 0.$$

By (44)

$$\exists n_0 \quad \forall n \ge n_0, \quad \Phi_{n,k}(u_1) < 0,$$

hence

$$\forall n \ge n_0 \quad \exists z_{1,n} = \frac{q^{n-2k+1} - 1}{\sqrt{q}} u_1 \in (q^{n-1} - q^{2k-2}, q^n - q^{2k-1})),$$

$$F_{n,k}(z_{1,n}, q) < 0.$$

By (43)

$$\forall n \ge n_0 \quad (-1)^n S_n(z_{1,n}, q) < 0. \tag{45}$$

We take $n_0 \ge 2k$ and will check that for all $n \ge n_0$

$$q^{n-1} - q^{2k-2} \ge \sqrt{(q^{n-1} - 1)(q^{n-2} - 1)} = \xi_{n-2}. \tag{46}$$

Really, this inequality is equivalent to

$$q^{2n-2} - 2q^{n+2k-3} + q^{4k-4} \ge q^{2n-3} - q^{n-1} - q^{n-2} + 1. \tag{47}$$

Since $q \ge 3$, we have $q^{2n-2} - q^{2n-3} \ge 2q^{2n-3}$, and, taking into account $n \ge n_0 \ge 2k$, we obtain

$$q^{2n-2} - q^{2n-3} - 2q^{n+2k-3} + q^{n-1} + q^{n-2} + q^{4k-4} - 1 \ge 2q^{2n-3} - 2q^{n+2k-3} + q^{n-1} + q^{n-2}$$

$$+ q^{4k-4} - 1 = 2q^{n+2k-3}(q^{n-2k} - 1) + q^{n-1} + q^{n-2} + q^{4k-4} - 1$$

$$\ge q^{n-1} + q^{n-2} + q^{4k-4} - 1 > 0.$$

So (47) holds, and hence (46) is true.

By Lemma 4, (46) and (43) there exist nonnegative numbers $0 = \xi_0 < \xi_1 < \xi_2 < \ldots < \xi_{n-2} < z_{1,n} =: \xi_{n-1}$ such that

$$(-1)^j S_n(\xi_j, q) > 0, \quad \lim_{x\to\infty}(-1)^n S_n(x,q) = +\infty.$$

Hence

$$P_{2k-2}(z,q) = \sum_{j=0}^{2k-2}(-1)^j \frac{z^j}{q^{j^2/2}} \in R^* \Rightarrow \exists N_0\ \forall n \ge N_0\ S_n(z,q) \in R^*.$$

Lemma 5 is proved.

To complete the proof of Theorem 3 we mention, that in [6] (see Theorem (2)) it is proved that

$$\exists N_0 \in \mathbf{N} \quad \forall k \ge N_0 \quad P_{2k-2}(z,q) \in R^* \ \Leftrightarrow\ q > q_\infty.$$

Theorem 3 is proved.

5. Some examples

Let $f(z)$ be an entire function of the form (14), namely

$$f(z) = Cz^m e^{-\alpha z^2+\beta z}\prod_{k=1}^{\infty}(1+\frac{z}{x_k})e^{-\frac{z}{x_k}}, \tag{48}$$

where $m \in \mathbf{N}\cup\{0\}$, $C,\beta \in \mathbf{R}$, $\alpha \ge 0$ and $0 < x_k \le \infty$, $\sum_{k=1}^{\infty}\frac{1}{x_k^2} < \infty$. In this section we will answer the following question: for which a does the function $\sum_{j=0}^{\infty} f(j)\frac{z^j}{a^{j^2}}$ has the sections with only real zeros?
To investigate this problem we need the following statement.

Statement 1. *For every function f of the form (48) the following limit exists:*

$$\lim_{j\to\infty}\frac{f^2(j)}{f(j-1)f(j+1)},$$

and

$$\lim_{j\to\infty}\frac{f^2(j)}{f(j-1)f(j+1)} = e^{2\alpha}. \tag{49}$$

Proof of Statement 1. We have $f = f_1 f_2$, where

$$f_1(z) := Cz^m e^{-\alpha z^2+\beta z},\ f_2(z) := \prod_{k=1}^{\infty}(1+\frac{z}{x_k})e^{-\frac{z}{x_k}}.$$

Obviously, there exists the limit

$$\lim_{j\to\infty} \frac{f_1^2(j)}{f_1(j-1)f_1(j+1)},$$

and

$$\lim_{j\to\infty} \frac{f_1^2(j)}{f_1(j-1)f_1(j+1)} = e^{2\alpha}.$$

It remains to prove that there exists the limit

$$\lim_{j\to\infty} \frac{f_2^2(j)}{f_2(j-1)f_2(j+1)},$$

and this limit is equal to 1. We have

$$\log \frac{f_2^2(j)}{f_2(j-1)f_2(j+1)} = \sum_{k=1}^{\infty} \Bigg((\log(1+\frac{j}{x_k})^2 - \frac{2j}{x_k})$$

$$-(\log(1+\frac{j-1}{x_k}) - \frac{j-1}{x_k}) - (\log(1+\frac{j+1}{x_k}) - \frac{j+1}{x_k}) \Bigg)$$

$$= \sum_{k=1}^{\infty} \log \frac{(1+\frac{j}{x_k})^2}{(1+\frac{j-1}{x_k})(1+\frac{j+1}{x_k})}. \tag{50}$$

Since

$$\frac{(1+\frac{j}{x_k})^2}{(1+\frac{j-1}{x_k})(1+\frac{j+1}{x_k})} = \frac{1+\frac{2j}{x_k}+\frac{j^2}{x_k^2}}{1+\frac{2j}{x_k}+\frac{j^2-1}{x_k^2}} = 1 + \frac{\frac{1}{x_k^2}}{1+\frac{2j}{x_k}+\frac{j^2-1}{x_k^2}}$$

we have

$$1 \le \frac{(1+\frac{j}{x_k})^2}{(1+\frac{j-1}{x_k})(1+\frac{j+1}{x_k})} \le 1 + \frac{1}{x_k^2},$$

and so series (50) is majorated by a convergent numerical series $\sum_{k=1}^{\infty} \frac{1}{x_k^2}$ and each term of (50) tends to 0 as $j \to \infty$. Thus, by the Lebesgue dominated theorem series (50) tends to 0 as $j \to \infty$. Hence

$$\lim_{j\to\infty} \frac{f_2^2(j)}{f_2(j-1)f_2(j+1)} = \exp\left(\lim_{j\to\infty} \sum_{k=1}^{\infty} \log \frac{(1+\frac{j}{x_k})^2}{(1+\frac{j-1}{x_k})(1+\frac{j+1}{x_k})} \right) = 1.$$

Statement 1 is proved.

Using Statement 1, we will prove the following statement.

Statement 2. *Denote by* $\varphi_{a,f}(z) := \sum_{j=0}^{\infty} f(j) \frac{z^j}{a^{j^2}}$, $a \ge 1$, *where* f *is of the form (48).*

(i) $(ae^{\alpha})^2 > q_\infty \Rightarrow \varphi_{a,f} \in A^*$;
(ii) $(ae^{\alpha})^2 = q_\infty \Rightarrow \varphi_{a,f} \in S^*$;
(iii) $(ae^{\alpha})^2 < q_\infty \Rightarrow \varphi_{a,f} \notin S^*$.

P r o o f o f S t a t e m e n t 2. We have $\varphi_{a,f}(z) = \sum_{j=0}^{\infty} f_0(j) \frac{z^j}{(ae^{\alpha})^{j^2}}$, where $f_0(z) = Cz^m e^{\beta z} \prod_{k=1}^{\infty}(1 + \frac{z}{x_k})e^{-\frac{z}{x_k}}$. By Theorem G the function $g_{ae^{\alpha}}(z) = \sum_{j=0}^{\infty} \frac{z^j}{(ae^{\alpha})^{j^2}} \in A^*$ provided $(ae^{\alpha})^2 > q_\infty$. Since by Laguerre's theorem the sequence $\{f_0(j)\}_{j=0}^{\infty} \in CZDS$ we obtain (i). Analogously by the first statement of Theorem G the function $g_{ae^{\alpha}} \in S^*$ provided $(ae^{\alpha})^2 = q_\infty$ and since the sequence $\{f_0(j)\}_{j=0}^{\infty} \in CZDS$ we obtain (ii). Corollary formulated after Theorems 1 and 2 and Statement 1 imply (iii).

Statement 2 is proved.

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