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The Haar system in L_1 is monotonically boundedly complete

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Answering a question posed by J.R. Holub we show that for the normalized Haar system $\{h_n\}$ in $L_1[0,1]$ whenever $\{a_n\}$ is a sequence of scalars with $|a_n|$ decreasing monotonically and with $\sup_N \|\sum_{n=1}^{\mathbb{N}} a_n h_n\| < \infty$, then $\sum_{n=1}^{\infty} a_n h_n$ converges in $L_1[0,1]$.

A basis $\{e_n\}_1^\infty$ of a Banach space X is said to be *boundedly complete* if for every sequence $\{a_n\}$ of scalars with $\sup_N \|\sum_{n=1}^{\mathbb{N}} a_n e_n\| < \infty$, the series $\sum_{n=1}^{\infty} a_n e_n$ converges. If a space possesses a boundedly complete basis, then the space is isomorphic to a dual space. In particular $C[0,1]$ and $L_1[0,1]$ do not have boundedly complete bases. Trying to find a weaker property, which may accrue for bases in nondual spaces, J.R. Holub introduced in [1] the following concept:

Definition 1. *A seminormalized basis $\{e_n\}_1^\infty$ of a Banach space X is said to be monotonically boundedly complete if whenever $\{a_n\}$ is a sequence of scalars which decreases monotonically to zero and for which $\sup_N \|\sum_{n=1}^{\mathbb{N}} a_n e_n\| < \infty$, then $\sum_{n=1}^{\infty} a_n e_n$ converges.*

He proved in particular that the Schauder's basis in $C[0,1]$ is monotonically boundedly complete and asked whether the Haar basis in $L_1[0,1]$ is monotonically boundedly complete as well. In this note using some martingale inequalities we give the affirmative answer to this question.

Allover the paper we are using the standard Banach space and probabilistic terminology. For the properties of the Haar system, its role in Banach space theory and connection with martingales we refer the reader to [3, v. 2, ch. 2.c].

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First of all let us recall some probabilistic results. Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space. All the random variables below are defined on this space. Symbol $\|f\|_p$ stands for norm of f in $L_p(\Omega, \Sigma, \mathbb{P})$. The following statement can be found for example in [2, Lemma 1.5].

Lemma 2. *Let $f \in L_1$ and let $f = \sum_1^N d_i$ be the sum of consecutive martingale differences d_i (with respect to an increasing sequence of σ-algebras on Ω) and let $\|d_i\|_\infty < \infty$. Set $a = \left(\sum_1^N \|d_i\|_\infty^2\right)^{1/2}$. Then for every $t > 0$*

$$\mathbb{P}\{|f| > t\} \leq 2\exp\{-t^2/2a^2\}.$$

From this in the same way as one proves the Khintchine inequalities (see the proof of Lemma 4.1 of [2]), one can deduce the following two statements (we give the proofs for reader's convenience).

Corollary 3. *For every $1 \leq p < \infty$ there is a constant B_p such that for every $f = \sum_1^N d_i$ satisfying the assumptions of the Lemma 2 the following inequality holds:*

$$\|f\|_p \leq B_p \left(\sum_1^N \|d_i\|_\infty^2\right)^{1/2}.$$

P r o o f. By homogeneity assume that $a = \left(\sum_1^N \|d_i\|_\infty^2\right)^{1/2} = 1$. Then

$$\|f\|_p^p = \int_0^\infty \mathbb{P}\{|f| > t\} dt^p \leq 2 \int_0^\infty \exp\{-t^2/2\} dt^p.$$

The last expression can serve as B_p^p. ■

Corollary 4. *There is a constant C such that for every $f = \sum_1^N d_i$ satisfying the assumptions of the Lemma 2 the following inequality holds:*

$$\|f\|_1 \geq C \frac{\sum_1^N \|d_i\|_2^2}{\sum_1^N \|d_i\|_\infty^2} \left(\sum_1^N \|d_i\|_2^2\right)^{1/2}.$$

P r o o f. Using the fact that $\{d_i\}_1^N$ form an orthogonal system in $L_2(\Omega, \Sigma, \mathbb{P})$ and using the Hölder inequality, we obtain

$$\sum_1^N \|d_i\|_2^2 = \|f\|_2^2 = \mathbb{E}(|f|^{2/3}|f|^{4/3}) \leq \|f\|_1^{2/3}\|f\|_4^{4/3}.$$

From this and the previous Corollary 3 follows that

$$\sum_1^N \|d_i\|_2^2 \le B_4^{4/3} \|f\|_1^{2/3} \left(\sum_1^N \|d_i\|_\infty^2 \right)^{2/3},$$

which implies the inequality we need. ■

Now let us turn to Holub's question. Recall that the Haar system in $L_1[0,1]$ consists of $h_0 = 1$ and functions

$$h_{k,j} = 2^{k-1} \left(\chi_{[(2j-2)2^{-k},(2j-1)2^{-k})} - \chi_{[(2j-1)2^{-k},2j2^{-k})} \right),$$

where $k \in \mathbb{N}, 1 \le j \le 2^{k-1}$. Written in the natural order: $h_0, h_1 = h_{1,1}, h_2 = h_{2,1}, h_3 = h_{2,2}, h_4 = h_{3,1}, \ldots$ the Haar system forms a normalized basis in $L_1[0,1]$. Answering Holub's question we are going to prove that this basis is monotonically boundedly complete, and in fact much more.

Theorem 5. *Let for a given sequence of numbers $a_{1,1}, a_{2,1}, a_{2,2}, a_{3,1}, \ldots$ having decreasing moduli*

$$\sup_n \left\| \sum_{i=1}^n \sum_{j=1}^{2^{i-1}} a_{i,j} h_{i,j} \right\|_1 = M < \infty. \tag{1}$$

Then the series $\sum_{i=1}^\infty \sum_{j=1}^{2^{i-1}} a_{i,j} h_{i,j}$ converges in $L_2[0,1]$ (and hence converges in L_1 as well).

P r o o f. We shall make use of the Corollary 4 for the sequence $d_i = \sum_{j=1}^{2^{i-1}} a_{i,j} h_{i,j}$ which evidently forms a sequence of martingale differences on $[0,1]$ with respect to the Lebesgue measure and to the sequence of σ-algebras $\mathcal{A}_n$, where $\mathcal{A}_n$ is generated by the partition of $[0,1)$ into dyadic intervals of length 2^{-n+1}. First of all the supports of summands $a_{i,j} h_{i,j}$ in the definition of d_i are disjoint and cover together the whole segment $[0,1)$. Hence for every $i \ge 2$

$$\|d_i\|_\infty = 2^{i-1} \max_{1 \le j \le 2^{i-1}} |a_{i,j}| = 2^{i-1} |a_{i,1}| \le 2^{i-1} |a_{i-1,2^{i-2}}| \le 2\|d_{i-1}\|_2.$$

It is sufficient to prove that $\sum_1^\infty \|d_i\|_2^2 < \infty$, since in such a case the series $\sum_1^\infty d_i$ converges in L_2 to a function g, and the series $\sum_{i=1}^\infty \sum_{j=1}^{2^{i-1}} a_{i,j} h_{i,j}$ under consideration would be just the expansion of g by the Haar system.

Assume contrary that $\sum_1^\infty \|d_i\|_2^2 = \infty$. Due to the Corollary 4 and to the previous inequality for every $N \in \mathbb{N}$ we have

$$\| \sum_1^N d_i \|_1 \ge \frac{C}{2} \frac{\sum_2^N \|d_i\|_\infty^2}{\sum_1^N \|d_i\|_\infty^2} \left(\sum_1^N \|d_i\|_2^2 \right)^{1/2}$$

$$= \frac{C}{2}\left(1 - \frac{\|d_1\|_\infty^2}{\sum_1^N \|d_i\|_\infty^2}\right)\left(\sum_1^N \|d_i\|_2^2\right)^{1/2}.$$

The last expression tends to infinity as $N \to \infty$, which contradicts the condition (1). ■

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