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# Zeroes of holomorphic functions with almost–periodic modulus

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We give necessary and sufficient conditions for a divisor in a tube domain to be the divisor of a holomorphic function with almost–periodic modulus.

*To I.V. Ostrovskii on the occasion of his 70-th birthday*

Zero distribution for various classes of holomorphic almost–periodic functions in a strip was studied by many authors (cf. [1, 4, 7–10, 17]). The notion of almost-periodic discrete set appeared in [9] and [17] in connection with these investigations. Its generalization to several complex variables was the notion of almost–periodic divisor, introduced by L.I. Ronkin (cf. [14]) and studied in his works and works of his disciples (cf. [5, 6, 15]). But these notions are not sufficient for a complete description of zero sets of holomorphic almost–periodic functions (cf. [18]): in addition, one needs some topological characteristic, namely, Chern class of the special (generated by an almost-periodic set or a divisor) line bundle over Bohr's compact set (cf. [2, 3]). On the other hand, the class of zero sets of holomorphic functions with almost-periodic modulus in a strip is just the class of almost-periodic discrete sets (cf. [4]). That's why it is natural to obtain a description of zeroes of holomorphic functions with the almost-periodic modulus for several complex variables without using topological terms. This problem is just solved in our paper.

By $T_S$ denote a tube set $\{z = x + iy : \, x \in \mathbb{R}^m, y \in S\}$, where the base $S$ is a subset of $\mathbb{R}^m$.

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**Definition 1.** *A continuous function $f$ on $T_S$ is called almost-periodic, if for each sequence $\{f(z+h_n)\}_{h_n\in\mathbb{R}^m}$ of shifts there exists a uniformly convergent on $T_S$ subsequence.*

In particular, for $S=\{0\}$ we obtain the definition of an almost-periodic function on $\mathbb{R}^m$.*

It follows easily that any almost-periodic function on a tube set with a compact base is bounded.

**Definition 2.** *Let $\Omega$ be a domain in $\mathbb{R}^m$. A continuous function $f$ on $T_\Omega$ is called almost-periodic, if its restriction to every tube set $T_K$ with compact base $K\subset\Omega$ is an almost–periodic function on $T_K$.*

**Definition 3 (cf. [14], for distributions from $\mathcal{D}'(\mathbb{R})$ cf. also [16]).** *A distribution $F(z)\in\mathcal{D}'(T_\Omega)$ is called almost-periodic, if for any test-function $\varphi(z)\in\mathcal{D}(T_\Omega)$ the function $\langle F(z),\varphi(z-t)\rangle$ is an almost-periodic function in $t\in\mathbb{R}^m$.*

The next assertion is valid.

**Theorem 1 (cf. [14]).** *A distribution $F\in\mathcal{D}'(T_\Omega)$ is almost periodic if and only if for each sequence $\{h^n\}\in\mathbb{R}^m$ there exists a subsequence $\{\tilde h^n\}$ such that the sequence of the distributions $F(z+\tilde h^n)$ converges uniformly on the sets $\{\kappa(z-t):\,t\in\mathbb{R}^m,\,\kappa\in\mathcal{K}\}$, where $\mathcal{K}$ is any compact family in $\mathcal{D}(T_\Omega)$.*

**Definition 4 (cf. [14]).** *The mean value (in the variable $x\in\mathbb{R}^m$) of an almost–periodic distribution $F$ is the distribution $c_F(y)\otimes dx$ with $c_F(y)\in\mathcal{D}'(\Omega)$ and the Lebesgue measure $dx$ on $\mathbb{R}^m$, defined for a test–functions $\varphi\in\mathcal{D}(T_\Omega)$ by the equality*

$$\langle c_F(y)\otimes dx,\varphi(z)\rangle=\lim_{N\to\infty}(2N)^{-m}\int\limits_{\max_j|t_j|<N}\langle F(z),\varphi(z-t)\rangle dt,$$

*where $t=(t_1,\dots,t_m)\in\mathbb{R}^m$.*

Note that if $F(z)$ is an almost-periodic function on $T_\Omega$, then $c_F(y)$ is a continuous function on $\Omega$. Further, if $F(z)$ is an almost-periodic complex measure on $T_\Omega$, then $c_F(y)$ is a complex measure on $\Omega$ as well, and $c_F(y)\otimes dx$ is the weak limit of the measures $F(tx+iy)dx\,dy$ as $|t|\to\infty$ (cf. [14]).

By $\mathcal{H}(G)$ denote the space of holomorphic functions on the domain $G\subset\mathbb{C}^m$ with respect to the topology of the uniform convergence on compact subsets of $G$.

The following assertion is true.

* This definition is equivalent to another one that makes use of the notion of an $\varepsilon$-almost period; for $m=1$ see, for example, [12], the extension to $m>1$ is trivial.

**Theorem 2 (cf. [14]).** *If a function $f \in \mathcal{H}(T_\Omega)$ is almost-periodic, then* $\log|f|$ *is an almost-periodic distribution on $T_\Omega$.*

The main part of the proof of this theorem is the following lemma.

**Lemma 1 (cf. [14]).** *If $f_n \in \mathcal{H}(G)$, $n = 1, 2, \ldots,$ and $f_n \to f \not\equiv 0$ in the space $\mathcal{H}(G)$, then* $\log|f_n| \to \log|f|$ *in the space $\mathcal{D}'(G)$.*

Now let

$$(i/\pi)\partial\bar\partial \log|f| = (2/\pi)\sum_{j,k=1}^{m} \frac{\partial^2 \log|f|}{\partial z_j \partial \bar z_k}(i/2)dz_j \wedge d\bar z_k \tag{1}$$

be the current of integration over the divisor $d_f$ of the function $f(z) \in \mathcal{H}(G)$, $z = (z_1, \ldots, z_m)$. In the case $m = 1$ this current corresponds to the discrete measure with integer masses equal to the multiplicities of the zeroes of the function $f$.

Note that all the coefficients of the current (1) are complex measures on $G$, and the "diagonal" coefficients $\frac{\partial^2 \log|f|}{\partial z_j \partial \bar z_j}$ are positive measures (cf. [11]).

Clearly, the differentiation keeps the almost periodicity of distributions. Therefore, it follows from Theorem 2 that all the coefficients of the current (1) are almost-periodic distributions for any holomorphic almost-periodic function on $T_\Omega$. If we replace $f$ by another holomorphic function on $T_\Omega$ with the same divisor, then the coefficients of the current (1) do not change. Hence an almost–periodicity of all the coefficients does not imply almost periodicity of the function $f$ itself.

**Definition 5 (cf. [5, 6]).** *The divisor $d_f$ of a function $f \in \mathcal{H}(T_\Omega)$ is called almost–periodic, if all the coefficients of the current (1) are almost–periodic distributions.*

Note that in [14] a divisor $d_f$ was called almost–periodic, if the measure $\sum_{j=1}^{m} \frac{\partial^2 \log|f|}{\partial z_j \partial \bar z_j}$ was almost–periodic on $T_\Omega$. But that definition is equivalent to the given above (cf. [6]).

There exist almost periodic divisors which cannot be generated by holomorphic almost periodic functions. For example, let $g(w)$ be an entire function on $\mathbb{C}$ with simple zeroes at the points of the standard integer–valued lattice, and let $d[\lambda, \mu]$, $\lambda, \mu \in \mathbb{R}^m$ be the divisor of the function $g(\langle z,\, \lambda\rangle + i\langle z,\, \mu\rangle)$. This divisor is periodic for vectors $\lambda$, $\mu$ that are linearly dependent over $\mathbb{Q}$ or linearly independent over $\mathbb{R}$ (with the periods $\frac{|\mu|^2\lambda - \langle\lambda,\, \mu\rangle\mu}{|\lambda|^2|\mu|^2 - \langle\lambda,\, \mu\rangle^2}$ and $\frac{|\lambda|^2\mu - \langle\lambda,\, \mu\rangle\lambda}{|\lambda|^2|\mu|^2 - \langle\lambda,\, \mu\rangle^2}$). Then $d[\lambda, \mu]$ is almost periodic for $\lambda$, $\mu$ linearly independent over $\mathbb{Q}$ and linearly dependent over $\mathbb{R}$ (for $m = 1$ cf. [18]; since a real linearly transform in $\mathbb{C}^m$ keeps almost–periodicity, the case $m > 1$ follows as well). Besides, the divisor $d[\lambda, \mu]$ for any linearly independent over $\mathbb{Q}$ vectors $\lambda$, $\mu$ is the divisor of no holomorphic almost periodic function (in the case $m = 1$, i.e., irrational $\lambda/\mu$ cf.[18], for $m > 1$ cf. [15]). A complete description of the divisors of holomorphic almost–periodic functions is contained in the following theorem.

**Theorem 3 (for $m = 1$ cf. [2], for $m > 1$ cf. [3]).** *A holomorphic bundle over Bohr's compactification $K_B$ of the space $\mathbb{R}^m$ is assigned to each almost–periodic divisor $d$ on a tube domain $T_\Omega$ with convex base $\Omega$ such that:*

*the map $d \mapsto c(d)$, $c(d)$ being the first Chern class of this bundle, is a homomorphism of the semigroup of positive almost–periodic divisors on $T_\Omega$ to the cohomology group $H^2(K_B, \mathbb{Z})$, the kernel of this homomorphism is just the set of all divisors of holomorphic almost–periodic functions on $T_\Omega$,*

*a finite family $\lambda^j, \mu^j \in \mathbb{R}^m$ corresponds to each cohomology class $c(d)$ such that $c(d) = \sum_j c(d[\lambda^j, \mu^j])$,*

*the mapping $W : (\lambda, \mu) \mapsto c(d[\lambda, \mu])$ is skew-symmetric and additive in variables $\lambda, \mu \in \mathbb{R}^m$.*

A description of zeroes for holomorphic functions of one variable with the almost–periodic modulus is given in the following theorem.

**Theorem 4 (cf. [4]; for divisors $d[\lambda, \mu]$, $\lambda, \mu \in \mathbb{R}$ cf. [18]).** *A divisor $d$ on a strip is the divisor of some holomorphic function on the strip with almost–periodic modulus if and only if $d$ is almost–periodic.*

Now consider the multidimensional case again. Note that for an almost-periodic divisor $d$ on $T_\Omega$ all the coefficients of the current (1) have mean values in $x$. The imaginary parts of these mean values, i.e., the mean values of the real measures $(2/\pi)\Im\frac{\partial^2 \log|f|}{\partial z_j \partial \bar{z}_k}$ have the form $a_{j,k} dy \otimes dx$, $a_{j,k} \in \mathbb{R}$ (cf. [6]). By $A(d)$ denote the matrix with the entries $a_{j,k}$. In the case $d = d_f$ for an almost-periodic function $f \in \mathcal{H}(T_\Omega)$ we have $A(d) = 0$ (cf. [13]).

**Theorem 5.** *A divisor $d$ on a tube domain $T_\Omega$ with convex base $\Omega$ is the divisor of a holomorphic function with almost-periodic modulus if and only if divisor $d$ is almost-periodic, and the skew-symmetric matrix $A(d)$ is zero.*

To prove this theorem we need the following improvement of Theorem 2.

**Theorem 6.** *A function $f \in \mathcal{H}(T_\Omega)$, $f \not\equiv 0$, has almost-periodic modulus if and only if the distribution $\log|f| \in \mathcal{D}'(T_\Omega)$ is almost-periodic.*

P r o o f o f T h e o r e m 6. Let $|f(z)|$ be an almost-periodic function on $T_\Omega$, and let $\{h^n\}$ be an arbitrary sequence from $\mathbb{R}^m$. In order to check that $\log|f|$ is an almost-periodic distribution on $T_\Omega$, we will prove that for any continuous function $\varphi$ with compact support in $T_\Omega$, the sequence of functions

$$\psi_n(t) = \int \log|f(z + h^n)|\varphi(z - t)dxdy \tag{2}$$

contains a convergent, uniformly on $\mathbb{R}^m$, subsequence. We will prove this assertion by contradiction.

First, since the function $|f(z)|$ is uniformly bounded on $T_K$ for every compact set $K \subset \Omega$, we may assume that the sequence of the functions $\{f(z+h^n)\}$ converges to some function $g(z)$ in the space $\mathcal{H}(T_\Omega)$. Further, since the function $|f(z)|$ is almost-periodic on $T_\Omega$, we may assume that the sequence of the functions $\{|f(z+h^n)|\}$ converges to some function $\Phi(z) \not\equiv 0$ uniformly on each $T_K$. If the sequence (2) does not converge uniformly on $\mathbb{R}^m$, then for some $\delta > 0$ and some subsequence of $n$ there exist $t^n \in \mathbb{R}^m$ with the property

$$|\psi_n(t^n) - \int \log|g(z)|\varphi(z-t^n)dxdy| \geq \delta. \tag{3}$$

The function $|g(z)| \equiv \Phi(z)$ is almost-periodic on $T_\Omega$, hence we may assume that the same subsequence of the functions $\{|g(z+t^n)|\}$ converges uniformly on each $T_K$ to some function $\Psi(z) \not\equiv 0$. Since the sequence of the functions $\{|f(z+h^n+t)|\}$ converges uniformly in $t \in \mathbb{R}^m$ and $z \in T_K$ to the function $|g(z+t)|$, we see that the subsequence $\{|f(z+h^n+t^n)|\}$ converges to $\Psi(z)$ uniformly on $T_K$. Also, the subsequences of the functions $\{f(z+h^n+t^n)\}$ and $\{g(z-t^n)\}$ are bounded uniformly on compact subsets of $T_\Omega$, therefore passing to a subsequence again, we get that $f(z+h^n+t^n) \to H_1(z)$ and $g(z+t^n) \to H_2(z)$ in the space $\mathcal{H}(T_\Omega)$, and $|H_1(z)| = \Psi(z) = |H_2(z)|$. Using Lemma 1, we obtain that the corresponding subsequences of the functions $\{\log|f(z+h^n+t^n)|\}$ and $\{\log|g(z+t^n)|\}$ converge, in the sense of distributions, to the same function $\log\Psi(z)$. The last assertion contradicts (3).

On the other hand, let $\log|f(z)|$ be an almost-periodic distribution on $T_\Omega$, and let $\varphi_\varepsilon(z)$ be a nonnegative, depending on $|z|$ smooth function such that $\varphi(z) = 0$ for $|z| > \varepsilon$ and $\int_{\mathbb{C}^m} \varphi_\varepsilon(z)dx\,dy = 1$. Evidently, the family of functions $\{\varphi_\varepsilon(z+iy)\}_{|y|\leq C}$ is a compact set in the space $\mathcal{D}(\mathbb{C}^m)$ for every $C < \infty$. Let $K$ be a compact set in $\Omega$ and $\varepsilon < \operatorname{dist}\{K, \partial\Omega\}$. Now Theorem 1 implies that the convolution $(\log|f| * \varphi_\varepsilon)(z)$ is an almost-periodic function on $T_K$. Hence this convolution is bounded on $T_K$, and the inequality $\log|f(z)| \leq (\log|f| * \varphi_\varepsilon)(z)$ shows that $|f(z)|$ is bounded on $T_K$ as well.

Suppose that $|f|$ is not an almost-periodic function on $T_\Omega$. Then there exists a sequence of functions $\{|f(z+h^n)|\}$, $h^n \in \mathbb{R}^m$, such that every its subsequence does not converge uniformly on $T_{K'}$ for some compact set $K' \subset \Omega$. Without loss of generality it can be assumed that the sequence of functions $\{f(z+h^n)\}$ converges in the space $\mathcal{H}(T_\Omega)$ to some function $g(z)$. It is clear that $g(z)$ is bounded on $T_K$ for every compact set $K \subset \Omega$. Further, by Lemma 1 we get $\log|f(z+h^n)| \to \log|g(z)|$ in the sense of distributions. Using Theorem 1 and passing to a subsequence, we obtain

$$\int (\log|f(z+h^n)| - \log|g(z)|)\varphi_\varepsilon(z-t-is)dxdy \to 0 \tag{4}$$

uniformly in $t \in \mathbb{R}^m$ and $s \in K'$. On the other hand, for some $\delta > 0$ and some subsequence of $n$ there exist points $z^n = x^n + iy^n \in T'_K$ such that

$$||f(h^n + x^n + iy^n)| - |g(x^n + iy^n)|| \geq \delta. \tag{5}$$

Passing to a subsequence if necessary, we may assume that $y^n \to y^0 \in K'$, and the sequences of the functions $\{f(z + h^n + z^n - iy^0)\}$ and $\{g(z + z^n - iy^0)\}$ converge in the space $\mathcal{H}(T_\Omega)$ to functions $H_1(z)$ and $H_2(z)$, respectively. Then Lemma 1 implies that $\log|f(z+h^n+z^n-iy^0)| \to \log|H_1(z)|$ and $\log|g(z+z^n-iy^0)| \to \log|H_2(z)|$ in the space $\mathcal{D}'(T_\Omega)$. Taking into account (4), we obtain

$$\int \log|H_1(z)|\varphi_\varepsilon(z-iy^0)dx\,dy = \int \log|H_2(z)|\varphi_\varepsilon(z-iy^0)dxdy.$$

Since $\varepsilon$ is arbitrary small, we get $|H_1(iy^0)| = |H_2(iy^0)|$. At the same time, by (5) we have $|H_1(iy^0)| \neq |H_2(iy^0)|$. This contradiction proves Theorem 6.

P r o o f o f t h e n e c e s s i t y i n T h e o r e m 5. It follows from Theorem 6 that every function $f \in \mathcal{H}(T_\Omega)$ with almost-periodic modulus has an almost-periodic divisor. Further, the mean value $c_{\log|f|}(y) \otimes dx$ of the function $\log|f|$ is the weak limit of the measures $\log|f(tx+iy)|dx \otimes dy$ as $|t| \to \infty$ in the space $\mathcal{D}'(T_\Omega)$, therefore for all $j \neq k$ the mean values of the distributions

$$\Im \frac{\partial^2 \log|f|}{\partial z_j \partial \bar{z}_k} = \frac{1}{4}\left(\frac{\partial^2}{\partial x_j \partial y_k} - \frac{\partial^2}{\partial x_k \partial y_j}\right)\log|f|$$

equal

$$\lim_{|t|\to\infty} \frac{1}{4}\left(\frac{\partial^2}{\partial x_j \partial y_k} - \frac{\partial^2}{\partial x_k \partial y_j}\right)\log|f(tx+iy)|dx \otimes dy$$
$$= \frac{1}{4}\left(\frac{\partial^2}{\partial x_j \partial y_k} - \frac{\partial^2}{\partial x_k \partial y_j}\right)c_{\log|f|}(y) \otimes dx = 0.$$

The necessity of the conditions in Theorem 5 is proved.

The proof of the sufficiency makes use of the following lemmas. As above, $d[\lambda, \mu]$, $\lambda, \mu \in \mathbb{R}^m$ is the divisor of the function $g(\langle z, \lambda\rangle + i\langle z, \mu\rangle)$, where $g(w)$ is an entire function on $\mathbb{C}$ with simple zeroes at the points of the standard integer-valued lattice.

**Lemma 2.** *The divisor $d[\lambda,\mu]$ with $\lambda = t\mu$, $\lambda \in \mathbb{R}^m$, $t \in \mathbb{R}$, is the divisor of an entire function on $\mathbb{C}^m$ with almost-periodic modulus.*

P r o o f o f L e m m a 2. After a suitable regular real linear transform we obtain the case $\mu = (1, 0, \ldots, 0)$, i.e., the case of a divisor depending only on one coordinate, therefore the assertion of our lemma is a consequence of Theorem 4.

Further, let $e^1, \ldots, e^m$ be the coordinate vectors in $\mathbb{C}^m$.

**Lemma 3.** *The entries $a_{j,k}$ of the matrix $A_0 = A(d[e^1, e^2])$ vanish for $(j,\, k) \neq (1,\, 2)$ or $(2,\, 1)$, and $a_{1,2} = -1$, $a_{2,1} = 1$.*

P r o o f o f L e m m a 3. The divisor of the function $g(z_1 + iz_2)$ does not depend on variables $z_j$ with $j > 2$, hence the distributions $\Im \frac{\partial^2 \log|g(z_1+iz_2)|}{\partial z_j \partial \bar z_k}$ vanish for $(j,\, k) \neq (1,\, 2)$ or $(2,\, 1)$.

Consider the expression

$$(L_z \log|g(z_1 + iz_2)|,\, \varphi(z_1 + t_1, z_2 + t_2)), \quad (t_1,\, t_2) \in \mathbb{R}^2, \tag{6}$$

for $L_z = \frac{2}{\pi} \Im \frac{\partial^2}{\partial \bar z_1 \partial z_2}$ and a function $\varphi(z) \geq 0$ from the space $\mathcal{D}(\mathbb{C}^2)$. In the coordinates $\zeta_1 = z_1 + iz_2$, $\zeta_2 = z_1 - iz_2$, it has a form

$$\frac{1}{4}(\tilde L_\zeta \log|g(\zeta_1)|,\, \varphi((\zeta_1 + \zeta_2)/2 + t_1, (\zeta_1 - \zeta_2)/2i + t_2))$$

with

$$\tilde L_\zeta = \frac{2}{\pi} \Re \left( \frac{\partial^2}{\partial \zeta_1 \partial \bar\zeta_1} - \frac{\partial^2}{\partial \zeta_2 \partial \bar\zeta_1} + \frac{\partial^2}{\partial \zeta_1 \partial \bar\zeta_2} - \frac{\partial^2}{\partial \zeta_2 \partial \bar\zeta_2} \right).$$

Using the definition of $g$ and properties of the Laplace operator, we get

$$\tilde L_\zeta \log|g(\zeta_1)| = \frac{2}{\pi} \frac{\partial^2}{\partial \zeta_1 \partial \bar\zeta_1} \log|g(\zeta_1)| = \sum_{q_1, q_2 \in \mathbb{Z}} \delta(\zeta_1 - q_1 - iq_2) \otimes d\xi d\eta,$$

where $\delta$ is the Dirac function on the plane, $\xi = \Re\zeta_2, \eta = \Im\zeta_2$. Therefore, (6) is equal to

$$\frac{1}{4} \sum_{q_1, q_2 \in \mathbb{Z}} \int_{\mathbb{C}} \varphi(t_1 + (q_1 + iq_2 + \xi + i\eta)/2, t_2 + (q_1 + iq_2 - \xi - i\eta)/2i) d\xi d\eta.$$

Substituting $\xi - q_1$, $\eta + q_2$ for $u$, $v$, respectively, we get

$$\frac{1}{4} \sum_{q_1, q_2 \in \mathbb{Z}} \int_{\mathbb{C}} \varphi(u/2 + iv/2 + t_1 + q_1, -v/2 + iu/2 + t_2 + q_2) du\, dv. \tag{7}$$

Since the divisor $d_{e^1,e^2}$ has period 1 in each variable, we see that the mean value of (6) is the integral of (7) over the square $0 \le t_1 \le 1,\ 0 \le t_2 \le 1$. Then it is equal to the integral

$$\frac{1}{4}\int\limits_{\mathbb{R}^4} \varphi(u/2 + iv/2 + t_1, -v/2 + iu/2 + t_2)du\,dv\,dt_1\,dt_2.$$

Finally, substituting $u/2 + t_1 = x_1,\ v/2 = y_1,\ t_2 - v/2 = x_2,\ u/2 = y_2$, we obtain the equality

$$\int\limits_0^1\int\limits_0^1 (L_z \log|g(z_1 + iz_2)|,\ \varphi(z_1 + t_1, z_2 + t_2))dt_1\,dt_2$$
$$= \int\limits_{\mathbb{R}^4} \varphi(x_1 + iy_1, x_2 + iy_2)dx_1\,dy_1\,dx_2\,dy_2,$$

hence the mean value of the distribution $L_z \log|g(z_1 + iz_2)|$ is the Lebesgue measure in $\mathbb{C}^2$. The lemma is proved.

By $(\lambda, \mu)$ denote the matrix product $(\lambda_j \mu_k)_{j,k=1}^m$ of the vectors $\lambda = (\lambda_1, \dots, \lambda_m)$, $\mu = (\mu_1, \dots, \mu_m) \in \mathbb{R}^m$.

**Lemma 4.** *For any $\lambda$, $\mu \in \mathbb{R}^m$, the matrix $A(d[\lambda, \mu])$ equals the difference $(\mu, \lambda) - (\lambda, \mu)$.*

P r o o f o f L e m m a 4. If $\lambda$, $\mu$ are linearly dependent over $\mathbb{R}$, then $(\mu, \lambda) - (\lambda, \mu) = 0$. On the other hand, it follows from Lemma 2 that in this case the divisor $d[\lambda, \mu]$ is the divisor of some holomorphic in $\mathbb{C}^m$ function with almost–periodic modulus. Using the proved part of Theorem 5, we have $A(d[\lambda, \mu]) = 0$.

Let $\lambda$, $\mu$ be linearly independent over $\mathbb{R}$. The divisor $d[\lambda, \mu]$ is the divisor $d[e^1, e^2]$ in the coordinates $\zeta = Bz$ for some real nondegenerate matrix $B$ with the first and second rows $\lambda$ and $\mu$, respectively. Note that the matrix $A(d)$ is the matrix of the mean values for the matrix $\frac{1}{2i}(D(z) - \bar{D}(z))$, where

$$D(z) = \left(\frac{\partial^2 \log|g(\langle z, \lambda\rangle + i\langle z, \mu\rangle)|}{\partial z_j \partial \bar{z}_k}\right),$$

$\bar{D}$ being the matrix with all the entries complex conjugated to the corresponding entries of the matrix $D$. Therefore $D(z) = B' D(\zeta) B$, $B'$ being the transpose matrix to $B$, and $A(d[\lambda, \mu]) = B' A_0 B$ for the matrix $A_0$ from the previous lemma. This completes the proof of Lemma 4.

**Lemma 5.** *If numbers $\alpha_j,\ \beta_j \in \mathbb{R},\ j=1,\dots,n,$ satisfy the condition $\sum_1^n \alpha_j\beta_j = 0$, then for some $\gamma_k \in \mathbb{R},\ \nu^k \in \mathbb{R}^m,\ k=1,\dots,N$, we get*

$$\sum_1^n W(\alpha_j e^1,\, \beta_j e^2) = \sum_1^N W(\gamma_k \nu^k,\, \nu^k), \tag{8}$$

*$W$ being the mapping from Theorem 3.*

P r o o f o f L e m m a 5. The case $n=1$ means that the left-hand side of (8) vanishes. For $n>1$ we have

$$W(\alpha_{n-1}e^1,\, \beta_{n-1}e^2) + W(\alpha_n e^1,\, \beta_n e^2) = W(\alpha_{n-1}e^1,\, \alpha_n e^1)$$
$$+W(\beta_n e^2,\, \beta_n\alpha_n/\alpha_{n-1}e^2) + W(\alpha_n e^1 + \alpha_n\beta_n/\alpha_{n-1}e^2,\, \alpha_{n-1}e^1 + \beta_n e^2)$$
$$+W(\alpha_{n-1}e^1,\, (\beta_{n-1} + \beta_n\alpha_n/\alpha_{n-1})e^2).$$

The first three terms of the right-hand side have the form $W(\gamma\nu,\, \nu)$, $\gamma\in\mathbb{R}, \nu\in\mathbb{R}^m$. Subtracting these terms from the left-hand side of (8), we get

$$\sum_1^{n-2} W(\alpha_j e^1,\, \beta_j e^2) + W(\alpha_{n-1}e^1,\, (\beta_{n-1} + \beta_n\alpha_n/\alpha_{n-1})e^2).$$

Hence the lemma can be proved by induction over $n$.

**Lemma 6.** *Let vectors $\lambda^j, \mu^j \in \mathbb{R}^m,\ j=1,\dots,n$ be such that the matrix $\sum_1^n(\lambda^j,\mu^j)$ is symmetric. Then*

$$\sum_1^n W(\lambda^j,\, \mu^j) = \sum_1^N W(\gamma_k\nu^k,\, \nu^k) \tag{9}$$

*for some $\gamma_k \in \mathbb{R},\ \nu^k \in \mathbb{R}^m,\ k=1,\dots,N$.*

P r o o f o f L e m m a 6. The vectors $\lambda^j, \mu^j$ are linear combinations of the vectors $e^1,\dots,e^m$, therefore the left-hand side of (9) has the form

$$\sum_{1\le p,q\le m}\left(\sum_{j=1}^{M(p,q)} W(\alpha_{j,p}e^p,\, \beta_{j,q}e^q)\right) \tag{10}$$

with $\alpha_{j,p},\ \beta_{j,q}\in\mathbb{R}$. The mapping $W$ is skew-symmetric, hence we may assume that all the terms in (10) vanish for $p>q$, and the entries of the corresponding matrix $\left(\sum_{j=1}^{M(p,q)}\alpha_{j,p}\beta_{j,q}\right)_{p,q=1}^m$ vanish for all $p>q$. Since this matrix coincides with the symmetric matrix $\sum_1^n(\lambda^j, \mu^j)$, we see that $\sum_{j=1}^{M(p,q)} \alpha_{j,p}\beta_{j,q} = 0$ for $p < q$ as well. Now it follows from Lemma 5 that for $p < q$ the sum

$$\sum_{j=1}^{M(p,q)} W(\alpha_{j,p}e^p,\, \beta_{j,q}e^q)$$

has the form of the right-hand side of (9). The terms of (10) with $p = q$ have already the form $W(\gamma\nu,\, \nu)$. The lemma is proved.

P r o o f o f t h e s u f f i c i e n c y i n T h e o r e m 5. Let $d$ be a divisor in $T_\Omega$ such that $A(d) = 0$. It follows from Theorem 3 that there exist $\lambda^j,\, \mu^j \in \mathbb{R}^m$, $j = 1, \dots, n$, such that the sum $d + \sum_j d[\lambda^j, \mu^j]$ is the divisor of a holomorphic almost-periodic function. Now, by [13], $A(d + \sum_1^n d[\lambda^j, \mu^j]) = 0$. Since the mapping $d \mapsto A(d)$ is a homomorphism, we get $\sum_1^n(\lambda^j,\, \mu^j) - (\mu^j,\, \lambda^j) = \sum_1^n A(d[\lambda^j, \mu^j]) = 0$, i.e., the matrix $\sum_1^n(\lambda^j,\, \mu^j)$ is symmetric. Using Lemma 6, we get (9) for some $\gamma_k \in \mathbb{R},\ \nu^k \in \mathbb{R}^m,\ k = 1, \dots, N$. Therefore,

$$c(d + \sum_1^N d[\gamma_k\nu^k, \nu^k]) = c(d) + \sum_1^N W(\gamma_k\nu^k,\, \nu^k)$$

$$= c(d) + \sum_1^n W(\lambda^j,\, \mu^j) = c(d + \sum_1^n d[\lambda^j, \mu^j]) = 0.$$

An application of Theorem 3 yields that there exists an almost–periodic function $F \in \mathcal{H}(T_\Omega)$ with the divisor $d + \sum_1^N d[\gamma_k\nu^k, \nu^k]$. Using Lemma 2, we can take functions $f_k \in \mathcal{H}(T_\Omega)$ with the divisors $d[\gamma_k\nu^k,\, \nu^k]$ and almost–periodic modula. The function $f(z) = F(z)(\prod_1^N f_k(z))^{-1}$ is holomorphic on $T_\Omega$ and has the divisor $d$. Then Theorem 6 implies that the distributions $\log|F|$ and $\log|f_k|$, $k = 1, \dots, N$, are almost–periodic. Hence the distribution $\log|f| = \log|F| - \sum_1^N \log|f_k|$ is almost–periodic as well. Using Theorem 6 again, we see that the function $|f|$ is almost–periodic. This completes the proof of Theorem 5.

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