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A dimension-reduced description of general Brownian motion by non-autonomous diffusion-like equations

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The Brownian motion of a classical particle can be described by a Fokker–Planck-like equation. Its solution is a probability density in phase space. By integrating this density w.r.t. the velocity, we get the spatial distribution or concentration. We reduce the $2n$-dimensional problem to an n-dimensional diffusion-like equation in a rigorous way, i.e., without further assumptions in the case of general Brownian motion, when the particle is forced by linear friction and homogeneous random (non-Gaussian) noise. Using a representation with pseudodifferential operators, we derive a reduced diffusion-like equation, which turns out to be non-autonomous and can become elliptic for long times and hyperbolic for short times, although the original problem was time homogeneous. Moreover, we consider some examples: the classical Brownian motion (Gaussian noise), the Cauchy noise case (which leads to an autonomous diffusion-like equation), and the free particle case.

1. Introduction

The Brownian motion of a classical particle in a space- and time-homogeneous medium is characterized by two forces acting on the particle: a deterministic linear friction and a random force — Gaussian or white noise. Thus, the velocity evolution of the particle is random, whereas the evolution of the spatial coordinate is deterministic. The trajectory $\big(v(t), x(t)\big)$ of the particle in phase space can be described by the random system

$$\begin{aligned} \dot{v}(t) &= -av(t) + \sqrt{2b}\,\frac{dw(t)}{dt}, \\ \dot{x}(t) &= v(t) \end{aligned}$$

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with a Wiener process $w(t)$ and is no longer deterministic, but rather a Markovian process with the probability density $W(v,x,t)$. The equation describing the time evolution of this density is the Fokker–Planck equation (see, e.g., [2])

$$\frac{\partial}{\partial t}W(v,x,t) = \frac{\partial}{\partial v}\big(avW(v,x,t)\big) + b\frac{\partial^2}{\partial v^2}W(v,x,t) - v\frac{\partial}{\partial x}W \tag{1}$$

with the initial data $W(v,x,0) = W_0(v,x)$ and decreasing boundary conditions. Dealing with probability densities, we demand normalization

$$\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} W_0(v,x)dvdx = \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} W(v,x,t)dvdx = 1$$

and positivity $W(v,x,t) \geq 0$.

The following can be considered in an arbitrary n-dimensional case. For simplicity, all formulas are written for the case $n = 1$. Thus, we consider a two dimensional phase space coordinate $(v,x) \in \mathbb{R}^2$.

Often our interest is only in the spatial distribution (concentration)

$$c(x,t) = \int\limits_{\mathbb{R}} W(v,x,t)dv \ . \tag{2}$$

Looking from a phenomenological point of view at the time evolution of $c(x,t)$, for long times it looks like a diffusion. So we can guess that $c(x,t)$ is the solution of a diffusion-like equation, say

$$\frac{\partial}{\partial t}c(x,t) = D\frac{\partial^2}{\partial x^2}c(x,t) \ . \tag{3}$$

This equation was derived by A. Einstein in 1905, by P. Langevin in 1908, and by others using phenomenological assumptions to describe the same physical problem — Brownian motion. Indeed, by some heuristic arguments it can be shown that $c(x,t)$, derived from the solution of (1) by (2), satisfies (3): by integrating equation (1) w.r.t. v, we get

$$\frac{\partial}{\partial t}c(x,t) = -\frac{\partial}{\partial x}\int\limits_{\mathbb{R}} vW(v,x,t)dv =: -\frac{\partial}{\partial x}j(x,t) \tag{4}$$

with the current $j(x,t) = \int_{\mathbb{R}} vW(v,x,t)dv$. To get an equation for $j(x,t)$, we multiply (1) by v and integrate w.r.t. v:

$$\frac{\partial}{\partial t}j(x,t) = -aj(x,t) - \frac{\partial}{\partial x}\sigma(x,t) \tag{5}$$

with the mean energy $\sigma(x,t) = \int_{\mathbb{R}} v^2 W(v,x,t)dv$. Multiplying (1) by v^2 and integrating w.r.t. v, we get

$$\frac{\partial}{\partial t}\sigma(x,t) = -2a\sigma(x,t) + 2bc(x,t) - \frac{\partial}{\partial x}\int_{\mathbb{R}} v^3 W(v,x,t)dv. \tag{6}$$

Now we assume that the x-derivative of the third moment vanishes and $\sigma(x,t)$ is not changing in time. We get from (6)

$$\sigma(x,t) = \frac{b}{a}c(x,t).$$

Assuming that $j(x,t)$ is not changing in time, either, we get from (5)

$$0 = -aj(x,t) - \frac{b}{a}\frac{\partial}{\partial x}c(x,t) \quad \Longrightarrow \quad j(x,t) = -\frac{b}{a^2}\frac{\partial}{\partial x}c(x,t).$$

Now we get from (4)

$$\frac{\partial}{\partial t}c(x,t) = \frac{b}{a^2}\frac{\partial^2}{\partial x^2}c(x,t), \tag{7}$$

i.e., equation (3) with the diffusion coefficient $D = \frac{b}{a^2}$.

Of course, equation (1) is more physical than (3), because it takes into account the real state of the particle (v,x) instead of the only spatial coordinate x. If our interest is only in the spatial distribution, it is very tempting to use the much simpler equation (3), but in the derivation shown (1) $\Longrightarrow$ (7) it is difficult to understand what we have done exactly. Moreover, equation (3) cannot be correct at least for short times, because, from a phenomenological point of view, it is clear that the evolution of $c(x,t)$ has to depend on the initial velocity and seems to look more like a solution of a hyperbolic equation than a parabolic one.

As shown, integrating equation (1) w.r.t. v, we get an unclosed equation (4) for $c(x,t)$. The method shown here is a way to close this equation by applying some heuristic assumptions. The goal of the present paper is to derive a diffusion-like closed equation for $c(x,t)$ in a rigorous way for a more general transport equation in phase space.

In Section 2 we show a general scheme to close equation (4). In Section 3 we consider the general Brownian motion. In this case, the general scheme can be calculated explicitly. It turns out that the reduced equation is non-autonomous. We investigate the time behavior of this equation in Section 4. Some examples in Section 5 complete the paper.

We will assume throughout that the considered evolution equations have classical norm and positivity conserving solutions in $L_1(\mathbb{R}^2)$ resp. $L_1(\mathbb{R})$.

2. A general scheme for the reduced equation

We will consider the motion of a classical particle with phase-space coordinate (v, x) in a random medium. In a space- and time-homogeneous medium, the forces acting on the particle are represented by a random function F not dependent on x, whereas the evolution of the spatial coordinate is determined by the velocity. We have the following random system

$$\begin{array}{rcl} \dot{v}(t) & = & F\big(v(t)\big), \\ \dot{x}(t) & = & v(t). \end{array} \tag{8}$$

Assuming that $(v(t), x(t))$ is Markovian with the probability density $W(v, x, t)$, the Kolmogorov equation describing the time evolution of this density has the structure

$$\frac{\partial}{\partial t} W(v, x, t) = \mathbf{A}W - v\frac{\partial}{\partial x} W, \quad W(v, x, 0) = W_0(v, x), \tag{9}$$

where $\mathbf{A}$ is a linear operator of the general form (see, e.g., [2, 6])

$$\big(\mathbf{A}f\big)(v) = \frac{\partial}{\partial v}\big(a(v)f(v)\big) + \frac{\partial^2}{\partial v^2}\big(b(v)f(v)\big) + \fint_{\mathbb{R}} \big[Q(v', v)f(v') - Q(v, v')f(v)\big]\,dv'$$

acting only on the parameter v. The first derivative comes from a deterministic part in F, the second and the integral operator come from a random part. In general, the kernel $Q(v, v')$ can become singular for $v = v'$. Therefore, the integral is to be understood as a mean (or principle) value integral. In this case, the corresponding operator is unbounded, but dominated by the second derivative. In x, equation (9) contains only the first derivative because $\dot{x}(t) = v(t)$ is a deterministic equation (nevertheless $x(t)$ is a random process because $v(t)$ is so).

With suitable boundary conditions, regularity conditions for the coefficients $a(v)$, $b(v)$, and $Q(v, v')$, and with positivity conditions $b(v) \geq 0$ and $Q(v, v') \geq 0$, the solution of equation (9) — if it exists — conserves positivity, $W_0(v, x) \geq 0 \Longrightarrow W(v, x, t) \geq 0$, and L_1-norm,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} W_0(v, x)\,dv\,dx = \int_{\mathbb{R}}\int_{\mathbb{R}} W(v, x, t)\,dv\,dx = 1.$$

In general, the existence of a solution to (9) in L_1 is a difficult problem and can be proved in some special cases, for instance, if $b(v)\frac{\partial^2}{\partial v^2}$ is strongly elliptic and the integral operator is dominated by the second derivative (see, e.g., [5]).

Integrating (9) w.r.t. x, we get a closed equation

$$\frac{\partial}{\partial t} w(v, t) = \mathbf{A}w \tag{10}$$

for the velocity distribution $w(v,t) = \int_{\mathbb{R}} W(v,x,t)dx$, whereas integrating (9) w.r.t. v, we get the unclosed equation (4) for $c(x,t)$. We will try to close this equation in a rigorous way for some suitable operators $\mathbf{A}$, i.e., we will try to write the right-hand side of (4) as a function of $c(x,t)$. Since equation (9) and the expression (2) to calculate $c(x,t)$ are linear, this function is linear, too. Therefore, we will look for an equation for $c(x,t)$ in the form

$$\frac{\partial}{\partial t}c(x,t) = \mathbf{R}\, c(x,t), \quad c(x,0) = c_0(x), \tag{11}$$

where $\mathbf{R}$ is some unknown linear operator. We will call this equation the reduced equation.

The only assumption we demand is

$$W_0(v,x) = w_0(v) \cdot c_0(x). \tag{12}$$

This assumption means that the initial velocity and spatial distributions are independent. This seems to be natural. When looking for a velocity-free description of the problem, we have to assume that the initial spatial distribution $c_0(x)$ is well defined and given. This can be made transparent in the following way. Let us assume that equations (9) and (11) have Green functions ω and h. The solutions of these equations can then be formally written as

$$W(v,x,t) \quad = \quad \int_{\mathbb{R}}\int_{\mathbb{R}} \omega(v,v',x-x',t)W_0(v',x')dv'dx' \tag{13}$$

and

$$c(x,t) \quad = \quad \int_{\mathbb{R}} h(x-x',t)c_0(x')dx'. \tag{14}$$

(Because of the homogeneity of the problem in x, the Green function depends only on the difference $x-x'$.) Integrating (13) w.r.t. v, we have to get a solution of type (14). Assuming (12), we have

$$W(v,x,t) \quad = \quad \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \omega(v,v',x-x',t)w_0(v')dv' \right) c_0(x')dx'$$

and so

$$h(x,t) = \int_{\mathbb{R}}\int_{\mathbb{R}} \omega(v,v',x,t)w_0(v')dv'dv.$$

In the following, we will use Fourier transforms of the solutions and initial data. Let

$$\begin{aligned}
\psi(\eta,t) &= \int_{\mathbb{R}} e^{i\eta x} c(x,t)dx, \\
\psi_0(\eta) &= \int_{\mathbb{R}} e^{i\eta x} c_0(x)dx, \\
\varphi(\mu,\eta,t) &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(\mu v+\eta x)} W(v,x,t)dvdx, \\
\varphi_0(\mu,\eta) &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(\mu v+\eta x)} W_0(v,x)dvdx, \\
e^{\beta(\mu)} &= \int_{\mathbb{R}} e^{i\mu v} w_0(v)dv \quad \Longleftrightarrow \quad \beta(\mu) = \log \int_{\mathbb{R}} e^{i\mu v} w_0(v)dv.
\end{aligned}$$

Note that $\psi(\eta,t) = \varphi(0,\eta,t)$ because of (2).

An operator $\mathbf{D}$ of the form

$$(\mathbf{D}\, f)(x) = \frac{1}{2\pi} \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\eta(x'-x)} \gamma(\eta) f(x')d\eta dx' \tag{15}$$

is called pseudodifferential operator (PDO) with the symbol γ. If γ is given, we can write $\mathbf{D}$ as an integro-differential operator calculating the inverse Fourier transform in a distribution sense. Now we are ready for the following

Theorem 1. *Let $W(v,x,t)$ be the solution of the equation*

$$\frac{\partial}{\partial t} W(v,x,t) = \mathbf{A}W(v,x,t) - v\frac{\partial}{\partial x} W(v,x,t) \tag{16}$$

with initial data

$$\begin{aligned}
&W(v,x,0) = w_0(v)\cdot c_0(x),\ w_0(x) \geq 0,\ c_0(x) \geq 0, \\
&\int_{\mathbb{R}} w_0(v)dv = \int_{\mathbb{R}} c_0(x)dx = 1,
\end{aligned} \tag{17}$$

and $\varphi(\mu,\eta,t)$ its Fourier transform. Then $c(x,t)$ defined by

$$c(x,t) = \int_{\mathbb{R}} W(v,x,t)dv \tag{18}$$

is the solution of the equation

$$\frac{\partial}{\partial t}c(x,t) = \mathbf{R}(t)c(x,t) \tag{19}$$

with initial data

$$c(x,0) = c_0(x), \tag{20}$$

where the (in general time-dependent) operator $\mathbf{R}(t)$ *is a PDO with the symbol*

$$r(\eta,t) \quad = \quad \frac{\partial}{\partial t}\big(\log\varphi(0,\eta,t)\big).$$

P r o o f. Let $\mathbf{C}_a$ be the shift operators $(\mathbf{C}_a f)(x) = f(x+a)$. Because of the spatial homogeneity of the medium, the operator $\mathbf{R}$ has to commute with the shift operators: $\mathbf{R}\mathbf{C}_a = \mathbf{C}_a\mathbf{R}$. The Fourier transform is the operator which diagonalizes $\mathbf{C}_a$. Since $\mathbf{R}$ commutes with $\mathbf{C}_a$, the Fourier transform diagonalizes $\mathbf{R}$, too. That is, operator $\mathbf{R}$ becomes a multiplication operator in Fourier space, and equation (19) is equivalent to equation

$$\frac{\partial}{\partial t}\psi(\eta,t) = r(\eta,t)\psi(\eta,t), \tag{21}$$

where $r(\eta,t)$ is the symbol of $\mathbf{R} = \mathbf{R}(t)$, in general depending on t. Knowing $\psi(\eta,t)$, we get $r(\eta,t)$ from (21) by

$$r(\eta,t) = \frac{\frac{\partial}{\partial t}\psi(\eta,t)}{\psi(\eta,t)} = \frac{\partial}{\partial t}\log\psi(\eta,t) = \frac{\partial}{\partial t}\log\varphi(0,\eta,t). \tag{22}$$

For $t=0$ we get $c(x,0) = \int_{\mathbb{R}} W(v,x,0)dv = c_0(x)\cdot\int_{\mathbb{R}} w_0(v)dv = c_0(x)$. ■

Thus, knowing $\varphi(0,\eta,t)$ from equation (16), we can calculate $r(\eta,t)$, i.e., $\mathbf{R}(t)$. We can say the problem is solved if we know $\mathbf{R}(t)$ explicitly. But this does not mean that each of the calculations denoted by the arrows in the following scheme has to be done explicitly.

$$\mathbf{A}, w_0(v) \implies \varphi(0,\eta,t) \implies r(\eta,t) \implies \mathbf{R}(t).$$

Sometimes it is enough to know the structure of the value. Unfortunately, even this seems to be a difficult problem in general, but there are some special cases for which this can be done. One of them is the important case of general Brownian motion.

3. General Brownian motion

Following A.M. Yaglom [7], we will call the motion of a random particle general Brownian motion, if the forces acting on the particle are the sum of a deterministic linear friction and random forces not depending on the velocity of the particle. In this case equation (9) has the following form:

$$\begin{aligned}\frac{\partial}{\partial t}W(v,x,t) &= \frac{\partial}{\partial v}\big(avW\big)+b\frac{\partial^2}{\partial v^2}W-v\frac{\partial}{\partial x}W\\ &+ \fint_{\mathbb{R}}\big(Q(v-v')W(v',x,t)-Q(v'-v)W(v,x,t)\big)dv', \quad (23)\\ W(v,x,0) &= w_0(v)c_0(x),\ \beta(\mu):=\log\int_{\mathbb{R}}e^{i\mu v}w_0(v)dv. \quad (24)\end{aligned}$$

The classical Fokker–Planck equation (1) is the special case $Q\equiv 0$.

Theorem 2. *Let $W(v,x,t)$ be the solution of equation (23) with initial data (24). Then $c(x,t)$ defined by (18) is the solution to the equation*

$$\begin{aligned}\frac{\partial}{\partial t}c(x,t) &= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}e^{i\eta(x'-x)}\eta e^{-at}\beta'\left(\frac{\eta}{a}\big(1-e^{-at}\big)\right)c(x',t)d\eta dx'\\ &+ \frac{b}{a^2}\big(1-e^{-at}\big)^2\frac{\partial^2}{\partial x^2}c(x,t)\\ &+ \fint_{\mathbb{R}}\frac{a}{1-e^{-at}}\,Q\left(\frac{ax'}{1-e^{-at}}\right)\big(c(x-x',t)-c(x,t)\big)dx' \quad (25)\end{aligned}$$

with initial data $c(x,0)=c_0(x)$.

P r o o f. Transforming equation (23) to Fourier space, we get

$$\frac{\partial}{\partial t}\varphi(\mu,\eta,t)=\alpha(\mu)\varphi(\mu,\eta,t)+(\eta-a\mu)\frac{\partial}{\partial\mu}\varphi(\mu,\eta,t), \quad (26)$$

where

$$\alpha(\mu)=-b\mu^2+\fint_{\mathbb{R}}\big(e^{i\mu v}-1\big)Q(v)dv \quad (27)$$

is the symbol of the random part of operator $\mathbf{A}$. Equation (26) is a first-order PDE and can be solved explicitly. Let

$$h(\mu,\eta,t)=\frac{\eta}{a}\big(1-e^{-at}\big)+\mu e^{-at}\ ,h(0,\eta,t)=\frac{\eta}{a}\big(1-e^{-at}\big).$$

We get

$$\varphi(\mu,\eta,t)=\varphi_0\big(h(\mu,\eta,t),\eta\big)\cdot\exp\left(\int_0^t\alpha\big(h(\mu,\eta,t')\big)dt'\right). \tag{28}$$

Taking into account the special initial value (24), we get

$$\begin{aligned}\varphi(\mu,\eta,t) &= \psi_0(\eta)w_0\big(h(\mu,\eta,t)\big)\cdot\exp\int_0^t\alpha\big(h(\mu,\eta,t')\big)dt',\\ \varphi(0,\eta,t) &= \psi_0(\eta)w_0\big(h(0,\eta,t)\big)\cdot\exp\int_0^t\alpha\big(h(0,\eta,t')\big)dt'.\end{aligned}$$

From (22) it follows

$$\begin{aligned}r(\eta,t) &= \frac{\partial}{\partial t}\big(\log\varphi(0,\eta,t)\big)=\frac{\partial}{\partial t}h(0,\eta,t)\beta'\big(h(0,\eta,t)\big)+\alpha\big(h(0,\eta,t)\big)\\ &= \eta e^{-at}\beta'\big(h(0,\eta,t)\big)+\alpha\big(h(0,\eta,t)\big)\\ &= \eta e^{-at}\beta'\left(\frac{\eta}{a}\big(1-e^{-at}\big)\right)+\alpha\left(\frac{\eta}{a}\big(1-e^{-at}\big)\right).\end{aligned} \tag{29}$$

Thus, the symbol of operator $\mathbf{R}(t)$ as a function of w_0 (via β) and $\mathbf{A}$ (via α) is calculated. Therefore, we have

$$\begin{aligned}&(\mathbf{R}(t)\,f)(x)\\ &=\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\left[e^{i\eta(x'-x)}\eta e^{-at}\beta'\left(\frac{\eta}{a}\big(1-e^{-at}\big)\right)+\alpha\left(\frac{\eta}{a}\big(1-e^{-at}\big)\right)\right]f(x')d\eta dx'.\end{aligned}$$

Using (27), we get (25). ■

Let

$$\begin{aligned}r_0(\eta,t) &= \eta e^{-at}\beta'\left(\frac{\eta}{a}\big(1-e^{-at}\big)\right),\\ r_{\mathbf{A}}(\eta,t) &= \alpha\left(\frac{\eta}{a}\big(1-e^{-at}\big)\right).\end{aligned}$$

Then, the operator $\mathbf{R}(t)$ can be written as a sum of two parts

$$\mathbf{R}(t)=\mathbf{R}_0(t)+\mathbf{R}_{\mathbf{A}}(t)\ ,$$

where $\mathbf{R}_0$ is a PDO with the symbol $r_0(\eta,t)$ and depends only on the initial velocity distribution $w_0(v)$, and $\mathbf{R}_{\mathbf{A}}$ is a PDO with the symbol $r_{\mathbf{A}}(\eta,t)$ depending only on $\mathbf{A}$, i.e., on the interaction of the particle with the medium.

Considering equation (25) instead of equation (23), we assume that the changing of the velocity is not interesting for us. This can be so, because the velocity density does not change in time and is known from the beginning. This means the initial velocity density $w_0(v)$ is the stationary one $w_\infty(v)$, i.e., a solution to the stationary variant of equation (10): $\mathbf{A}w_\infty = 0$. We will assume that the stationary solution is unique. In this case, the operator of the reduced equation $\mathbf{R}(t)$ is determined by the operator $\mathbf{A}$ and the following theorem holds.

Theorem 3. *Let $W(v,x,t)$ be the solution of equation (23) with initial data*

$$W(v,x,0) = w_\infty(v) \cdot c_0(x) \ , \tag{30}$$

where $w_\infty(v)$ is the unique solution (with $\int\limits_{\mathbb{R}} w_\infty(v)dv = 1$) to the equation

$$\begin{aligned} 0 &= \frac{\partial}{\partial v}\big(avw_\infty(v)\big) + b\frac{\partial^2}{\partial v^2}w_\infty(v) \\ &+ \fint_{\mathbb{R}} \big(Q(v-v')w_\infty(v') - Q(v'-v)w_\infty(v)\big)dv'. \end{aligned} \tag{31}$$

Then $c(x,t)$ defined by (18) is the solution to the equation

$$\begin{aligned} \frac{\partial}{\partial t}c(x,t) &= \frac{b}{a^2}\big(1-e^{-at}\big)\frac{\partial^2}{\partial x^2}c(x,t) + \\ &+ \fint_{\mathbb{R}} \frac{a}{(1-e^{-at})^2}\, Q\left(\frac{ax'}{1-e^{-at}}\right)\big(c(x-x',t) - c(x,t)\big)dx' \end{aligned} \tag{32}$$

with initial data $c(x,0) = c_0(x)$.

P r o o f. We will use formula (29) and therefore have to calculate $\beta'(\mu)$. Because of $w_\infty(v) = \int_{\mathbb{R}} W(v,x,t)dx$ we have $\varphi_0(\mu,0) = \varphi(\mu,0,\infty)$, where $\varphi(\mu,0,\infty)$ is the solution of the stationary variant of equation (26) for $\eta = 0$:

$$0 = \alpha(\mu)\varphi(\mu,0,\infty) - a\mu\frac{\partial}{\partial\mu}\varphi(\mu,0,\infty). \tag{33}$$

Using this, we get

$$\beta'(\mu) = \frac{\partial}{\partial\mu}\log\varphi_0(\mu,\eta)\bigg|_{\eta=0} = \frac{\frac{\partial}{\partial\mu}\varphi_0(\mu,0)}{\varphi_0(\mu,0)} = \frac{\frac{\partial}{\partial\mu}\varphi(\mu,0,\infty)}{\varphi(\mu,0,\infty)} = \frac{\alpha(\mu)}{a\mu}\,.$$

Now, from (29) it follows

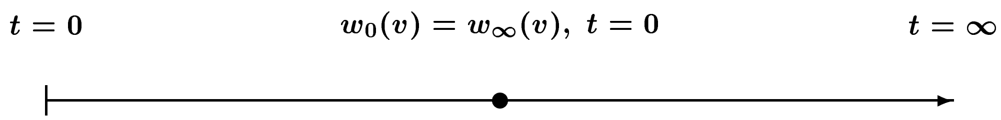
$$
\begin{aligned}
r(\eta,t) &= \eta e^{-at}\frac{\alpha\left(\frac{\eta}{a}\left(1-e^{-at}\right)\right)}{a\frac{\eta}{a}\left(1-e^{-at}\right)}+\alpha\left(\frac{\eta}{a}\left(1-e^{-at}\right)\right)\\
&= \frac{1}{1-e^{-at}}\alpha\left(\frac{\eta}{a}\left(1-e^{-at}\right)\right). \qquad (34)
\end{aligned}
$$

Thus, the symbol of operator $\mathbf{R}(t)$ as a function of $\mathbf{A}$ is calculated. Using (15) and (27), we get (32). ■

4. Time dependence of the operator $\mathbf{R}(t)$

It seems to be strange that the operator $\mathbf{R}(t)$ depends on time explicitly, whereas the original physical problem is time-independent. The reason is that the original problem has various time scales, where the solution behaves differently. For short times, the initial velocity distribution $w_0(v)$ is the dominating influence on the particle. It moves like a free particle with given velocity. From (8) we see that the velocity changes independently of x. Thus for intermediate times, the probability density of the velocity relaxes and becomes stationary. This stationary density $w_\infty(v)$ is the solution of the stationary equation related to equation (10): $\mathbf{A}w_\infty = 0$. This middle time behavior can be investigated setting $w_0(v) = w_\infty(v)$ and then $t = 0$, assuming the velocity was relaxed from the beginning. For long times, the particle moves with equilibrium velocity.

Let us consider three cases:



4.1. The asymptotic behavior for $t \longrightarrow 0$

Setting $t = 0$ in (29), we get, because of $e^{-at}\big|_{t=0} = 1$, that

$$r(\eta,0) = \eta\beta'(0) + \alpha(0).$$

From (27) we conclude $\alpha(0) = 0$ and from the definition (24) of β

$$\beta'(\mu) = \frac{i\int_{\mathbb{R}} e^{i\mu v} v w_0(v)dv}{\int_{\mathbb{R}} e^{i\mu v} w_0(v)dv}.$$

Therefore,

$$\beta'(0) = \frac{i\int_{\mathbb{R}} v w_0(v)dv}{\int_{\mathbb{R}} w_0(v)dv} = i\bar{v},$$

where $\bar{v}$ is the average velocity at $t = 0$. Finally, we have $r(\eta, 0) = i\eta\bar{v}$ and so

$$\frac{\partial}{\partial t}c(x,t) \quad = \quad \mathbf{R}(0)c(x,t) = -\bar{v}\frac{\partial}{\partial x}c(x,t).$$

This equation shows that for $t \longrightarrow 0$ the particle moves like a free particle with average velocity $\bar{v}$. The equation is first-order hyperbolic.

4.2. The asymptotic behavior for $t \longrightarrow \infty$

For $t \longrightarrow \infty$, we have $\mathbf{R}_0(t) \longrightarrow 0$ (the influence of the initial velocity vanishes) and it follows

$$r(\eta, \infty) = \alpha\left(\frac{\eta}{a}\right),$$

and so we get

$$\begin{aligned}\frac{\partial}{\partial t}c(x,t) &= \mathbf{R}(\infty)c(x,t) \\ &= \frac{b}{a^2}\frac{\partial^2}{\partial x^2}c(x,t) + \fint_{\mathbb{R}} aQ(ax')\big(c(x-x',t) - c(x,t)\big)\,dx'.\end{aligned}$$

For $Q = 0$, this is the diffusion equation (a parabolic one).

4.3. The $w_0(v) = w_\infty(v)$, $t \longrightarrow 0$ case

Expanding $r(\eta, t)$ from (34) in Taylor series for $t = 0$ and taking into account $\alpha(0) = 0$, we get formally

$$r(\eta,t) = \frac{\eta}{a}\alpha'(0) + \eta^2 t\frac{1}{2a}\alpha''(0).$$

Setting

$$q_1 = \fint_{\mathbb{R}} vQ(v)dv, \quad q_2 = \int_{\mathbb{R}} v^2 Q(v)dv,$$

we have

$$r(\eta,t) = \frac{q_1}{a}i\eta - \eta^2 t\frac{b+q_2}{2a}.$$

This leads to the reduced equation

$$\frac{\partial}{\partial t}c(x,t) = -\frac{q_1}{a}\frac{\partial}{\partial x}c(x,t) + \frac{b+q_2}{2a}t\frac{\partial^2}{\partial x^2}c(x,t).$$

The special case of symmetric $Q(v) = Q(-v)$ leads to $q_1 = 0$ and to the reduced equation

$$\frac{\partial}{\partial t}c(x,t) = \frac{b+q_2}{2a}t\frac{\partial^2}{\partial x^2}c(x,t). \tag{35}$$

From this we get $\frac{\partial}{\partial t}c(x,t)\big|_{t=0} = 0$ and therefore for $t \to 0$

$$\frac{\partial^2}{\partial t^2}c(x,t) = \lim_{t\to 0}\frac{1}{t}\left(\frac{\partial}{\partial t}c(x,t) - \frac{\partial}{\partial t}c(x,t)\bigg|_{t=0}\right) = \lim_{t\to 0}\frac{1}{t}\frac{\partial}{\partial t}c(x,t).$$

This shows that (35) is for $t \to 0$ a second-order hyperbolic equation:

$$\frac{1}{t}\frac{\partial}{\partial t}c(x,t) = \frac{\partial^2}{\partial t^2}c(x,t) = \frac{b+q_2}{2a}\frac{\partial^2}{\partial x^2}c(x,t).$$

Equations (25) or (32) arise in a natural way as a strong derivation from a phase-space equation and in some sense interpolate between parabolic and hyperbolic equations. This can be an alternative to fractional time derivatives considered, for instance, in [1].

5. Some examples

5.1. The classical Brownian motion

For $Q \equiv 0$, equation (23) is the classical Fokker–Planck equation (1). We have $\alpha(\mu) = -b\mu^2$ and the Fourier transform of the stationary velocity distribution is (see (33))

$$\varphi(\mu, 0, \infty) = e^{-\frac{b}{2a}\mu^2}$$

or

$$w_\infty(v) = \frac{1}{\sqrt{2\pi\frac{b}{a}}}e^{-\frac{a}{2b}v^2}.$$

This is the well-known Maxwell distribution. If we take $w_0(v) = w_\infty(v)$, we get from (32) the equation

$$\frac{\partial}{\partial t}c(x,t) = \frac{b}{a^2}\left(1 - e^{-at}\right)\frac{\partial^2}{\partial x^2}c(x,t), \tag{36}$$

which is indeed similar to equation (7) and tends to it for $t \longrightarrow \infty$. So we can state: the classical diffusion equation describes the Brownian motion of a particle if the initial velocity distribution is Maxwellian and is independent of the initial spatial distribution (assumption (24)) and only for long times, i.e., near equilibrium.

From (34), we can conclude that this is the only case where we get a parabolic PDE for $c(x,t)$.

We have the following limit cases:

$$\begin{aligned} \frac{\partial^2}{\partial t^2}c(x,t) &\sim \frac{1}{t}\frac{\partial}{\partial t}c(x,t) = \frac{b}{a}\frac{\partial^2}{\partial x^2}c(x,t), \quad t \longrightarrow 0, \\ \frac{\partial}{\partial t}c(x,t) &= \frac{b}{a^2}\frac{\partial^2}{\partial x^2}c(x,t), \quad t \longrightarrow \infty. \end{aligned}$$

5.2. The Cauchy case

An interesting question is, in which case, we get an equation for $c(x,t)$ with a time-independent operator $\mathbf{R}(t) = \mathbf{R}$. In the general case (29) this is only possible for the noninteresting case $w_0(v) = \delta(v)$ and $\alpha(\mu) = 0$. In the $w_0(v) = w_\infty(v)$ case, this is possible if $b = 0$ and

$$Q(v) = \frac{d}{\pi v^2}.$$

Then we have $\alpha(\mu) = -d|\mu|$ and

$$\varphi(\mu, 0, \infty) = \exp\left(\int_0^\mu \frac{\alpha(\mu')}{a\mu'}d\mu'\right) = \exp\left(-\frac{d}{a}\int_0^\mu \operatorname{sign}\mu' d\mu'\right) = e^{-\frac{d}{a}|\mu|},$$

or, after inverse Fourier transform,

$$w_\infty(v) = \frac{1}{\pi}\frac{ad}{d^2 + a^2v^2}.$$

This is the well-known Cauchy distribution. To get the reduced equation, we have from (34)

$$r(\eta,t) = \frac{1}{1-e^{-at}}\alpha\left(\frac{\eta}{a}\big(1-e^{-at}\big)\right) = \frac{1}{1-e^{-at}}(-d)\left|\frac{\eta}{a}\big(1-e^{-at}\big)\right| = -\frac{d}{a}|\eta|.$$

So we can state: Let $W(v,x,t)$ be the solution of the equation

$$\frac{\partial}{\partial t}W(v,x,t) = \frac{\partial}{\partial v}\big(avW\big) + \frac{d}{\pi}\fint_{\mathbb{R}} \frac{W(v',x,t) - W(v,x,t)}{|v-v'|^2}dv' - v\frac{\partial}{\partial x}W$$

with initial value

$$W(v,x,0) = \frac{d}{\pi a v^2}\cdot c_0(x).$$

Then, $c(x,t)$ is the solution to the equation

$$\frac{\partial}{\partial t}c(x,t) = \frac{d}{a}\fint_{\mathbb{R}} \big(c(x-x',t) - c(x,t)\big)\frac{dx'}{|x'|^2}$$

with initial value $c(x,0) = c_0(x)$.

5.3. A free particle

An important question in applications is: what can we say about the evolution of $c(x,t)$, if we know that the velocity distribution is given and does not change in time? This means that $w_0(v) = w_\infty(v)$ and $\mathbf{A}w_\infty = 0$. Often we know how the velocity is distributed (e.g., Maxwell distributed), but we do not know the real interaction in the medium. This means that we know $w_\infty(v)$, but we do not know $\mathbf{A}$. Does $c(x,t)$ depend on $\mathbf{A}$?

Let us assume that $w_0(v) = w_\infty(v) = \delta(v - v_0)$, i.e., the particle moves all the time with determined velocity v_0, then we will expect that

$$c(x,t) = c_0(x - v_0 t).$$

If $w_0(v) = w_\infty(v)$ is arbitrary, in heuristic derivations sometimes the seemingly obvious assumption

$$c(x,t) = \int_{\mathbb{R}} w_\infty(v) c_0(x - vt)dv \tag{37}$$

is used. We will show that this is wrong in general. From (37) we get for $t > 0$

$$c(x,t) = \int_{\mathbb{R}} w_\infty(v)c_0(x-vt)dv = \int_{\mathbb{R}} \frac{1}{t}w_\infty\left(\frac{x-x'}{t}\right)c_0(x')dx'. \tag{38}$$

Thus, $\frac{1}{t}w_\infty\left(\frac{x-x'}{t}\right)$ is the Green function of some operator $\mathbf{R}(t)$. Taking as an example the Maxwell distribution

$$w_0(v) = \frac{1}{\sqrt{2\pi\sigma}}e^{-\frac{v^2}{2\sigma}}, \tag{39}$$

a simple calculation shows that the corresponding reduced equation is

$$c_t(x,t) = \sigma t c_{xx}(x,t). \tag{40}$$

On the other hand, we know that, for the classical Fokker–Planck equation, the Maxwell distribution is the equilibrium distribution (with $\sigma = \frac{b}{a}$), but the corresponding reduced equation is (36), which is similar to (40) only for short times.

Equation (38) — or (40) taken for special $w_\infty(v)$ — is the reduced equation for a free particle, i.e., for the case without interaction, where $\mathbf{A} = \mathbf{O}$ is the zero operator. In this case, equation (23) reads as

$$\frac{\partial}{\partial t} W(v,x,t) = -v \frac{\partial}{\partial x} W(v,x,t). \tag{41}$$

This is a model for the motion of a particle (or a swarm of noninteracting particles) with given initial velocity distribution $w_0(v) = w_\infty(v)$ (obviously every distribution $w_\infty(v)$ is an equilibrium distribution satisfying $\mathbf{O}w_\infty = 0$). Simple calculation shows that the solution of (41) is $W(v,x,t) = W_0(v, x - vt) = w_\infty(v)c_0(x - vt)$, and so we get (38). This shows that, making assumption (37), we assume no interaction between the particle and the medium.

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