Matematicheskaya fizika, analiz, geometriya
2005, v. 12, No. 2, p. 218–229

On the separated maximum modulus points of meromorphic functions

E. Ciechanowicz

Szczecin University, Institute of Mathematics
15 Wielkopolska Str., Szczecin, 70451, Poland
E-mail:ewa.ciechanowicz@poczta.onet.pl

I.I. Marchenko

Department of Mechanics and Mathematics, V.N. Karazin Kharkov National University
4 Svobody Sq., Kharkov, 61077, Ukraine
Szczecin University, Institute of Mathematics
15 Wielkopolska Str., Szczecin, 70451, Poland
E-mail:iim@ukr.net
marchenko@wmf.univ.szczecin.pl

Received April 15, 2004
Communicated by I.V. Ostrovskii

We consider the relationship between the number of separated maximum modulus points and the Eremenko's value $b(\infty, f)$ for meromorphic functions.

Let $\nu(r,g)$ denote the number of maximum modulus points of an entire function $g(z)$ on the circle $|z|=r$. In 1964 P. Erdös set up the question whether it is possible to find an entire function $g(z)\neq cz^m$ with $\nu(r,g)$ unbounded. In 1968 F. Herzog and G. Piranian [10] gave a positive answer to this question. They constructed an entire function $g(z)$ with $\nu(r,g)\to\infty$ for $r\to\infty$.

In this paper we present an upper estimate of the number of separated maximum modulus points for meromorphic functions. We shall use the standard notations of value distribution theory: $m(r,a,f)$, $N(r,a,f)$ and $T(r,f)$ [8]. Let $f(z)$ be a meromorphic function.

Let's set $\mathcal{L}(r,\infty,f)=\max\limits_{|z|=r}\log^+|f(z)|$, $\mathcal{L}(r,a,f)=\mathcal{L}(r,\infty,\frac{1}{f-a})$. The quantity

$$\beta(a,f)=\liminf_{r\to\infty}\frac{\mathcal{L}(r,a,f)}{T(r,f)}$$

Mathematics Subject Classification 2000: 30D35 (primary); 30D30 (secondary).
Key words and phrases: meromorpic function, subharmonic function, maximum modulus points.
This research was partly supported by the grant INTAS-99-0089.

is called *Petrenko's magnitude of deviation of meromorphic function $f(z)$ at point a.* V.P. Petrenko in [13] obtained a sharp upper estimate of the magnitude of deviation of meromorphic functions of finite lower order $\lambda = \liminf_{r\to\infty}\frac{\ln T(r,f)}{\ln r}$.

Theorem A. *If $f(z)$ is a meromorphic function of finite lower order λ, then for each $a \in \overline{\mathbb{C}}$*

$$\beta(a,f) \le \begin{cases} \frac{\pi\lambda}{\sin \pi\lambda} & \text{if} \quad \lambda \le 0.5, \\ \pi\lambda & \text{if} \quad \lambda > 0.5. \end{cases}$$

We now introduce the quantities which count the number of separated maximum modulus points of a meromorphic function $f(z)$ on the circle $|z| = r$. For $0 < \eta \le 1$ and $r > 0$ we denote by $p_\eta(r,\infty,f)$ the number of component intervals of the set

$$\{\theta : \ln|f(re^{i\theta})| > (1-\eta)T(r,f)\}$$

possessing at least one maximum modulus point of the meromorphic function $f(z)$. Moreover, we set $p_\eta(\infty,f) = \liminf_{r\to\infty} p_\eta(r,\infty,f)$ and $p(\infty,f) = \sup_{\{\eta\}} p_\eta(\infty,f)$.

In [3] the authors obtained the following estimate of the value $p(\infty,f)$ through Petrenko's magnitude of deviation $\beta(\infty,f)$.

Theorem B. *For meromorphic functions $f(z)$ of finite lower order λ the following inequality is true:*

$$p(\infty,f) \le \max\left(\left[2\frac{\pi\lambda}{\beta(\infty,f)}\right],1\right),$$

where $[x]$ means the entire part of the number x.

For entire functions $\beta(\infty,g) \ge 1$, which leads us to the following conclusion.

Corollary B. *For entire functions $g(z)$ of finite lower order λ we have*

$$p(\infty,g) \le \max\left([2\pi\lambda],1\right).$$

In case of meromorphic functions of infinite lower order the quantity $\beta(a,f)$ may be infinite, so we apply the following result of Bergweiler and Bock [2].

Theorem C. *If $f(z)$ is a meromorphic function of infinite lower order, then*

$$\liminf_{r\to\infty}\frac{\mathcal{L}(r,\infty,f)}{rT'_-(r,f)} \le \pi,$$

where $T'_-(r,f)$ is the left derivative of Nevanlinna's characteristic function.

We have $rT'_{-}(r,f) = A(r,f) + O(1)$, where $A(r,f)$ means the spherical area covered by the image of the disc $\{z : |z| \le r\}$ under $f(z)$, divided by the area of the Riemann's sphere. In connection with this equality and the above theorem A. Eremenko introduced the quantity

$$b(a,f) = \liminf_{r\to\infty} \frac{\mathcal{L}(r,a,f)}{A(r,f)}.$$

In [5] he proved the following estimate for $b(a,f)$.

Theorem D. *For a meromorphic function $f(z)$ of lower order λ, $0 < \lambda \le \infty$, and for $a \in \overline{\mathbb{C}}$ we have*

$$b(a,f) \le \begin{cases} \pi & \text{if } \ \frac{1}{2} \le \lambda \le \infty, \\ \frac{\pi}{\sin \pi\lambda} & \text{if } \ 0 < \lambda < \frac{1}{2}. \end{cases}$$

In case of $\eta = 1$ one of the authors in [12] obtained the upper estimate of $p_1(\infty,f)$ through $b(\infty,f)$. Our main result is the upper estimate of $p(\infty,f)$ through $b(\infty,f)$ for meromorphic functions.

Theorem 1. *For a meromorphic function $f(z)$ of lower order λ, where $0 < \lambda \le \infty$, and for $0 < \eta \le 1$ we have*

$$p_\eta(\infty,f) \le \max\left\{1, \left[(2-\eta)\frac{\pi}{b(\infty,f)}\right]\right\}.$$

Corollary 1. *For a meromorphic function of lower order λ, $0 < \lambda \le \infty$ we have*

$$p(\infty,f) \le \max\left\{1, \left[2\frac{\pi}{b(\infty,f)}\right]\right\}.$$

1. Auxiliary results

For $0 < \eta \le 1$ let's consider the function

$$u_\eta(z) = \max(\log|f(z)|, (1-\eta)T(|z|,f)),$$

where $f(z)$ is a meromorphic function in $\mathbb{C}$.

Lemma 1. *The function $u_\eta(z)$ is a δ-subharmonic function in $\mathbb{C}$.*

P r o o f. Let $g_1(z)$ and $g_2(z)$ be entire functions without common zeros such that $f(z) = \frac{g_1(z)}{g_2(z)}$. Then we can write

$$u_\eta(z) = \max(\log|g_1(z)| - \log|g_2(z)|, (1-\eta)T(|z|,f))$$

$$= \max(\log|g_1(z)|, (1-\eta)T(|z|, f) + \log|g_2(z)|) - \log|g_2(z)|.$$

The characteristic function $T(r, f)$ is a nondecreasing and convex function of $\log r$ for $r > 0$, hence the function $T(|z|, f)$ is a subharmonic function in $\mathbb{C}$ [14]. Therefore $u_\eta(z)$ is a difference of two subharmonic functions: $U_1(z) = \max(\log|g_1(z)|, (1-\eta)T(|z|, f) + \log|g_2(z)|)$ and $U_2(z) = \log|g_2(z)|$. This completes the proof of Lemma 1.

For a complex number $z = re^{i\theta}$ let's put [1]

$$m^*(r, \theta, u_\eta) = \sup_{|E|=2\theta} \frac{1}{2\pi} \int\limits_E u_\eta(re^{i\varphi}) d\varphi,$$

$$T^*(r, \theta, u_\eta) = T^*(re^{i\theta}) = m^*(r, \theta, u_\eta) + N(r, \infty, f),$$

where $r \in (0, \infty)$, $\theta \in [0, \pi]$, $|E|$ is the Lebesgue's measure of the set E and $N(r, \infty, f)$ is the Nevanlinna's counting function. Let's put $\tilde{u}_\eta(z)$ for the circular symmetrization of the function $u_\eta(z)$ [9]. The function $\tilde{u}_\eta(re^{i\varphi})$ is nonnegative and nonincreasing on the interval $[0, \pi]$, even in φ and for each fixed r equimeasurable with $u_\eta(re^{i\varphi})$. Moreover, it satisfies the relations:

$\tilde{u}_\eta(r) = \max(\log \max\limits_{|z|=r} |f(z)|, (1-\eta)T(r, f))$,

$\tilde{u}_\eta(re^{i\pi}) = \tilde{u}_\eta(-r) = \max(\log \min\limits_{|z|=r} |f(z)|, (1-\eta)T(r, f))$,

$m^*(r, \theta, u_\eta) = \sup\limits_{|E|=2\theta} \frac{1}{2\pi} \int_E u_\eta(re^{i\varphi})\, d\varphi = \frac{1}{\pi} \int_0^\theta \tilde{u}_\eta(re^{i\varphi}) d\varphi$.

From Baernstein's theorem [1] the function $T^*(r, \theta, u_\eta)$ is subharmonic in

$$D = \{re^{i\theta} : 0 < r < \infty, 0 < \theta < \pi\},$$

continuous in $D \cup (-\infty, 0) \cup (0, +\infty)$ and logarithmically convex in $r > 0$ for each fixed $\theta \in [0, \pi]$. Furthermore:

$T^*(r, 0, u_\eta) = N(r, \infty, f)$,

$T^*(r, \pi, u_\eta) \le (2-\eta)T(r, f)$,

$\frac{\partial}{\partial\theta} T^*(r, \theta, u_\eta) = \frac{\tilde{u}_\eta(re^{i\theta})}{\pi}$ for $0 < \theta < \pi$,

where $T(r, f)$ is the Nevanlinna's characteristic function of $f(z)$.

Let $\alpha(r)$ be a real-valued function of a real variable r and

$$L\alpha(r) = \liminf_{h\to 0} \frac{\alpha(re^h) + \alpha(re^{-h}) - 2\alpha(r)}{h^2}.$$

When $\alpha(r)$ is twice differentiable in r, then

$$L\alpha(r) = r\frac{d}{dr} r \frac{d}{dr} \alpha(r).$$

In [3] the authors obtained the following result.

Lemma 2. *For all $0 < \eta \le 1$ and for almost all $\theta \in [0, \pi]$ and for all $r > 0$ such that on the set $\{z : |z| = r\}$ the meromorphic function $f(z)$ has neither zeros nor poles we have*

$$LT^*(r, \theta, u_\eta) \ge -\frac{p_\eta^2(r, \infty, f)}{\pi} \frac{\partial \tilde{u}_\eta(re^{i\theta})}{\partial \theta}.$$

W. Bergweiler and H. Bock in [2] introduced a generalization of Polya peaks [4] to functions of infinite lower order. Let's remind the basic facts of this construction.

For all sequences $M_j \to \infty$, $\varepsilon_j \to 0$ there exist sequences $\rho_j \to \infty$ and $\mu_j \to \infty$ such that, for all r's fulfilling the inequality $|\log(\frac{r}{\rho_j})| \le \frac{M_j}{\mu_j}$, we have

$$T(r, f) \le (1 + \varepsilon_j) \left(\frac{r}{\rho_j}\right)^{\mu_j} T(\rho_j, f). \tag{1}$$

We can choose the sequences μ_j and M_j such that

$$\mu_j = o(\log^{\frac{3}{2}} T(\rho_j, f)), \quad M_j = o(\log T(\rho_j, f)), \quad j \to \infty.$$

Let's put

$$P_j = \rho_j e^{-\frac{M_j}{\mu_j}}, \quad Q_j = \rho_j e^{\frac{M_j}{\mu_j}}.$$

Then the inequality (1) is true for all $r \in [P_j, Q_j]$. We shall assume that $M_j > 1$.

Let's consider the sets

$$A_j = \left\{ r \in [\rho_j, Q_j] : T(r, f) \le \frac{1}{\sqrt{\mu_j}} \left(\frac{r}{\rho_j}\right)^{\mu_j} T(\rho_j, f) \right\},$$

$$B_j = \left\{ r \in [P_j, \rho_j] : T(r, f) \le \frac{1}{\sqrt{\mu_j}} \left(\frac{r}{\rho_j}\right)^{\mu_j} T(\rho_j, f) \right\}.$$

Let's put

$$R_j = \begin{cases} \min A_j, & \text{if } A_j \ne \emptyset, \\ Q_j, & \text{if } A_j = \emptyset, \end{cases} \quad t_j = \begin{cases} \max B_j, & \text{if } B_j \ne \emptyset, \\ P_j, & \text{if } B_j = \emptyset, \end{cases} \tag{2}$$

$$S_j = e^{-\frac{1}{\mu_j}} R_j, \quad T_j = e^{-\frac{2}{\mu_j}} R_j.$$

Then

$$t_j < \rho_j < T_j < S_j < R_j.$$

In [2] it is shown that

$$\frac{T(R_j,f)}{R_j^{\mu_j}}+\frac{T(t_j,f)}{t_j^{\mu_j}}=o\left(\mu_j\int\limits_{t_j}^{T_j}\frac{T(r,f)}{r^{\mu_j+1}}dr\right),\quad j\to\infty. \tag{3}$$

Apart from that, it follows from the inequality (19) in [2] that

$$T(\rho_j,f)\le T^{\frac{3}{2}}(t_j,f),\quad j\to\infty.$$

In order to prove our main results we shall need several additional lemmas.

Lemma A [13]. *Let $f(z)$ be a meromorphic function of finite lower order λ. Then there exist sequences S_k, R_k tending to infinity such that $\lim\limits_{k\to\infty}\frac{S_k}{R_k}=0$ and for each $\varepsilon>0$, for all $k\ge k_0(\varepsilon)$ we have*

$$\frac{T(2R_k,f)}{R_k^{\lambda}}+\frac{T(2S_k,f)}{S_k^{\lambda}}<\varepsilon\int\limits_{2S_k}^{R_k}\frac{T(r,f)}{r^{\lambda+1}}dr.$$

Let's define new quantities

$$h(r,\lambda,p):=\mathcal{L}(r,\infty,f)\cos\frac{\lambda\psi}{p}-\frac{\pi\lambda}{p}T^*(r,\alpha,u_\eta)\sin\frac{\lambda(\alpha+\psi)}{p}$$

$$+\frac{\pi\lambda}{p}N(r,\infty,f)\sin\frac{\lambda\psi}{p}-\tilde{u}_\eta(r,\alpha)\cos\frac{\lambda(\alpha+\psi)}{p},$$

$$h_\eta(r,\lambda):=h(r,\lambda,p_\eta(\infty,f)).$$

The inequality, that we present as a lemma below, was proved in [3].

Lemma B. *Let $f(z)$ be a meromorphic function of finite lower order λ. Then for $0<\alpha\le\min(\pi,\frac{\pi p_\eta(\infty,f)}{2\lambda})$ and $-\frac{\pi p_\eta(\infty,f)}{2\lambda}\le\psi\le\frac{\pi p_\eta(\infty,f)}{2\lambda}-\alpha$, we have the asymptotic inequality*

$$\int\limits_{2S_k}^{R_k}\frac{h_\eta(r,\lambda)}{r^{\lambda+1}}dr<\varepsilon\int\limits_{2S_k}^{R_k}\frac{T(r,f)}{r^{\lambda+1}}dr,\quad k\to\infty,$$

where S_k and R_k are the sequences from lemma A.

The following lemma is an analogue of lemma B for meromorphic functions of infinite lower order.

Lemma 3. *Let $f(z)$ be a meromorphic function of infinite lower order. Then for such numbers p that $1 \le p \le \max\{1, p_\eta(\infty, f)\}$, $0 < \alpha \le \min\{\pi, \frac{\pi p}{2\mu_j}\}$, $-\frac{\pi p}{2\mu_j} \le \psi \le \frac{\pi p}{2\mu_j} - \alpha$ we have*

$$\int_{t_j}^{T_j} \frac{h(r, \mu_j, p)}{r^{\mu_j+1}} dr < \varepsilon \mu_j \int_{t_j}^{T_j} \frac{T(r, f)}{r^{\mu_j+1}} dr \ , \quad j \to \infty, \tag{4}$$

where T_j and t_j were defined in (2).

P r o o f. Let's put [11, 6, 7]

$$\sigma(r) = \int_0^\alpha T^*(r, \theta, u_\eta) \cos \frac{\mu_j(\theta + \psi)}{p} d\theta.$$

Applying Lemma 2, the fact that $LT^*(r, \theta, u_\eta) \ge 0$ and Fatou's lemma, we obtain that for almost all $r \ge r_0$

$$r \frac{d}{dr} r\sigma'_-(r) \ge - \int_0^\alpha \frac{p_\eta^2(r\infty, f)}{\pi} \frac{\partial \tilde{u}_\eta(r, \theta)}{\partial \theta} \cos \frac{\mu_j(\theta + \psi)}{p} d\theta.$$

After applying integration by parts to the right side of the above inequality we have

$$r \frac{d}{dr} r\sigma'_-(r) \ge p^2 h(r, \mu_j, p) + \mu_j^2 \sigma(r).$$

We divide this inequality by r^{μ_j+1} and integrate it over an interval $[t_j, T_j]$.

$$\int_{t_j}^{T_j} \frac{1}{r^{\mu_j}} \frac{d}{dr} r\sigma'_-(r) dr \ge p^2 \int_{t_j}^{T_j} \frac{h(r, \mu_j, p)}{r^{\mu_j+1}} dr + \mu_j^2 \int_{t_j}^{T_j} \frac{\sigma(r)}{r^{\mu_j+1}} dr. \tag{5}$$

Integrating by parts the left side of (5) and applying the monotonicity of $r\sigma'_-(r)$, we obtain

$$p^2 \int_{t_j}^{T_j} \frac{h(r, \mu_j, p)}{r^{\mu_j+1}} dr \le \left(\frac{\sigma'_-(r)}{r^{\mu_j-1}} + \mu_j \frac{\sigma(r)}{r^{\mu_j}} \right) \Bigg|_{t_j}^{T_j} . \tag{6}$$

The definition of $\sigma(r)$ implies that

$$\sigma(r) \le \frac{(2 - \eta)p}{\mu_j} T(r, f).$$

Since $r\sigma'_-(r)$ is monotonically increasing on $[t_j, T_j]$, we have

$$\sigma(S_j) - \sigma(T_j) = \int\limits_{T_j}^{S_j} \sigma'_-(r)dr \geq T_j\sigma'_-(T_j)\log\frac{S_j}{T_j} = \frac{1}{\mu_j}T_j\sigma'_-(T_j).$$

Hence

$$T_j\sigma'_-(T_j) \leq \mu_j\sigma(S_j) \leq (2-\eta)pT(S_j, f).$$

Apart from that, for all $r \geq 1$ we have $r\sigma'_-(r) \geq \sigma'_-(1)$. Now, applying (6) and (3), we obtain

$$p^2\int\limits_{t_j}^{T_j}\frac{h(r,\mu_j,p)}{r^{\mu_j+1}}dr \leq \frac{2(2-\eta)pT(S_j,f)}{T_j^{\mu_j}} - \frac{\sigma'_-(1)}{t_j^{\mu_j}}$$

$$< \frac{2(2-\eta)pe^2T(R_j,f)}{R_j^{\mu_j}} + \frac{T(t_j,f)}{t_j^{\mu_j}} < \varepsilon\mu_j\int\limits_{t_j}^{T_j}\frac{T(r,f)}{r^{\mu_j+1}}dr, \quad j\to\infty.$$

This completes the proof of Lemma 3.

2. Main result

In this section we present the proof of Theorem 1.

If $b(\infty, f) = 0$ or $p_\eta(\infty, f) = 0$ then the statement is obviously true. Therefore let's take $b(\infty, f) > 0$. Then also $p(\infty, f) > 0$.

First we shall prove the statement for meromorphic functions of finite lower order λ. We consider the case when $p(\infty, f) < \infty$. For $\lambda > 0$ we have

$$\int\limits_{2S_k}^{R_k}\frac{T(r,f)}{r^{\lambda+1}}dr = \frac{T(2S_k,f)}{\lambda 2^\lambda S_k^\lambda} - \frac{T(R_k,f)}{\lambda R_k^\lambda} + \frac{1}{\lambda}\int\limits_{2S_k}^{R_k}\frac{rT'_-(r,f)}{r^{\lambda+1}}dr.$$

Thus, applying lemma A, we obtain

$$\int\limits_{2S_k}^{R_k}\frac{T(r,f)}{r^{\lambda+1}}dr < \frac{1+\varepsilon}{\lambda}\int\limits_{2S_k}^{R_k}\frac{A(r,f)}{r^{\lambda+1}}dr, \quad k\to\infty. \tag{7}$$

Let's first assume that $\frac{\lambda}{p_\eta(\infty,f)} > \frac{1}{2}$. Then $\frac{\pi p_\eta(\infty,f)}{2\lambda} < \pi$. In Lemma B we put $\alpha = \frac{\pi p_\eta(\infty,f)}{2\lambda}$, $\psi = 0$. Then, as $k\to\infty$

$$\int\limits_{2S_k}^{R_k}\frac{\mathcal{L}(r,\infty,f)}{r^{\lambda+1}}dr < \left(\frac{\pi\lambda}{p_\eta(\infty,f)}(2-\eta)+\varepsilon\right)\int\limits_{2S_k}^{R_k}\frac{T(r,f)}{r^{\lambda+1}}dr.$$

Inserting (7) into this inequality, we obtain

$$\int\limits_{2S_k}^{R_k} \frac{\mathcal{L}(r,\infty,f)}{r^{\lambda+1}}dr < \frac{1}{\lambda}(1+\varepsilon)\left(\frac{\pi\lambda}{p_\eta(\infty,f)}(2-\eta)+\varepsilon\right)\int\limits_{2S_k}^{R_k}\frac{A(r,f)}{r^{\lambda+1}}dr, \quad k\to\infty.$$

Therefore there exists a sequence $r_k \in [2S_k, R_k]$ such that

$$\mathcal{L}(r_k,\infty,f) < \frac{1}{\lambda}\left[\frac{\pi\lambda}{p_\eta(\infty,f)}(2-\eta)+\varepsilon\right](1+\varepsilon)A(r_k,f), \quad k\to\infty.$$

Passing to the limit with $k\to\infty$ and $\varepsilon\to 0$, we obtain

$$b(\infty,f) \le \frac{\pi}{p_\eta(\infty,f)}(2-\eta).$$

This leads us to the statement in this case, as $p_\eta(\infty,f)$ takes only integral values.

Let's now assume that $\frac{\lambda}{p_\eta(\infty,f)} \le \frac{1}{2}$. Then $\pi \le \frac{\pi p_\eta(\infty,f)}{2\lambda}$. In the definition of $h_\eta(r,\lambda)$ we put $\alpha=\pi$ and $\psi=0$. Thus

$$h_\eta(r,\lambda) = \mathcal{L}(r,\infty,f) - \frac{\pi\lambda}{p_\eta(\infty,f)}T^*(r,\pi,u_\eta)\sin\frac{\pi\lambda}{p_\eta(\infty,f)} - \tilde{u}_\eta(r,\pi)\cos\frac{\pi\lambda}{p_\eta(\infty,f)}.$$

If $p_\eta(\infty,f)=1$ then the statement is obvious. Let then $p_\eta(\infty,f)\ge 2$. Then we have

$$h_\eta(r,\lambda)$$
$$= \mathcal{L}(r,\infty,f) - \frac{\pi\lambda}{p_\eta(\infty,f)}T^*(r,\pi,u_\eta)\sin\frac{\pi\lambda}{p_\eta(\infty,f)} - (1-\eta)T(r,f)\cos\frac{\pi\lambda}{p_\eta(\infty,f)}.$$

This leads us to inequality

$$\int\limits_{2S_k}^{R_k}\frac{\mathcal{L}(r,\infty,f)}{r^{\lambda+1}}dr$$

$$\le \int\limits_{2S_k}^{R_k}\frac{h_\eta(r,\lambda)+(2-\eta)\frac{\pi\lambda}{p_\eta(\infty,f)}T(r,f)\sin\frac{\pi\lambda}{p_\eta(\infty,f)}+(1-\eta)T(r,f)\cos\frac{\pi\lambda}{p_\eta(\infty,f)}}{r^{\lambda+1}}dr.$$

Applying lemma B, we get

$$\int\limits_{2S_k}^{R_k}\frac{\mathcal{L}(r,\infty,f)}{r^{\lambda+1}}dr$$

$$< [(2-\eta)\frac{\pi\lambda}{p_\eta(\infty,f)}\sin\frac{\pi\lambda}{p_\eta(\infty,f)} + (1-\eta)\cos\frac{\pi\lambda}{p_\eta(\infty,f)}+\varepsilon]\int\limits_{2S_k}^{R_k}\frac{T(r,f)}{r^{\lambda+1}}dr.$$

Inserting (7) into this inequality, we obtain

$$\int_{2S_k}^{R_k} \frac{\mathcal{L}(r,\infty,f)}{r^{\lambda+1}}dr$$

$$< \frac{(1+\varepsilon)}{\lambda}[(2-\eta)\frac{\pi\lambda}{p_\eta(\infty,f)}\sin\frac{\pi\lambda}{p_\eta(\infty,f)} + (1-\eta)\cos\frac{\pi\lambda}{p_\eta(\infty,f)} + \varepsilon]\int_{2S_k}^{R_k}\frac{A(r,f)}{r^{\lambda+1}}dr.$$

Therefore there exists a sequence $r_k \in [2S_k, R_k]$ such that

$$\lambda\mathcal{L}(r_k,\infty,f) < (1+\varepsilon)[(2-\eta)\frac{\pi\lambda}{p_\eta(\infty,f)} + (1-\eta)\cos\frac{\pi\lambda}{p_\eta(\infty,f)} + \varepsilon]A(r_k,f).$$

As the above inequality holds for any $\lambda > 0$ such that $\frac{\lambda}{p_\eta(\infty,f)} \le \frac{1}{2}$ we have

$$\lambda\frac{\mathcal{L}(r_k,\infty,f)}{A(r_k,f)} < (1+\varepsilon)[(2-\eta)\frac{\pi\lambda}{p_\eta(\infty,f)} + \varepsilon].$$

Passing to the limit with $k \to \infty$ and $\varepsilon \to 0$, we obtain the statement in this case. The proof for $p(\infty,f) = \infty$ can be conducted similarly [11].

We now consider the case when $f(z)$ is a meromorphic function of infinite lower order. Let $p_\eta(\infty,f) \ge 1$ and let p be the number from Lemma 3. We take j_0 such that for $j \ge j_0$ we have (4) and $\frac{p}{\mu_j} < 1$. In Lemma 3 we put $\psi = 0$ and $\alpha = \frac{\pi p}{2\mu_j}$. Then we have

$$h(r,\mu_j,p) = \mathcal{L}(r,\infty,f) - \frac{\pi\mu_j}{p}T^*(r,\alpha,u_\eta),$$

and

$$\int_{t_j}^{T_j}\frac{\mathcal{L}(r,\infty,f)}{r^{\mu_j+1}}dr = \int_{t_j}^{T_j}\frac{h(r,\mu_j,p) + \frac{\pi\mu_j}{p}T^*(r,\alpha,u_\eta)}{r^{\mu_j+1}}dr.$$

Since $T^*(r,\theta,u_\eta) \le (2-\eta)T(r,f)$ for all $\theta \in [0,\pi]$

$$\int_{t_j}^{T_j}\frac{\mathcal{L}(r,\infty,f)}{r^{\mu_j+1}}dr \le \int_{t_j}^{T_j}\frac{h(r,\mu_j,p) + \frac{\pi\mu_j}{p}(2-\eta)T(r,f)}{r^{\mu_j+1}}dr.$$

Hence, on the basis of Lemma 3

$$\int_{t_j}^{T_j}\frac{\mathcal{L}(r,\infty,f)}{r^{\mu_j+1}}dr < \left[\frac{\pi}{p}(2-\eta)+\varepsilon\right]\mu_j\int_{t_j}^{T_j}\frac{T(r,f)}{r^{\mu_j+1}}dr, \quad j \to \infty.$$

Using integration by parts and applying (3), we obtain

$$\mu_j \int_{t_j}^{T_j} \frac{T(r,f)}{r^{\mu_j+1}} dr = \frac{T(t_j,f)}{t_j^{\mu_j}} - \frac{T(T_j,f)}{T_j^{\mu_j}} + \int_{t_j}^{T_j} \frac{rT'_-(r,f)}{r^{\mu_j+1}} dr$$

$$< (1+\varepsilon) \int_{t_j}^{T_j} \frac{A(r,f)}{r^{\mu_j+1}} dr, \quad j \to \infty.$$

Thus

$$\int_{t_j}^{T_j} \frac{\mathcal{L}(r,\infty,f)}{r^{\mu_j+1}} dr < \left[\frac{\pi}{p}(2-\eta)+\varepsilon\right](1+\varepsilon) \int_{t_j}^{T_j} \frac{A(r,f)}{r^{\mu_j+1}} dr, \quad j \to \infty.$$

Therefore there is such a sequence $r_j \in [t_j, T_j]$ that

$$\mathcal{L}(r_j,\infty,f) < \left[\frac{\pi}{p}(2-\eta)+\varepsilon\right](1+\varepsilon)A(r,f). \tag{8}$$

The definition of the sequence (t_j) implies that $t_j \geq P_j = \rho_j e^{-\frac{M_j}{\mu_j}}$ where $\rho_j \to \infty$, $\frac{M_j}{\mu_j} \to 0$. The sequence $P_j \to \infty$ as $j \to \infty$. Thus $t_j \to \infty$ and $r_j \to \infty$ as $j \to \infty$. From the definition of $b(\infty, f)$ and from (8) we get

$$b(\infty,f) \leq \left[\frac{\pi}{p}(2-\eta)+\varepsilon\right](1+\varepsilon).$$

As it is true for any $\varepsilon > 0$, therefore for all numbers p such that $1 \leq p \leq p_\eta(\infty, f)$ we have

$$b(\infty,f) \leq \frac{\pi}{p}(2-\eta). \tag{9}$$

If $p_\eta(\infty,f) < \infty$ then, putting in (9) $p = p_\eta(\infty,f)$, we obtain the statement. If, on the other hand, $p_\eta(\infty,f) = \infty$ then the inequality (9) is true for all numbers $p \geq 1$. Hence in this case $b(\infty,f) = 0$. This completes the proof of Theorem 1.

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