Matematicheskaya fizika, analiz, geometriya
2004, v. 11, No. 4, p. 470–483

On conditionally convergent series

Vladimir Logvinenko

Mathematics Department, De Anza College
21250 Stevens Creek Blvd., Cupertino, Ca 95014-5793, US
E-mail:logvinenkovladimir@deanza.edu

Received September 23, 2004

The most interesting result of the paper is that for any two complementary subsets A and B of the set of positive odd integers there exists such a sequence $\{\alpha_k\}_{k=1}^{\infty} \subset [-1, 1]$ that

$$\forall m \in A: \text{ the series } \sum_{k=1}^{\infty} \alpha_k^m \text{ is convergent and}$$

$$\forall m \in B: \text{ the series } \sum_{k=1}^{\infty} \alpha_k^m \text{ is divergent.}$$

Using the map $\vec{x} \longmapsto \|\vec{x}\|^{\lambda} \frac{\vec{x}}{\|\vec{x}\|}$ as a substitute of the power function, one can prove similar results for vectors and positive not necessarily integer exponents λ.

To Iossif Vladimirovich Ostrovskii on the occasion of his 70-th birthday

Introduction

Properties of powers of series with positive terms have been quite thoroughly studied. It is enough to mention, for instance, the theory of l^p-spaces (often disguised as L^p-spaces on spaces with measure) [2] or M. Riesz–Thorin's Interpolation Theorem, also known as Riesz's Convexity Theorem [2, 3]. In contrast, there is no paper talking about powers of sign-changing series, at least to the best of the author's knowledge. In this paper, we study convergence and divergence of powers of sign-changing series.

The most interesting, in the author's opinion, and rather unexpected result of this paper is the following theorem:

Mathematics Subject Classification 2000: 40A05.

Theorem 1. *For any complementary subsets A and B of the set of positive odd integers there exists such a sequence $\{\alpha_k\}_{k=1}^{\infty} \subset [-1,1]$ that*

$$\forall m \in A : \textit{the series } \sum_{k=1}^{\infty} \alpha_k^m \textit{ is convergent and}$$

$$\forall m \in B : \textit{the series } \sum_{k=1}^{\infty} \alpha_k^m \textit{ is divergent.}$$

This theorem is an immediate corollary of the following result that is formulated in terms of the following "power with sign": $x \longmapsto |x|^{\lambda}\, sign\,(x)$.

Theorem 2. *For any complementary subsets A and B of the set of positive integers there exists such a sequence $\{\alpha_k\}_{k=1}^{\infty} \subset [-1,1]$ that*

$$\forall m \in A : \textit{the series } \sum_{k=1}^{\infty} |\alpha_k|^m\, sign\,(\alpha_k) \textit{ is convergent and}$$

$$\forall m \in B : \textit{the series } \sum_{k=1}^{\infty} |\alpha_k|^m\, sign\,(\alpha_k) \textit{ is divergent.}$$

As we'll see later, the proofs of Theorem 1 and Theorem 2 (as well as the proof of more general statement of Theorem 3 from which they are derived) are based on the possibility to construct a certain "atom" series whose all positive odd powers, or all positive integer powers with sign, but one given converge and this single one diverge to ∞. The straightforward complex analog of Theorem 1 — as well as of Theorem 2 — is way too simple. The reason is that in this case, it is very easy to build such an atom series using a well-known property of complex roots of one.

Example. *Let p be a positive odd integer. An atom series may be defined as follows:*

$$\alpha_1 = \exp\{1 \cdot 2\pi i/p\}, \alpha_2 = \exp\{2 \cdot 2\pi i/p\}, \ldots, \alpha_p = \exp\{p \cdot 2\pi i/p\} = 1,$$

$$\alpha_{p+1} = \frac{\exp\{1 \cdot 2\pi i/p\}}{\sqrt[p]{2}}, \alpha_{p+2} = \frac{\exp\{2 \cdot 2\pi i/p\}}{\sqrt[p]{2}}, \ldots, \alpha_{2p} = \frac{1}{\sqrt[p]{2}}, \ldots,$$

$$\alpha_{lp+1} = \frac{\exp\{1 \cdot 2\pi i/p\}}{\sqrt[p]{l+1}}, \alpha_{lp+2} = \frac{\exp\{2 \cdot 2\pi i/p\}}{\sqrt[p]{l+1}}, \ldots, \alpha_{(l+1)p} = \frac{1}{\sqrt[p]{l+1}}, \ldots.$$

It is evident that the series $\sum \alpha_k^{2m+1}$, $m \in \mathbb{Z}_+$, converges to 0 when $2m+1 < p$, diverges to ∞ when $2m+1 = p$, and absolutely converges when $2m+1 > p$.

This example implies that any reasonable vectorial analog of Theorem 2 should contain some restrictions on the range of directions of terms as the following result, which is in a sense the best possible, does.

Theorem 3. *Let $\overrightarrow{\omega}_1, \overrightarrow{\omega}_2, \ldots, \overrightarrow{\omega}_q$ be such a finite set of unit vectors of $\mathbb{R}^d$ that the convex cone*

$$cone\left(\overrightarrow{\omega}_1, \ldots, \overrightarrow{\omega}_q\right) = \left\{ \overrightarrow{x} = \sum_{j=1}^{q} \lambda_j \overrightarrow{\omega}_j, \lambda_j \geq 0 \right\}$$

contains at least one straight line. For any complementary subsets A and B of the set of positive integers there exists such a sequence of vectors $\left\{\overrightarrow{f}_k\right\}_{k=1}^{\infty} \subset \mathbb{R}^d$ that, first,

$$\frac{\overrightarrow{f}_k}{\left\|\overrightarrow{f}_k\right\|} = \overrightarrow{\omega}_{j(k)} \in \left\{\overrightarrow{\omega}_1, \ldots, \overrightarrow{\omega}_q\right\}, k = 1, 2, \ldots ,$$

and, second, the series

$$\sum_{k=1}^{\infty} \left\|\overrightarrow{f}_k\right\|^m \overrightarrow{\omega}_{j(k)}$$

converges for all $m \in A$ and diverges for all $m \in B$

If the cone $cone\left(\overrightarrow{\omega}_1, \ldots, \overrightarrow{\omega}_q\right)$ does not contain any straight line, then such sequence does not exist if at least one element of A precedes some element of B.

The proof of Theorem 3 is based on the following lemma:

Lemma 1. *Let $\overrightarrow{\omega}_1, \overrightarrow{\omega}_2, \ldots, \overrightarrow{\omega}_q$ be such a finite set of unit vectors of $\mathbb{R}^d$ that for some positive $\lambda_1, \ldots, \lambda_q$ the following zero property*

$$\sum_{j=1}^{q} \lambda_j \overrightarrow{\omega}_j = \overrightarrow{0}$$

is valid. Let $\varepsilon > 0$ and let $p \in \mathbb{N}$. Let also

$$\overrightarrow{g}_k = \|\overrightarrow{g}_k\| \, \overrightarrow{\omega}_{j(k)}, j(k) \in \{1, \ldots, q\}, k \in \mathbb{N}, , \|\overrightarrow{g}_k\| \to 0 \text{ as } k \to \infty.$$

*Then there exist such a sequence of vectors**

$$\overrightarrow{h}_l = \left\|\overrightarrow{h}_l\right\| \overrightarrow{\omega}_{j(l)}, j(l) \in \{1, \ldots, q\}, l \in \mathbb{N},$$

* We take the liberty to use the same notation $j(\cdot)$ for three different functions that share the domain and the range, and perform the similar task. The first function $j(k)$ determines the direction of vectors $\overrightarrow{g}_k$ of the given sequence $\{\overrightarrow{g}_k\}$, the function $j(l)$ does the same for vectors $\overrightarrow{h}_l$ we are going to construct in the lemma, and the function $j(l')$ determines the direction of auxiliary vectors $\overrightarrow{e}_r$ we use to prove the lemma.

and such a sequence $\{l_k\}_{k=1}^{\infty} \subset \mathbb{N}, l_k \uparrow \infty$ *as* $k \to \infty$, *that the following statements are true:*

$$(i)\, \forall k \in \mathbb{N} : \overrightarrow{h}_{l_k} = \overrightarrow{g}_k;$$

$$(ii) \max\left\{\left\|\overrightarrow{h}_l\right\| : l \in \mathbb{N}\right\} = \max\left\{\|\overrightarrow{g}_k\| : k \in \mathbb{N}\right\};$$

$$(iii) \text{ For any } m \in \{1, \ldots, p\} \text{ the series } \sum_{l=1}^{\infty} \left\|\overrightarrow{h}_l\right\|^m \overrightarrow{\omega}_{j(l)} \text{ is convergent;}$$

$$(iv) \text{ For any } m \in \{1, \ldots, p\} \text{ and any } L \in \mathbb{N} \text{ the norm}$$

$$\left\|\sum_{l=1}^{L} \left\|\overrightarrow{h}_l\right\|^m \overrightarrow{\omega}_{j(l)}\right\| \leq (1+\varepsilon) \max\left\{\|\overrightarrow{g}_k\|^m : k \in \mathbb{N}\right\};$$

$$(v) \text{ For any } m \geq p+1 \text{ the inequality}$$

$$\sum_{l \in \mathbb{N} \setminus \{l_k\}_{k=1}^{\infty}} \left\|\overrightarrow{h}_l\right\|^m < \varepsilon \text{ is valid.}$$

According to this lemma, one always can add relatively small new terms to the sequence $\{\vec{g}_k\}_{k=1}^{\infty}$ so that a given number of powers with sign of the corresponding series converge. Carefully choosing $\{\vec{g}_k\}_{k=1}^{\infty}$, it is possible to obtain an "atom" series we talked about above.

R e m a r k 1. It is possible to chase out ε from statement (iv) but it makes the proof much longer.

R e m a r k 2. The statement of Lemma 1 with a little weaker estimates remains true if the initial sequence $\{\overrightarrow{g}_k\}$ has a wider range, namely, any converging to $\vec{0}$ sequence of the whole cone *cone* $(\overrightarrow{\omega}_1, \ldots, \overrightarrow{\omega}_q)$.

The concept of power with sign allows to consider not only integer but also any positive power λ of terms of a sign-changing series. Analyzing such noninteger powers of the series

$$\sum_{k=1}^{\infty} \alpha_k$$

constructed in Theorem 2, one can see that if B is infinite, then the powers with sign of this series converge only for $\lambda \in A$. It means that for each given $\lambda_0 > 0$ there exist only a finite number of such $\lambda \in (0, \lambda_0)$ that the series

$$\sum_{k=1}^{\infty} |\alpha_k|^{\lambda} \, sign\,(\alpha_k)$$

converge. The following question arises quite naturally: Is it possible that for some positive λ_0 the series $\sum |\alpha_k|^{\lambda_0} \, sign\,(\alpha_k)$ is divergent while for some infinite set Λ of $\lambda \in (0, \lambda_0)$ the series $\sum |\alpha_k|^\lambda \, sign\,(\alpha_k)$ is convergent? The positive answer to this question is given by the following result:

Theorem 4. *There exists such a sequence* $\{\alpha_k\}_{k=1}^\infty \subset [-1, 1]$ *that the series* $\sum_{k=1}^\infty |\alpha_k|^\lambda \, sign\,(\alpha_k)$ *is convergent for all* λ *of some continuum* $\Lambda \subset (0, 1.5)$ *and is divergent for* $\lambda = 2$.

The author believes but cannot prove at this moment that it is not the case if the set of the exponents of convergence Λ contains some interval or even if it is of positive Lebesgue's measure.

In the next section we prove Lemma 2. In the last section there are proofs of Theorem 3 and Theorem 4.

The author would like to express his gratitude to Professor Farshod Mosh whose question "Let a series $\sum_{k=1}^\infty \gamma_k$ with real terms be convergent. Should the series

$$\sum_{k=1}^\infty \frac{\gamma_k}{1 + |\gamma_k|}$$

also converge?" arose the author's interest to the topic. By the way, the answer to Professor Mosh's question is negative. It follows immediately from Lemma 1 if one takes $d = 1$, $g_k = \frac{1}{\sqrt{k}}$, and $p = 1$.

Proof of Lemma 1

We need the following two simple statements about convex sets of vectors. The reader can find its proof in any book on Convex Analysis, for instance, in [1].

Lemma 2. *Let* $\vec{\omega}_1, \vec{\omega}_2, \ldots, \vec{\omega}_q$ *be a finite set of unit vectors of* $\mathbb{R}^d$. *Then the convex cone*

$$cone\,(\vec{\omega}_1, \vec{\omega}_2, \ldots, \vec{\omega}_q) = \left\{ \vec{x} = \sum_{j=1}^q \lambda_j \vec{\omega}_j, \lambda_j \geq 0 \right\}$$

contains at least one straight line if, and only if, for some $\lambda_j \geq 0$ *where at least two of* λ_j *are positive*

$$\sum_{j=1}^q \lambda_j \vec{\omega}_j = \vec{0}.$$

P r o o f o f L e m m a 1. We will use induction by p. Let $p = 1$ and let $\{l_k\}_{k=1}^{\infty}, l_1 = 1$, be an increasing sequence of positive integers with the following properties:

$$(a)\,\forall k \in \mathbb{N} : l_{k+1} - l_k = 0 \,(\mathrm{mod}\ q);$$

$$(b)\,\forall k \in \mathbb{N} : \frac{\max \lambda_j}{\min \lambda_j} \cdot \frac{q^2}{l_{k+1} - l_k} < \varepsilon;$$

$$(c)\sum_{k=1}^{\infty} \left(\frac{q\, \|\overrightarrow{g}_k\| \max \lambda_j}{\min \lambda_j} \right)^2 \cdot \frac{1}{l_{k+1} - l_k} < \varepsilon.$$

Let us define $\overrightarrow{h}_0 = \overrightarrow{0}$. Assume that all $\overrightarrow{h}_l$ with $l < l_k$ are already defined and satisfy an equation

$$(d) \sum_{l=1}^{l_k - 1} \overrightarrow{h}_l = \overrightarrow{0}.$$

Let us denote by $a\,(\mathrm{mod}\ q)$ such a number $b \in \{0, \ldots, q-1\}$ that $b = a\,(\mathrm{mod}\ q)$. The next cycle of $\overrightarrow{h}_l$ is defined as follows:

$$\overrightarrow{h}_{l_k} = \overrightarrow{g}_k = \|\overrightarrow{g}_k\|\, \omega_{j(k)};$$

$$\overrightarrow{h}_{l_k+1} = \frac{\lambda_{(j(k)+1)\,(\mathrm{mod}\ q)}}{\lambda_{j(k)}} \cdot \frac{q\, \|\overrightarrow{g}_k\|}{l_{k+1} - l_k} \omega_{(j(k)+1)\,(\mathrm{mod}\ q)}; \ldots ;$$

$$\overrightarrow{h}_{l_k+q-1} = \frac{\lambda_{(j(k)-1)\,(\mathrm{mod}\ q)}}{\lambda_{j(k)}} \cdot \frac{q\, \|\overrightarrow{g}_k\|}{l_{k+1} - l_k} \omega_{(j(k)-1)\,(\mathrm{mod}\ q)};$$

$$\overrightarrow{h}_{l_k+q} = \overrightarrow{0};$$

$$\overrightarrow{h}_{l_k+q+1} = \frac{\lambda_{(j(k)+1)\,(\mathrm{mod}\ q)}}{\lambda_{j(k)}} \cdot \frac{q\, \|\overrightarrow{g}_k\|}{l_{k+1} - l_k} \omega_{(j(k)+1)\,(\mathrm{mod}\ q)}; \ldots ;$$

$$\overrightarrow{h}_{l_{k+1}-1} = \frac{\lambda_{(j(k)-1)\,(\mathrm{mod}\ q)}}{\lambda_{j(k)}} \cdot \frac{q\, \|\overrightarrow{g}_k\|}{l_{k+1} - l_k} \omega_{(j(k)-1)\,(\mathrm{mod}\ q)}.$$

Due to the zero property, condition (d) is satisfied — so, we can keep going. Assuming that $\varepsilon < 1$, we see that (i) and (ii) are evident. For $n = 1, 2, \ldots, (l_{k+1} - l_k)/q$,

$$\sum_{l=1}^{l_k+qn-1} \overrightarrow{h}_l = \left(1 - \frac{qn}{l_{k+1} - l_k}\right) \overrightarrow{h}_{l_k} = \left(1 - \frac{qn}{l_{k+1} - l_k}\right) \overrightarrow{g}_k.$$

So, for each such n

$$\sum_{l=1}^{l_k+qn-1} \overrightarrow{h}_l \to \overrightarrow{0} \text{ as } k \to \infty.$$

Let L be any integer of the interval $[l_k + q(n-1), l_k + qn - 1)$. Then, due to (b), the inequality

$$\left\| \sum_{l=1}^{L} \vec{h}_l \right\| \leq \left\| \sum_{l=1}^{l_k+q(n-1)-1} \vec{h}_l \right\| + \sum_{l=l_k+q(n-1)}^{L} \left\| \vec{h}_l \right\|$$

$$\leq \left(1 - \frac{q(n-1)}{l_{k+1} - l_k}\right) \|\vec{g}_k\| + \frac{q^2 \max \lambda_j}{\min \lambda_j} \cdot \frac{\|\vec{g}_k\|}{l_{k+1} - l_k} < (1+\varepsilon) \|\vec{g}_k\|$$

is valid. It means that (iii) and (iv) are also true. Without loss of generality, we can assume that $\max\{\|\vec{g}_k\| : k \in \mathbb{N}\} \leq 1$. Therefore, by (c), for any $m > 1$

$$\sum_{l \in \mathbb{N} \setminus \{l_k\}_{k=1}^{\infty}} \left\| \vec{h}_l \right\|^m \leq \sum_{l \in \mathbb{N} \setminus \{l_k\}_{k=1}^{\infty}} \left\| \vec{h}_l \right\|^2$$

$$\leq \sum_{k=1}^{\infty} \sum_{l=l_k+1}^{l_{k+1}-1} \left\| \vec{h}_l \right\|^2 \leq \sum_{k=1}^{\infty} \left(\frac{q \max \lambda_j \|\vec{g}_k\|}{\min \lambda_j} \right)^2 \frac{1}{l_{k+1} - l_k} < \varepsilon$$

and we conclude that (v) is true as well.

Assume now that the statements of Lemma 1 are true for all $p < P$ and prove that, in this case, they are true for $p = P$. Let $\{l'_k\}_{k=1}^{\infty}, l'_1 = 1$, be an increasing sequence of positive integers with the following properties:

$$(a')\ \forall k \in \mathbb{N} : l'_{k+1} - l'_k = 0 \,(\mathrm{mod}\ q);$$

$$(b')\ \forall k \in \mathbb{N} : \frac{\max \lambda_j}{\min \lambda_j} \cdot \frac{q^2}{l'_{k+1} - l'_k} < \varepsilon;$$

$$(c')\ \sum_{k=1}^{\infty} \frac{q \|\vec{g}_k\|^{P+1} \max \lambda_j}{\min \lambda_j} \cdot \sqrt[P]{\frac{q \max \lambda_j}{(l'_{k+1} - l'_k) \min \lambda_j}} < \varepsilon.$$

We define $\vec{e}_0 = \vec{0}$. Let for all $l' < l'_k$ vectors $\vec{e}_{l'}$ are already defined and let

$$(d')\ \sum_{l'=1}^{l'_k - 1} \|\vec{e}_{l'}\|^P \vec{\omega}_{j(l')} = \vec{0}.$$

The next cycle is defined as follows:

$$\vec{e}_{l'_k} = \vec{g}_k = \|\vec{g}_k\| \,\omega_{j(k)};$$

$$\vec{e}_{l'_k+1} = \sqrt[P]{\frac{\lambda_{(j(k)+1)\,(\mathrm{mod}\ q)}}{\lambda_{j(k)}} \cdot \frac{q}{(l'_{k+1} - l'_k)}} \,\|\vec{g}_k\| \,\omega_{(j(k)+1)\,(\mathrm{mod}\ q)}; \ldots;$$

$$\overrightarrow{e}_{l'_k+q-1} = \sqrt[P]{\frac{\lambda_{(j(k)-1)\,(\mathrm{mod}\ q)}}{\lambda_{j(k)}} \cdot \frac{q}{\left(l'_{k+1}-l'_k\right)}}\, \|\overrightarrow{g}_k\|\, \omega_{(j(k)-1)\,(\mathrm{mod}\ q)};$$

$$\overrightarrow{e}_{l'_k+q} = \overrightarrow{0};$$

$$\overrightarrow{e}_{l'_k+q+1} = \sqrt[P]{\frac{\lambda_{(j(k)+1)\,(\mathrm{mod}\ q)}}{\lambda_{j(k)}} \cdot \frac{q}{\left(l'_{k+1}-l'_k\right)}}\, \|\overrightarrow{g}_k\|\, \omega_{(j(k)+1)\,(\mathrm{mod}\ q)}; \dots;$$

$$\overrightarrow{e}_{l'_{k+1}-1} = \sqrt[P]{\frac{\lambda_{(j(k)-1)\,(\mathrm{mod}\ q)}}{\lambda_{j(k)}} \cdot \frac{q}{\left(l'_{k+1}-l'_k\right)}}\, \|\overrightarrow{g}_k\|\, \omega_{(j(k)-1)\,(\mathrm{mod}\ q)}.$$

Due to the zero property, condition (d') is satisfied — so, the construction keeps going. (i) and (ii) are again evident. Reasoning as above, we obtain, first, that the series

$$\sum_{l'=1}^{\infty} \|\overrightarrow{e}_{l'}\|^P\, \overrightarrow{\omega}_{j(l')}$$

is convergent, second, that for any $L \in \mathbb{N}$

$$\left\| \sum_{l'=1}^{L} \|\overrightarrow{e}_{l'}\|^P\, \overrightarrow{\omega}_{j(l')} \right\| \le (1+\varepsilon) \max\left\{ \|\overrightarrow{g}_k\|^P : k \in \mathbb{N} \right\},$$

and, third, that for any $m > P$

$$\sum_{l' \in \mathbb{N} \setminus \{l'_k\}_{k=1}^{\infty}}^{\infty} \|\overrightarrow{e}_{l'}\|^m < \varepsilon,$$

i.e., the fulfillment of (iii), (v), and (iv) for $m = P$.

According to the inductive assumption, for the sequence $\{\overrightarrow{e}_{l'}\}_{l'=1}^{\infty}$, $p = P-1$, and given $\varepsilon > 0$ there exist such a sequence $\left\{ \overrightarrow{h}_l = \left\|\overrightarrow{h}_l\right\| \overrightarrow{\omega}_{j(l)} \right\}_{l=1}^{\infty} \subset \mathbb{R}^d$ and such an increasing sequence of integers $\{l_{l'}\}_{l'=1}^{\infty}$ that the following statements are valid:

$$(i')\ \forall l' \in \mathbb{N} : \overrightarrow{h}_{l_{l'}} = \overrightarrow{e}_{l'} \Rightarrow \forall k \in \mathbb{N} : \overrightarrow{h}_{l_{l'_k}} = \overrightarrow{g}_k;$$

$$(ii')\ \max\left\{ \left\|\overrightarrow{h}_l\right\| : l \in \mathbb{N} \right\} = \max\left\{ \|\overrightarrow{e}_{l'}\| : l' \in \mathbb{N} \right\} = \max\left\{ \|\overrightarrow{g}_k\| : k \in \mathbb{N} \right\};$$

$$(iii')\ \text{For any } n \in \{1, \dots, P-1\}\ \sum_{l=1}^{\infty} \left\|\overrightarrow{h}_l\right\|^m \overrightarrow{\omega}_{j(l)} \text{ is convergent;}$$

(iv') For any $n \in \{1, \dots, P-1\}$ and for any $L \in \mathbb{N}$

$$\left\| \sum_{l=1}^{L} \left\| \overrightarrow{h}_l \right\|^m \overrightarrow{\omega}_{j(l)} \right\| \le (1+\varepsilon) \max \left\{ \| \overrightarrow{g}_k \|^m : k \in \mathbb{N} \right\} ;$$

(v') For any $n \ge P$ the inequality $\displaystyle\sum_{l \in \mathbb{N} \setminus \{l_{l'}\}_{l'=1}^{\infty}} \left\| \overrightarrow{h}_l \right\|^m < \varepsilon$ is valid.

Statements (i') and (ii') are equivalent to (i) and (ii), respectively. Statements (iii') and (v') imply (iii). Statements (iv') and (v') imply (iv). At last, for any $n \ge P+1$

$$\sum_{l \in \mathbb{N} \setminus \left\{ l_{l'_k} \right\}_{k=1}^{\infty}} \left\| \overrightarrow{h}_l \right\|^m \le \sum_{l \in \mathbb{N} \setminus \{l_{l'}\}_{l'=1}^{\infty}} \left\| \overrightarrow{h}_l \right\|^m + \sum_{l' \in \mathbb{N} \setminus \{l_k\}_{k=1}^{\infty}}^{\infty} \| \overrightarrow{e}_{l'} \|^m < 2\varepsilon.$$

Lemma 1 is proved. ■

Proofs of Theorem 3 and Theorem 4

P r o o f o f T h e o r e m 3. Let, first, the cone *cone* $(\overrightarrow{\omega}_1, \dots, \overrightarrow{\omega}_q)$ contain a straight line. Then, by Lemma 2, there exist such nonnegative real numbers $\lambda_1, \dots, \lambda_q$, $\lambda_1^2 + \dots + \lambda_q^2 > 0$, that the zero property

$$\sum_{j=1}^{q} \lambda_j \overrightarrow{\omega}_j = \overrightarrow{0}$$

is valid. If some of λ_j are zeros, we skip the corresponding $\overrightarrow{\omega}_j$. So, without loss of generality, we assume that all λ_j are positive. Let $B = \{m_i\}_{i=1}^{\omega} \subset \mathbb{N}$. Only the case of $\omega = \infty$ and $B \ne \mathbb{N}$ is of interest for us. For each $i \in \mathbb{N}$ let

$$\forall k \in \mathbb{N} : \overrightarrow{g}_k^{(i)} = \frac{e_i}{k^{1/m_i}} \overrightarrow{\omega}_1 ,$$

where constants $e_i > 0$ are chosen so that

$$\sum_{k=1}^{\infty} \left\| \overrightarrow{g}_k^{(i)} \right\|^{m_i+1} = 1.$$

Let $\left\{ \overrightarrow{h}_l^{(i)} \right\}_{l=1}^{\infty} \subset \mathbb{R}^d$ and $\left\{ l_k^{(i)} \right\}_{k=1}^{\infty}$ be those two sequences whose existence is guaranteed by Lemma 1 for these $\overrightarrow{g}_k^{(i)}$, $k = 1, 2, \dots, p = m_i - 1$, and $\varepsilon = 1$. Let, at last, positive numbers η_i, $i = 1, 2, \dots$, satisfy $\sum_{i=1}^{\infty} \eta_i = 1$.

Applying Cantor's "sweeping from a corner" procedure, we define vectors $\left\{ \overrightarrow{f}_\nu \right\}_{\nu=1}^{\infty}$ as follows:

$$\overrightarrow{f}_1 = \eta_1 \overrightarrow{h}_1^{(1)}, \overrightarrow{f}_2 = \eta_1 \overrightarrow{h}_2^{(1)}, \overrightarrow{f}_3 = \eta_2 \overrightarrow{h}_1^{(2)}, \overrightarrow{f}_4 = \eta_1 \overrightarrow{h}_3^{(1)}, \overrightarrow{f}_5 = \eta_2 \overrightarrow{h}_2^{(2)},$$

$\overrightarrow{f}_6 = \eta_3 \overrightarrow{h}_1^{(3)}, \overrightarrow{f}_7 = \eta_1 \overrightarrow{h}_4^{(1)}, \overrightarrow{f}_8 = \eta_2 \overrightarrow{h}_3^{(2)}, \overrightarrow{f}_9 = \eta_3 \overrightarrow{h}_2^{(3)}, \overrightarrow{f}_{10} = \eta_4 \overrightarrow{h}_1^{(4)}, \dots$.

Let n be any element of A, let $i \in \mathbb{N}$, and let

$$\overrightarrow{H}_n^{(i)} = \sum_{l=1}^{\infty} \left\| \overrightarrow{h}_l^{(i)} \right\|^n \frac{\overrightarrow{h}_l^{(i)}}{\left\| \overrightarrow{h}_l^{(i)} \right\|}.$$

All vectors $\overrightarrow{H}_n^{(i)}$ are well-defined. Besides, they are uniformly bounded as well as partial sums of corresponding series. Given any $\eta > 0$, let $r \in N$ be so large that

$$\sum_{i=r+1}^{\infty} \eta_i^n \left\| \overrightarrow{H}_n^{(i)} \right\| < \eta \text{ and}$$

$$\sum_{i=r+1}^{\infty} \eta_i^n \sup \left\{ \sum_{l=1}^{L} \left\| \overrightarrow{h}_l^{(i)} \right\|^n \frac{\overrightarrow{h}_l^{(i)}}{\left\| \overrightarrow{h}_l^{(i)} \right\|} : L \in \mathbb{N} \right\} < \eta.$$

Let also $s \in \mathbb{N}$ be so large that for all $i \in \{1, 2, \dots, r\}$

$$\sup_{L > s} \left\| H_n^{(i)} - \sum_{l=1}^{L} \left\| \overrightarrow{h}_l^{(i)} \right\|^n \frac{\overrightarrow{h}_l^{(i)}}{\left\| \overrightarrow{h}_l^{(i)} \right\|} \right\| < \eta.$$

Let us fix these r and s. Let also number $N \in \mathbb{N}$ be so large that the numbers $L_1, L_2, \dots, L_r$ of those of vectors $\overrightarrow{f}_\nu, \nu \in \{1, 2, \dots, N\}$, that are generated by $\left\{ \overrightarrow{h}_l^{(1)} \right\}, \left\{ \overrightarrow{h}_l^{(2)} \right\}, \dots, \left\{ \overrightarrow{h}_l^{(r)} \right\}$ respectively are greater than s. This choice guarantees that the inequality

$$\left\| \sum_{\nu=1}^{N} \left\| \overrightarrow{f}_\nu \right\|^n \frac{\overrightarrow{f}_\nu}{\left\| \overrightarrow{f}_\nu \right\|} - \sum_{i=1}^{\infty} \eta_i^n \overrightarrow{H}_n^{(i)} \right\|$$

$$\leq \sum_{i=1}^{r} \eta_i^n \left\| \sum_{l=1}^{L_i} \left\| \overrightarrow{h}_l^{(i)} \right\|^n \frac{\overrightarrow{h}_l^{(i)}}{\left\| \overrightarrow{h}_l^{(i)} \right\|} - \overrightarrow{H}_n^{(i)} \right\| + 2\eta < 4\eta$$

is valid. Therefore the series

$$\sum_{\nu=1}^{\infty} \left\| \overrightarrow{f}_\nu \right\|^n \frac{\overrightarrow{f}_\nu}{\left\| \overrightarrow{f}_\nu \right\|}$$

converges to

$$\sum_{i=1}^{\infty} \eta_i^n \overrightarrow{H}_n^{(i)}.$$

Let now $n = n_{i_0} \in B$. The same reasoning as above proves that the series

$$\sum \left\| \overrightarrow{f}_\nu \right\|^n \frac{\overrightarrow{f}_\nu}{\left\| \overrightarrow{f}_\nu \right\|},$$

where total only those of terms that are not generated by $\left\{ \overrightarrow{h}_l^{(i_0)} \right\}_{l=1}^\infty$ is convergent to

$$\sum_{i=1, i \neq i_0}^\infty \eta_i^n \overrightarrow{H}_n^{(i)}$$

while the series

$$\sum_{l=1}^\infty \left\| \overrightarrow{h}_l^{(i_0)} \right\|^n \frac{\overrightarrow{h}_l^{(i_0)}}{\left\| \overrightarrow{h}_l^{(i_0)} \right\|}$$

is divergent. It means that the series

$$\sum_{\nu=1}^\infty \left\| \overrightarrow{f}_\nu \right\|^n \frac{\overrightarrow{f}_\nu}{\left\| \overrightarrow{f}_\nu \right\|}$$

is divergent. The first statement of Theorem 3 is proved.

If *cone* $(\overrightarrow{\omega}_1, \ldots, \overrightarrow{\omega}_q)$ does not contain any straight line, then there exists such a hyperplane in $\mathbb{R}^d$ that passes through the origin and separates *cone* $(\overrightarrow{\omega}_1, \ldots, \overrightarrow{\omega}_q)$ and *cone* $(-\overrightarrow{\omega}_1, \ldots, -\overrightarrow{\omega}_q)$. Let $\overrightarrow{n}$ be a normal vector to this hyperplane. Assume, without loss of generality, that all dot products $\overrightarrow{\omega}_j \cdot \overrightarrow{n} > 0$ and, therefore, $\min\{\overrightarrow{\omega}_j \cdot \overrightarrow{n} : j = 1, \ldots, q\} = \delta > 0$. If for some $n \in \mathbb{N}$ and some sequence $\overrightarrow{f}_\nu = \left\| \overrightarrow{f}_\nu \right\| \overrightarrow{\omega}_{j(\nu)}, \nu \in \mathbb{N}$, the series $\sum_{\nu=1}^\infty \left\| \overrightarrow{f}_\nu \right\|^n \overrightarrow{\omega}_{j(\nu)}$ is convergent, then

$$\infty > \sum_{\nu=1}^\infty \left\| \overrightarrow{f}_\nu \right\|^n \overrightarrow{\omega}_{j(\nu)} \cdot \overrightarrow{n} \geq \delta \sum_{\nu=1}^\infty \left\| \overrightarrow{f}_\nu \right\|^n.$$

Therefore all of the series $\sum_{\nu=1}^\infty \left\| \overrightarrow{f}_\nu \right\|^m \overrightarrow{\omega}_{j(\nu)}$ with $m \geq n$ are convergent absolutely. Theorem 3 is proved. ■

P r o o f o f T h e o r e m 4. For any $j \in \mathbb{Z}_+$, let $\left\{ a_k^{(j)} \right\}_{k=1}^\infty$ be one of the series such that for $l = 1, \ldots, 2^j$ the series $\sum_{k=1}^\infty \left| a_k^{(j)} \right|^{l/2^j} sign\left(a_k^{(j)} \right)$ are convergent and the series $\sum_{k=1}^\infty \left| a_k^{(j)} \right|^2 sign\left(a_k^{(j)} \right)$ is divergent to ∞. The simplest way to get such a series is to use the one-dimensional case of the sequence $\left\{ f_\nu^{(j)} \right\}_{\nu=1}^\infty$ we have used in the first part of the proof of Theorem 3 when $B = \left\{2^{j+1}\right\}$ and to define $a_k^{(j)} = \left|f_k^{(j)}\right|^{2^j} sign\left(f_k^{(j)}\right)$, $k = 1, 2, \ldots$. Let η_j, $j = 1, 2, \ldots$, be such positive numbers that $\sum_{j=1}^{\infty} \eta_j = 1$. The sequence $\{a_k\}_{k=1}^{\infty}$ is defined in the following way.

Let $N_0 = 0$ an let N_1 be such a large number that

$$\sum_{k=1}^{N_1} \left|a_k^{(1)}\right|^2 sign\left(a_k^{(1)}\right) \geq 1 \text{ and}$$

$$\sup\left\{\left|\sum_{k=L}^{\infty} \left|a_k^{(2)}\right|^r sign\left(a_k^{(2)}\right)\right| : L \geq N_1 + 1, r \in \left\{\frac{1}{2}, \frac{2}{2}\right\}\right\} < \eta_1,$$

let $N_2 > N_1$ be such a large number that

$$\sum_{k=N_1+1}^{N_2} \left|a_k^{(2)}\right|^2 sign\left(a_k^{(2)}\right) \geq 1 \text{ and}$$

$$\sup\left\{\left|\sum_{k=L}^{\infty} \left|a_k^{(3)}\right|^r sign\left(a_k^{(3)}\right)\right| : L \geq N_2 + 1, r \in \left\{\frac{1}{2^2}, \frac{2}{2^2}, \frac{3}{2^2}, \frac{2^2}{2^2}\right\}\right\} < \eta_2,$$

and so on. Then for $N_j + 1 \leq k \leq N_{j+1}$, $j = 0, 1, \ldots$, we define $a_k = a_k^{(j)}$.

Let $r \in \left\{\frac{1}{2^j}, \ldots, \frac{2^j}{2^j}\right\}$ where j is any nonnegative integer. The series $\sum_{k=1}^{\infty} |a_k|^r sign(a_k)$ is convergent. Indeed, for $N_j + 1 \leq L < M \leq N_J$ — here j is the greatest and J is the smallest possible — we have

$$\left|\sum_{k=L}^{M} |a_k|^r sign(a_k)\right| \leq \left|\sum_{k=L}^{N_{j+1}} \left|a_k^{(j)}\right|^r sign\left(a_k^{(j)}\right)\right| + \sum_{i=j+1}^{J-2} \left|\sum_{k=N_i+1}^{N_{i+1}} \left|a_k^{(j)}\right|^r sign\left(a_k^{(j)}\right)\right|$$

$$+ \left|\sum_{k=N_{J-1}+1}^{M} \left|a_k^{(j)}\right|^r sign\left(a_k^{(j)}\right)\right| < 2\left(\eta_{j-1} + \cdots + \eta_{J-1}\right).$$

and convergence is implied by Weierstrass's criterion. On the other hand,

$$\begin{aligned}\sum_{k=1}^{\infty} |a_k|^2 sign(a_k) &\geq \sum_{k=1}^{N_1} |a_k|^2 sign(a_k) + \sum_{k=N_1+1}^{N_2} |a_k|^2 sign(a_k) + \cdots \\ &\geq 1 + 1 + \cdots = \infty.\end{aligned}$$

For the just constructed sequence $\{a_k\}_{k=1}^{\infty}$ the series $\sum_{k=1}^{\infty} a_k^2 sign(a_k) = \infty$ while all series $\sum_{k=1}^{\infty} a_k^{\lambda} sign(a_k) = \infty$, where λ is any binary rational number of the interval $(0,1]$ are convergent. In fact, they are convergent for all λ that belong to a certain continuum $\Lambda \subset [0,1.5]$. Let us describe this Λ.

Let ε_1 be such a number that

$$\forall\lambda \in \left(\frac{1}{2}-\varepsilon_1, \frac{1}{2}+\varepsilon_1\right) \cup (1-\varepsilon_1, 1+\varepsilon_1) \quad \forall L \in [N_1+1, N_2]:$$

$$\left|\sum_{k=N_1+1}^{L} |a_k|^\lambda \, sign\,(a_k)\right| = \left|\sum_{k=N_1+1}^{L} \left|a_k^{(2)}\right|^\lambda sign\left(a_k^{(2)}\right)\right| < 4\eta_1.,$$

let ε_2 be such a number that

$$\forall\lambda \in \bigcup_{l=1}^{2^2} \left(\frac{l}{2^2}-\varepsilon_2, \frac{l}{2^2}+\varepsilon_2\right) \quad \forall L \in [N_2+1, N_3]:$$

$$\left|\sum_{k=N_2+1}^{L} |a_k|^\lambda \, sign\,(a_k)\right| = \left|\sum_{k=N_2+1}^{L} \left|a_k^{(3)}\right|^\lambda sign\left(a_k^{(3)}\right)\right| < 4\eta_2.,$$

and so on. Λ can be defined as follows:

$$\Lambda = \bigcup_{i=1}^{\infty}\left(\bigcap_{j=i}^{\infty}\left(\bigcup_{l=1}^{2^j}\left(\frac{l}{2^j}-\varepsilon_j, \frac{l}{2^j}+\varepsilon_j\right)\right)\right).$$

The declared convergence for each $\lambda \in \Lambda$ is obvious and all we have to do is to prove that Λ is a continuum.

Let integers l_j be so large that

$$\varepsilon_j > 2^{-l_j+1}, j = 1, 2, \dots .$$

Consider the set of all numbers

$$1.00\dots0d_100\dots0d_200\dots0d_300\dots,$$

where $d_j = 0 \vee 1$ and there are l_1 zeros between the decimal point and d_1, l_2 zeros between d_1 and d_1, and so on. Each such number belongs to Λ and their set is equivalent to the set of all numbers

$$0.d_1d_2d_3\dots, d_j = 0 \vee 1,$$

that is binary representation of $[0,1]$. Theorem 4 is proved. ■

References

[1] *T. Bonnesen and W. Fenchel*, Theorie der Konvexen Körper. Chelsea Publishing Co., Bronx, New York (1971).

[2] *N. Danford and J.T. Schwartz*, Linear Operators. Part 1: General Theory. Interscience Publishers, New York, London (1958).

[3] *G.O. Thorin*, Convexity theorems generalizing those of M. Riesz and Hadamard with some applications. — *Comm. Sém. Math. Univ. Lund* (1948), No. 9, p. 1–58.